A Monte-Carlo Option-Pricing Algorithm for Log-Uniform Jump-Diffusion Model

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***Abstract*— A reduced European call option pricing formula by risk-neutral valuation is given. It is shown that the European call and put options for jump-diffusion models are worth more than that for the Black-Scholes (diffusion) model with the common parameters. Due to the complexity of the jump-diffusion models, obtaining a closed option pricing formula like that of Black-Scholes is not viable. Instead, a Monte Carlo algorithm is used to compute European option prices. Monte Carlo variance reduction techniques such as both antithetic and control variates are used. The numerical results show that this is a practical, efficient and easily implementable algorithm.**

I. Background

The model the dynamics of the asset price $S(t)$ is the stochastic differential equation (SDE) :

$$dS(t) = S(t)\left(\mu dt + \sigma dW(t) + J(Q)dN(t)\right), \qquad (1)$$

where $S_0 = S(0) > 0$, μ is the drift coefficient, σ is the diffusive volatility, $W(t)$ is a Wiener process, $J(Q)$ is the jump-amplitude, Q is an underlying amplitude mark process such that $Q = \ln(J(Q)+1)$, $N(t)$ is the standard Poisson jump counting process with joint mean and variance $\mathrm{E}[N(t)] = \lambda t = \mathrm{Var}[N(t)]$. The jump term in (1) is a symbol for $S(t)J(Q)dN(t) = \sum_{k=1}^{dN(t)} S(T_k^-)J(Q_k)$, where T_k is the kth jump time, Q_k is the kth mark and $S(T_k^-) = \lim_{t\uparrow T_k} S(t)$.

Let the jump-amplitude mark density be uniform:

$$\phi_Q(q) = \frac{1}{b-a}\left\{\begin{array}{ll} 1, & a \le q \le b \\ 0, & \text{else} \end{array}\right\}, \qquad (2)$$

where $a < 0 < b$. The mark Q has mean $\mu_j \equiv \mathrm{E}_Q[Q] = 0.5(b+a)$ and variance $\sigma_j^2 \equiv \mathrm{Var}_Q[Q] = (b-a)^2/12$. The jump-amplitude J has mean

$$\bar{J} \equiv \mathrm{E}[J(Q)] = (\exp(b) - \exp(a))/(b-a) - 1. \qquad (3)$$

Note that in absence of any special explanation, $\overline{X}$ will denote the mean of random variable X, that is, $\overline{X} = E[X]$. For more details, see [8] and [10].

By the Itô chain rule [9] for jump-diffusions, the log-return process $\ln(S(t))$ satisfies the constant coefficient SDE

$$d\ln(S(t)) = (\mu - \sigma^2/2)dt + \sigma dW(t) + QdN(t),$$

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which can be immediately integrated and the logarithm inverted to yield the stock price solution

$$S(t) = S_0\exp((\mu - \sigma^2/2)t + \sigma W(t) + QN(t)), \qquad (4)$$

where $QN(t) = \sum_{k=1}^{N(t)} Q_k$, but is zero if $N(t) = 0$, and the Q_k here are independent identically uniformly distributed jump-amplitude marks See the jump-diffusion book [9, Chapter 5].

Our objective is to derive a reduced formula and practical algorithm for the discounted, expected European call option price $\mathcal{C}(S_0, T)$, a function of the current stock price S_0, the option expiration time T, the strike price K, the stock volatility σ, the risk-free interest rate r, but for jump-diffusions also depends on the jump rate λ and the mean jump amplitude $\bar{J}$. In contrast to the Black-Scholes [3] hedge for constructing a portfolio to eliminate the diffusion in the case of a pure diffusion process, Merton [17] argued that such hedging was not possible in the case of the jump-diffusion model, but the risk-neutral part of the Black-Scholes strategy could preserve the no arbitrage strategy to ensure that the discounted, expected return would be at the market rate r. This strategy can be formulated in terms of a change of the drift of jump-diffusion to a risk-neutral drift at rate r or more abstractly in terms of an equivalent change of measure to a risk-neutral measure, say $\mathcal{M}$. Consequently, the European call option price can be formulated as the discounted expectation of the terminal claim $\max[S(T) - K, 0]$,

$$\mathcal{C}(S_0, T) \equiv e^{-rT}\mathrm{E}_{\mathcal{M}}[\max[S(T) - K, 0]]. \qquad (5)$$

It is sufficient to know that such a risk-neutral measure exists. See the readable accounts in Baxter and Rennie [2] or Hull [12]for the pure diffusions, else Cont and Tankov [6] for the more general jump-diffusion cases. For statistical evidence of jumps in various financial markets see Ball and Torous [1], Jarrow and Rosenfeld [13] or Jorion [14].

II. Risk-Neutral Constant-Coefficient SDE

By the equation (4), the expected stock price at expiration time T is found in the following theorem:

Theorem 2.1: The **Expected Stock Price** is

$$\mathrm{E}[S(t)] = S_0 e^{(\mu + \lambda\bar{J})t}. \qquad (6)$$

Proof: Using the stock price solution (4), the IID property of Q_k given a jump in $N(t)$ and iterated expectations,

$$
\begin{aligned}
\mathrm{E}[S(t)] &= S_0 e^{(\mu-\sigma^2/2)t}\mathrm{E}\left[e^{\sigma W(t)}e^{\sum_{i=1}^{N(t)}Q_i}\right]\\
&= S_0 e^{(\mu-\sigma^2/2)t}\mathrm{E}_W\left[e^{\sigma W(t)}\right]\mathrm{E}_{N,Q}\left[\prod_{i=0}^{N(t)}e^{Q_i}\right]\\
&= S_0 e^{(\mu-\sigma^2/2)t}e^{\sigma^2 t/2}\mathrm{E}_{N,Q}\left[\prod_{i=1}^{N(t)}e^{Q_i}\right]\\
&= S_0 e^{\mu t}\mathrm{E}_N\left[\mathrm{E}_{Q|N}\left[\prod_{i=1}^{N(t)}e^{Q_i}\middle| N(t)\right]\right]\\
&= S_0 e^{\mu t}\sum_{k=0}^{\infty}p_k\mathrm{E}\left[\prod_{i=1}^{k}e^{Q_i}\right] = S_0 e^{\mu t}\sum_{k=0}^{\infty}p_k\prod_{i=1}^{k}\mathrm{E}\left[e^{Q}\right]\\
&= S_0 e^{\mu t}\sum_{k=0}^{\infty}p_k\mathrm{E}^k[J(Q)+1]\\
&= S_0 e^{\mu t}\sum_{k=0}^{\infty}e^{-\lambda t}\frac{\left(\lambda t\left(\bar{J}+1\right)\right)^k}{k!} = S_0 e^{(\mu+\lambda\bar{J})t},
\end{aligned}
$$

where the Poisson distribution $p_k(\lambda t) \equiv e^{-\lambda t}(\lambda t)^k/k!$ has been used. □

Assume the source of the jumps is due to extraordinary changes in the firm's specifics, such as the loss of a court suit or bankruptcy, but not from external events such as war. Thus, such jump components in the jump-diffusion model represent only non-systematic risks. The market-stock return correlation *beta* of the portfolio for non-systematic risk is constructed by *delta* hedging as in Black-Scholes and is zero (see [17]). Under this assumption, the jump-diffusion model (1) is arbitrage-free. In the risk-neutral world, $\mathrm{E}[S(t)] = S_0 e^{rt}$, so $S_0 e^{(\mu+\lambda\bar{J})t} = S_0 e^{rt}$ and solving for μ, yields the risk-neutral appreciation rate, $\mu = \mu_{\mathrm{rn}} = r-\lambda\bar{J}$. In the more general case with time-dependent coefficients, the expected instant rate is the risk-free rate and $\mathrm{E}[dS(t)/S(t)] = (\mu(t)+\mathrm{E}[J(Q,t)]\lambda(t))dt = r(t)dt$, leading to the risk-neutral mean rate relationship $\mu(t) = \mu_{\mathrm{rn}}(t) = r(t)-\mathrm{E}[J(Q,t)]\lambda(t)$.

Back to the constant coefficient case and substituting $\mu = r-\lambda\bar{J}$ into (1), we get the risk-neutral SDE under the risk-neutral measure $\mathcal{M}$ as the following:

$$
\begin{aligned}
dS(t)/S(t) &= \left(r-\lambda\bar{J}\right)dt+\sigma dW(t)+\sum_{k=1}^{dN(t)}J(Q_k)\\
&= rdt+\sigma dW(t)+\sum_{k=1}^{dN(t)}\left(J(Q_k)-\bar{J}\right)\\
&\quad +\bar{J}\left(dN(t)-\lambda dt\right),
\end{aligned}
$$

where the jump terms are separated into the zero-mean forms of the compound Poisson process.

III. Risk-Neutral Option Price Solutions

The risk-neutral property means that the asset grows at the market risk-less rate, here r in a constant market environment, so that the expected, discounted price of an asset $S(t)$ satisfies $\mathrm{E}[e^{-rt}S(t)] = S(0)$. In order to achieve this, the mean growth rate $\mathrm{E}[dS(t)/S(t)] = (\mu-\lambda\bar{J})dt$ is changed to the risk-neutral growth rdt, or equivalently the original probability measure needs to be changed to the risk-neutral measure $\mathcal{M}$. Using risk-neutral valuation of the payoff for the European call option in (5) with the stock price solution (4) and risk-neutral drift,

$$
\begin{aligned}
\mathcal{C}(S_0,T) &\equiv e^{-rT}\mathrm{E}_{\mathcal{M}}[\max(S(T)-K,0)]\\
&= \frac{e^{-rT}}{\sqrt{2\pi}}\sum_{k=0}^{\infty}p_k(\lambda T)\int_{ka}^{kb}\int_{Z_0(s_k)}^{\infty}\left(S_0 e^{DJ(z,s_k)}-K\right)\\
&\quad \cdot e^{-z^2/2}\phi_{\widetilde{\mathcal{S}}_k}(s_k)dz ds_k\\
&= \frac{1}{\sqrt{2\pi}}\sum_{k=0}^{\infty}p_k(\lambda T)\mathrm{E}_{\widetilde{\mathcal{S}}_k}\Big[\int_{Z_0(\widetilde{\mathcal{S}}_k)}^{\infty}\Big(S_0 e^{DJ(z,\widetilde{\mathcal{S}}_k)-rT}\\
&\quad -Ke^{-rT}\Big)e^{-z^2/2}dz\Big],
\end{aligned}
$$

where $DJ(z,s_k) \equiv (r-\lambda\bar{J}-\sigma^2/2)T+\sigma\sqrt{T}z+s_k$, $Z_0(s) \equiv (\ln(K/S_0)-(r-\lambda\bar{J}-\sigma^2/2)T-s)/(\sigma\sqrt{T})$ is the *at-the-money* value of the normal variable of integration z and $\widetilde{\mathcal{S}}_k = \sum_{i=1}^{k}Q_i$ is the sum of k jump amplitudes, such that Q_i are uniformly distributed IID random variables over the interval $[a,b]$ but $\widetilde{\mathcal{S}}_0 = \sum_{i=1}^{0}Q_i \equiv 0$. Splitting up the integral term, let

$$
\begin{aligned}
A(s) &\equiv \frac{1}{\sqrt{2\pi}}\int_{Z_0(s)}^{\infty}S_0 e^{-(\lambda\bar{J}+\sigma^2/2)T+\sigma\sqrt{T}z+s}e^{-z^2/2}dz\\
&= S_0 e^{s-\lambda\bar{J}T}\Phi\left(d_1\left(S_0 e^{s-\lambda\bar{J}T}\right)\right)
\end{aligned}
$$

and

$$
\begin{aligned}
B(s) &\equiv \frac{1}{\sqrt{2\pi}}\int_{Z_0(s)}^{\infty}Ke^{-rT}e^{-z^2/2}dz\\
&= Ke^{-rT}\Phi\left(d_2\left(S_0 e^{s-\lambda\bar{J}T}\right)\right),
\end{aligned}
$$

where $d_1(x) \equiv (\ln(x/K)+(r+\sigma^2/2)T)/(\sigma\sqrt{T})$ and $d_2(x) \equiv d_1(x)-\sigma\sqrt{T}$ are the usual Black-Scholes normal distribution argument functions, while $\Phi(y) \equiv \int_{-\infty}^{y}e^{-z^2/2}dz/\sqrt{2\pi}$ is the standardized normal distribution. Therefore,

$$
\begin{aligned}
\mathcal{C}(S_0,T) &= \sum_{k=0}^{\infty}p_k(\lambda T)\mathrm{E}_{\widetilde{\mathcal{S}}_k}[A(\widetilde{\mathcal{S}}_k)-B(\widetilde{\mathcal{S}}_k)]\\
&= \sum_{k=0}^{\infty}p_k(\lambda T)\mathrm{E}_{\widetilde{\mathcal{S}}_k}\Big[S_0 e^{\widetilde{\mathcal{S}}_k-\lambda\bar{J}T}\Phi\Big(d_1\Big(S_0 e^{\widetilde{\mathcal{S}}_k-\lambda\bar{J}T}\Big)\Big)\\
&\quad -Ke^{-rT}\Phi\Big(d_2\Big(S_0 e^{\widetilde{\mathcal{S}}_k-\lambda\bar{J}T}\Big)\Big)\Big].
\end{aligned}
$$

Alternatively,

$$
\begin{aligned}
\mathcal{C}(S_0,T) &= \sum_{k=0}^{\infty}p_k(\lambda T) \qquad (7)\\
&\quad \cdot\mathrm{E}_{\widetilde{\mathcal{S}}_k}\Big[\mathcal{C}^{(BS)}\Big(S_0 e^{\widetilde{\mathcal{S}}_k-\lambda\bar{J}T},T;K,\sigma^2,r\Big)\Big],
\end{aligned}
$$

where

$$
\mathcal{C}^{(BS)}(x,T;K,\sigma^2,r) \equiv x\Phi(d_1(x))-Ke^{-rT}\Phi(d_2(x))
$$

or briefly $\mathcal{C}^{(BS)}(x,T)$, is the Black-Scholes formula [3], but with the stock price argument shifted by a jump factor

$\exp(\widetilde{\mathcal{S}}_k - \lambda\bar{J}T)$. The above equation agrees with Merton's formula (16) in [17].

The next step is to compute

$$\mathrm{E}_{\widetilde{\mathcal{S}}_k}\left[\mathcal{C}^{(BS)}\left(S_0 e^{\widetilde{\mathcal{S}}_k-\lambda\bar{J}T}, T, K, \sigma^2, r\right)\right].$$

However, producing a simple analytical solution is difficult, since the probability density of the partial sums $\widetilde{\mathcal{S}}_k$ for the log-uniform model is very complicated, so this problem will be solved by high-level simulation techniques.

A. Put-Call Parity

Put-call parity is founded on basic maximum function properties (Merton [16], Hull [12] and Higham [11]), so is independent of the particular process and

$$\mathcal{C}(S_0,T) + Ke^{-rT} = \mathcal{P}(S_0,T) + S_0 \tag{8}$$

or solving for the European put option price,

$$\mathcal{P}(S_0,T) = \mathcal{C}(S_0,T) + Ke^{-rT} - S_0, \tag{9}$$

in absence of dividends.

IV. A Monte Carlo Algorithm

From (7), the European call option price formulae can be equivalently written as

$$\mathcal{C}(S_0,T) = \mathrm{E}_{\widehat{\mathcal{S}}(T)}\Big[\mathcal{C}^{(BS)}\Big(S_0 e^{\widehat{\mathcal{S}}(T)-\lambda\bar{J}T}, T\Big)\Big], \tag{10}$$

where $\widehat{\mathcal{S}}(T) = \sum_{i=1}^{N(T)} Q_i$, Q_i are uniformly distributed IID random variables from $[a,b]$. Note if $\widehat{\widehat{\mathcal{S}}}(T) \equiv \widehat{\mathcal{S}}(T) - \lambda T\bar{J}$, then $\exp(\widehat{\widehat{\mathcal{S}}}(T))$ is an exponential compound Poisson process with the exponential martingale property on $[0,T]$ that $\mathrm{E}[\exp(\widehat{\widehat{\mathcal{S}}}(T))] = \exp(\widehat{\widehat{\mathcal{S}}}(0)) = 1$. The Monte Carlo method may be a good choice to compute it numerically. For the treatment of Monte Carlo methods, see, e.g., [5], [7] or [11].

Let N_i be a sample point taken from the same Poisson distribution as $N(T)$, so that N_i for $i = 1:n$ sample points form a set of IID Poisson variates. Given an N_i jump, let the $U_{i,j}$ for $j = 1:N_i$ be jump amplitude sample points, so that they are IID uniformly generated on [0, 1], then

$$\widehat{\mathcal{S}}_i = \sum_{j=1}^{N_i}(a + (b-a)U_{i,j}) = aN_i + (b-a)\sum_{j=1}^{N_i} U_{i,j}$$

for $i = 1:n$ will be a set of IID random variables on [a, b] having the same compound Poisson distribution with uniformly distributed jump amplitudes as $\widehat{\mathcal{S}}(T)$. Based upon (10), an elementary Monte Carlo estimate for $\mathcal{C}(S_0,T)$ is

$$\widehat{\mathcal{C}}_n = \frac{1}{n}\sum_{i=1}^{n}\mathcal{C}^{(BS)}\left(S_0 e^{\widehat{\mathcal{S}}_i-\lambda\bar{J}T}, T\right) \equiv \frac{1}{n}\sum_{i=1}^{n}\mathcal{C}_i^{(BS)},$$

such that the $\mathcal{C}_i^{(BS)}$ are IID random variables based on $\widehat{\mathcal{S}}_i$. Then, by the strong law of large numbers,

$$\widehat{\mathcal{C}}_n \to \mathcal{C}(S_0,T) \quad \text{with probability one as} \quad n \to \infty,$$

and by the IID property of $\mathcal{C}_i^{(BS)}$, the standard deviation $\sigma_{\widehat{\mathcal{C}}_n} = \sigma^{(BS)}/\sqrt{n}$, where

$$\sigma^{(BS)} = \sqrt{\mathrm{Var}\Big[\mathcal{C}^{(BS)}(S_0 e^{\widehat{\mathcal{S}}(T)-\lambda\bar{J}T}, T)\Big]} = \sqrt{\mathrm{Var}\Big[\mathcal{C}_i^{(BS)}\Big]},$$

but may be estimated by the unbiased sample variance

$$s^{(BS)} = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}\left(\mathcal{C}_i^{(BS)} - \widehat{\mathcal{C}}_n\right)^2}.$$

In order to reduce the standard deviation $\sigma_{\widehat{\mathcal{C}}_n}$ by a factor of ten, the number of simulations n has to be increased one hundredfold. However, there are alternative approaches to reduce the size of $\sigma^{(BS)}$ by variance reduction techniques.

Thus, the Monte Carlo simulations will be used with antithetic variate and control variate variance reduction techniques. Let

$$\begin{aligned} X_i &= \frac{1}{2}\Big(\mathcal{C}^{(BS)}\Big(S_0 e^{\widehat{\mathcal{S}}_i-\lambda\bar{J}T}, T\Big) + \mathcal{C}^{(BS)}\Big(S_0 e^{\widehat{\mathcal{S}}_i^{(a)}-\lambda\bar{J}T}, T\Big)\Big) \\ &\equiv 0.5\Big(\mathcal{C}_i^{(BS)} + \mathcal{C}_i^{(aBS)}\Big), \end{aligned}$$

for $i = 1:n$ is the thetic-antithetic averaged, Black-Scholes risk-neutral, discounted payoff and

$$Y_i = 0.5\left(\exp(\widehat{\mathcal{S}}_i) + \exp\left(\widehat{\mathcal{S}}_i^{(a)}\right)\right)$$

is the thetic-antithetic averaged jump factors and a variance reducing control variate. The control adjusted payoff is

$$Z_i(\alpha) = X_i - \alpha\cdot(Y_i - \exp(\lambda T\bar{J})),$$

where $(Y_i - \exp(\lambda T\bar{J}))$ is the control deviation and α is an adjustable control parameter. The sample mean of $Z_i(\alpha)$ produces the Monte Carlo estimator for $\mathcal{C}(S_0,T)$: $\overline{Z}_n(\alpha) = \sum_{i=1}^{n} Z_i(\alpha)/n = \sum_{i=1}^{n} X_i/n - \alpha\sum_{i=1}^{n}(Y_i - \exp(\lambda T\bar{J}))/n = \overline{X}_n - \alpha(\overline{Y}_n - \exp(\lambda T\bar{J}))$, an unbiased estimation since $\mathrm{E}[\overline{Z}_n(\alpha)] = \mathcal{C}(S_0,T)$ using IID mean properties $\mathrm{E}[\overline{X}_n] = \mathrm{E}[X_i] = \mathcal{C}(S_0,T)$ by (10) and $\mathrm{E}[\overline{Y}_n] = \mathrm{E}[Y_i] = \exp(\lambda T\bar{J})$ from the proof of Thm. 2.1.

The variance of the sample mean $\overline{Z}_n(\alpha)$ is

$$\sigma^2_{\overline{Z}_n(\alpha)} \equiv \mathrm{Var}\left[\overline{Z}_n(\alpha)\right] = \mathrm{Var}[Z_i(\alpha)]/n,$$

following from IID property of the $Z_i(\alpha)$. However,

$$\mathrm{Var}[Z_i(\alpha)] = \mathrm{Var}[X_i] - 2\alpha\mathrm{Cov}[X_i,Y_i] + \alpha^2\mathrm{Var}[Y_i].$$

So, the optimal parameter α^* to minimize $\mathrm{Var}[Z_i(\alpha)]$ is

$$\alpha^* = \mathrm{Cov}[X_i,Y_i]/\mathrm{Var}[Y_i]. \tag{11}$$

Using this optimal parameter α^*,

$$\begin{aligned} \mathrm{Var}[Z_i^*] \equiv \mathrm{Var}[Z_i(\alpha^*)] &= \mathrm{Var}[X_i] - \frac{\mathrm{Cov}^2[X_i,Y_i]}{\mathrm{Var}[Y_i]} \\ &\equiv \left(1 - \rho^2_{X_i,Y_i}\right)\mathrm{Var}[X_i], \end{aligned}$$

where ρ_{X_i,Y_i} is the correlation coefficient between X_i and Y_i. We also know that

$$\begin{aligned}\mathrm{Var}[X_i] &= \frac{1}{4}\left(\mathrm{Var}\left[\mathcal{C}_i^{(BS)}\right]+2\mathrm{Cov}\left[\mathcal{C}_i^{(BS)},\mathcal{C}_i^{(aBS)}\right]\right.\\ &\quad\left.+\mathrm{Var}\left[\mathcal{C}_i^{(aBS)}\right]\right)\\ &= \frac{1}{2}\left(1+\rho_{\mathcal{C}_i^{(BS)},\mathcal{C}_i^{(aBS)}}\right)\mathrm{Var}\left[\mathcal{C}_i^{(BS)}\right]\end{aligned}$$

because $\mathrm{Var}\left[\mathcal{C}_i^{(aBS)}\right]=\mathrm{Var}\left[\mathcal{C}_i^{(BS)}\right]$. Therefore,

$$\begin{aligned}\mathrm{Var}[Z_i^*] &= \frac{1}{2}\left(1-\rho^2_{X_i,Y_i}\right)\left(1+\rho_{\mathcal{C}_i^{(BS)},\mathcal{C}_i^{(aBS)}}\right)\\ &\quad\cdot\mathrm{Var}\left[\mathcal{C}_i^{(BS)}\right]\le\frac{1}{2}\mathrm{Var}\left[\mathcal{C}_i^{(BS)}\right]\end{aligned}\qquad(12)$$

because $\rho^2_{X_i,Y_i}\ge 0$ and provided $\rho_{\mathcal{C}_i^{(BS)},\mathcal{C}_i^{(aBS)}}\le 0$. From (12), $\sigma^2_{\widehat{Z}_n}\le\mathrm{Var}[\mathcal{C}_i^{(BS)}]/(2n)=(\sigma_{\widehat{\mathcal{C}}_n})^2/2$. This says the variance of the Monte Carlo estimate with antithetic and control variates techniques is at most the half as the variance of the elementary Monte Carlo estimate if $\rho_{\mathcal{C}_i^{(BS)},\mathcal{C}_i^{(aBS)}}\le 0$.

Remark: In a real market, the ratio a/b will be close to -1, that is $b+a$ will be very small since the skewness of the daily return distribution is not far away from 0 and the skewness is generated by the jump part of the jump-diffusion model. For example, the skewness is -0.1952 for 1988-2003 S&P 500 daily return market data and $a/b=-1.08$ and $a+b=-0.002$ [18]. In fact, in our Monte-carlo algorithm, the $\rho_{\mathcal{C}_i^{(BS)},\mathcal{C}_i^{(aBS)}}$ is about -0.83. So, we can get a lot of benefit from the antithetic variate variance reduction method by equation (12). In fact, our simulations using uniformly distributed jump amplitudes confirms that $\mathrm{Cov}[\mathcal{C}_i^{(BS)},\mathcal{C}_i^{(aBS)}]<0$ in the range of the ratio $-3.75<a/b<-0.25$ with $a=-0.028$ which is well within the range of market data. However, if b/a is far away from -1, the correlation coefficient $\mathrm{Cov}[\mathcal{C}_i^{(BS)},\mathcal{C}_i^{(aBS)}]$ can be positive which will worsen the variance, though this range is not realistic.

In general, we do not know the parameter α^* exactly, so some estimation is needed for it and we need the following Lemma.

Lemma 4.1:

$$\mathrm{Var}\left[e^{\widehat{\mathcal{S}}_i}+e^{\widehat{\mathcal{S}}_i^{(a)}}\right]=2\left(e^{\lambda T\hat{J}}-2e^{2\lambda T\bar{J}}+e^{\lambda T(e^{a+b}-1)}\right),$$

where $\hat{J}=(\exp(2b)-\exp(2a))/(2(b-a))-1$ and $\bar{J}=(\exp(b)-\exp(a))/(b-a)-1$ from (3).

Proof: Using the properties of the antithetic pair $(\widehat{\mathcal{S}}_i,\widehat{\mathcal{S}}_i^{(a)})$,

$$\begin{aligned}\mathrm{Cov}\left[e^{\widehat{\mathcal{S}}_i},e^{\widehat{\mathcal{S}}_i^{(a)}}\right] &= \mathrm{E}\left[e^{\widehat{\mathcal{S}}_i}e^{\widehat{\mathcal{S}}_i^{(a)}}\right]-\mathrm{E}\left[e^{\widehat{\mathcal{S}}_i}\right]E\left[e^{\widehat{\mathcal{S}}_i^{(a)}}\right]\\ &= \mathrm{E}\left[e^{(a+b)N(T)}\right]-\mathrm{E}^2\left[e^{\widehat{\mathcal{S}}_i}\right]\\ &= e^{\lambda T(e^{a+b}-1)}-e^{2\lambda T\bar{J}}\end{aligned}$$

and $\mathrm{Var}[e^{\widehat{\mathcal{S}}_i}]=\mathrm{E}[e^{2\widehat{\mathcal{S}}_i}]-\mathrm{E}^2[e^{\widehat{\mathcal{S}}_i}]=e^{\lambda T\hat{J}}-e^{2\lambda T\bar{J}}=\mathrm{Var}[e^{\widehat{\mathcal{S}}_i^{(a)}}]$. Thus, $\mathrm{Var}[e^{\widehat{\mathcal{S}}_i}+e^{\widehat{\mathcal{S}}_i^{(a)}}]=\mathrm{Var}[e^{\widehat{\mathcal{S}}_i}]+2\mathrm{Cov}[e^{\widehat{\mathcal{S}}_i},e^{\widehat{\mathcal{S}}_i^{(a)}}]+\mathrm{Var}[e^{\widehat{\mathcal{S}}_i^{(a)}}]=2\mathrm{Var}[e^{\widehat{\mathcal{S}}_i}]+2\mathrm{Cov}[e^{\widehat{\mathcal{S}}_i},e^{\widehat{\mathcal{S}}_i^{(a)}}]=2(e^{\lambda T\hat{J}}-2e^{2\lambda T\bar{J}}+e^{\lambda T(e^{a+b}-1)})$. □

From Lemma 4.1, $\sigma_Y^2\equiv\mathrm{Var}[Y_i]=\mathrm{Var}[0.5(\exp(\widehat{\mathcal{S}}_i)+\exp(\widehat{\mathcal{S}}_i^{(a)}))]=0.5(\exp(\lambda T\hat{J})-2\exp(2\lambda T\bar{J})+\exp(\lambda T(\exp(a+b)-1)))$.

Proposition 4.1: An unbiased estimator for α^* is

$$\begin{aligned}\widehat{\alpha} &= \left(\frac{1}{n-1}\sum_{i=1}^n X_iY_i-\frac{1}{n(-1)}\sum_{i=1}^n\sum_{j=1}^n X_iY_j\right)\frac{1}{\sigma_Y^2}\\ &= \frac{n}{n-1}\frac{\overline{XY}_n-\overline{X}_n\overline{Y}_n}{\sigma_Y^2},\end{aligned}\qquad(13)$$

where $\overline{X}_n=\sum_{i=1}^n X_i/n$ is the sample mean, $\overline{XY}_n$ and $\overline{Y}_n$ have the similar meaning.

Proof: It is necessary to show the condition for an unbiased estimate $E[\widehat{\alpha}]=\alpha^*$ is true. Splitting the common part out of the double sum and the IID property of the random variables at different compound Poisson sample points for $i=1:n$,

$$\begin{aligned}\mathrm{E}[\widehat{\alpha}] &= \mathrm{E}\left[\frac{1}{n-1}\sum_{i=1}^n\left(X_iY_i-\frac{1}{n}\sum_{j=1}^n X_iY_j\right)\frac{1}{\sigma_Y^2}\right]\\ &= \frac{1}{n-1}\sum_{i=1}^n\mathrm{E}\left[\left(1-\frac{1}{n}\right)X_iY_i-\frac{1}{n}\sum_{j=1,j\neq i}^n X_iY_j\right]\frac{1}{\sigma_Y^2}\\ &= \frac{1}{n(n-1)}\sum_{i=1}^n\left((n-1)\mathrm{E}[X_iY_i]-\sum_{j=1,j\neq i}^n\mathrm{E}[X_i]\mathrm{E}[Y_j]\right)\frac{1}{\sigma_Y^2}\\ &= \left(\mathrm{E}[XY]-\mathrm{E}[X]\mathrm{E}[Y]\right)/\sigma_Y^2=\mathrm{Cov}[X,Y]/\sigma_Y^2=\alpha^*.\end{aligned}$$

□

Since $\widehat{\alpha}$ depends on Y_i for $i=1:n$, the estimate $\widehat{\alpha}$ of α^* introduces a bias into the estimate

$$\widehat{Z}_n=\frac{1}{n}\sum_{i=1}^n X_i-\widehat{\alpha}\left(\frac{1}{n}\sum_{i=1}^n Y_i-e^{\lambda T\bar{J}}\right).\qquad(14)$$

Fortunately, we can compute the bias which asymptoticly goes to zero at the rate $O(1/n)$ as shown in the following theorem.

Theorem 4.1: The estimate $\widehat{Z}_n$ of $\mathcal{C}(S_0,T)$ has bias

$$\mathcal{B}\equiv\mathrm{E}[\widehat{Z}_n]-\mathcal{C}(S_0,T)=\mathrm{Cov}[X,(2\mu_Y-Y])Y]]/(n\sigma_Y^2),$$

where $\mu_Y=\mathrm{E}[Y_i]=E[Y]=\exp(\lambda T\bar{J})$, $\sigma_Y^2=\mathrm{Var}[Y_i]=\mathrm{Var}[Y]$, Y has the same distribution as Y_i, for $i=1{:}n$.

Proof: Set $\eta_k=\sigma_Y^2\widehat{\alpha}(Y_k-\mu_Y)$. Then,

$$\begin{aligned}\eta_k &= \left(\frac{\sum_{i=1}^n X_iY_i}{n-1}-\frac{\sum_{i=1}^n\sum_{j=1}^n X_iY_j}{n(n-1)}\right)(Y_k-\mu_Y)\\ &= \frac{1}{n}\sum_{i=1}^n X_iY_iY_k-\frac{\sum_{i=1}^n\sum_{j\neq i}X_iY_jY_k}{n(n-1)}\\ &\quad-\frac{\mu_Y\sum_{i=1}^n X_iY_i}{n}+\frac{\mu_Y\sum_{i=1}^n\sum_{j\neq i}X_iY_j}{n(n-1)}\\ &= \frac{X_kY_k^2+\sum_{i\neq k}X_iY_iY_k}{n}-\\ &\quad\frac{\sum_{j\neq k}X_kY_jY_k+\sum_{i\neq k}X_iY_k^2+\sum_{i\neq k}\sum_{j\neq i,k}X_iY_jY_k}{n(n-1)}\\ &\quad-\frac{\mu_Y\sum_{i=1}^n X_iY_i}{n}+\frac{\mu_Y\sum_{i=1}^n\sum_{j\neq i}X_iY_j}{n(n-1)}.\end{aligned}$$

By the independence of $\{X_i,Y_i\}$ and $\{X_j,Y_j\}$ for $j\neq i$ but with identical distributions, $\mathrm{E}[\eta_k]=(\overline{XY^2}+(n-1)\overline{XY}\mu_Y)/n-$

$((n-1)\overline{XY}\mu_Y+(n-1)\mu_X\overline{Y^2}+(n-1)(n-2)\mu_X{\mu_Y}^2)/(n(n-1))-\mu_Y\overline{XY}+\mu_Y^2\mu_X=(\overline{XY^2}-2\overline{XY}\mu_Y-\mu_X\overline{Y^2}+2\mu_X\mu_Y^2)/n=\mathrm{Cov}[X,Y^2]-2\mu_Y\mathrm{Cov}[X,Y]/n=\mathrm{Cov}[X,Y(Y-2\mu_Y)]/n$, where $\mu_X=\mathrm{E}[X_i]$, $\mu_Y=\mathrm{E}[Y_i]$, $\overline{XY}=\mathrm{E}[X_iY_i]$, $\overline{Y^2}=\mathrm{E}[Y_i^2]$ and $\overline{XY^2}=\mathrm{E}[X_iY_i^2]$. Therefore, the bias $\mathcal{B}\equiv\mathrm{E}[\widehat{Z}_n]-\mathcal{C}(S_0,T)=\mathrm{E}[-\widehat{\alpha}(Y_k-\mu_Y)]=-\mathrm{E}[\sigma_Y^2\widehat{\alpha}(Y_k-\mu_Y)]/\sigma_Y^2=-\mathrm{E}[\eta_k]/\sigma_Y^2=\mathrm{Cov}[X,Y(2\mu_Y-Y)]/(n\sigma_Y^2)$. □

Remark: From Theorem 4.1, the corrected estimate to $\widehat{Z}_n$ is $\widehat{\mathcal{Z}}\equiv\widehat{Z}_n-\widehat{\mathcal{B}}$, where $\widehat{\mathcal{B}}$ is an estimate of $\mathcal{B}$ similar to $\widehat{\alpha}$ in (13) ,

$$\widehat{\mathcal{B}}=\left(\frac{1}{n(n-1)}\sum_{i=1}^{n}X_iY_i^{'}-\frac{1}{n^2(n-1)}\sum_{i=1}^{n}\sum_{j=1}^{n}X_iY_j^{'}\right)\frac{1}{\sigma_Y^2}$$
$$=\frac{1}{n-1}\frac{\overline{XY'_n}-\overline{X}_n\overline{Y'}_n}{\sigma_Y^2}\,, \tag{15}$$

where $Y_i^{'}=Y_i(2\mu_Y-Y_i)$, for $i=1\!:\!n$, $\overline{XY'_n}$, $\overline{X}_n$ and $\overline{Y'}_n$ are sample means. Then, the estimate $\widehat{\mathcal{Z}}$ is an unbiased estimate of $\mathcal{C}(S_0,T)$.

Finally, our Monte Carlo algorithm with antithetic and control variates variance reduction techniques is:

The Monte Carlo Algorithm:

for $i=1\!:\!n$

  Randomly generate N_i;

  Randomly generate IID $U_{i,j}$, $j=1\!:\!N_i$;

  Set $\widehat{\mathcal{S}}_i=aN_i+(b-a)\sum_{j=1}^{N_i}U_{i,j}$;

  Set $\widehat{\mathcal{S}}_i^{(a)}=(a+b)N_i-\widehat{\mathcal{S}}_i$;

  Set $\mathcal{C}_i^{(BS)}=\mathcal{C}^{(BS)}\left(S_0\exp\left(\widehat{\mathcal{S}}_i-\lambda T\bar{J}\right),T\right)$;

  Set $\mathcal{C}_i^{(aBS)}=\mathcal{C}^{(BS)}\left(S_0\exp\left(\widehat{\mathcal{S}}_i^{(a)}-\lambda T\bar{J}\right),T\right)$;

  Set $X_i=0.5\left(\mathcal{C}_i^{(BS)}+\mathcal{C}_i^{(aBS)}\right)$;

  Set $Y_i=0.5\left(\exp(\widehat{\mathcal{S}}_i)+\exp\left(\widehat{\mathcal{S}}_i^{(a)}\right)\right)$;

end for i

Compute $\widehat{\alpha}$ according to (13);

Set $\widehat{Z}_n=\frac{1}{n}\sum_{i=1}^{n}X_i-\widehat{\alpha}(\frac{1}{n}\sum_{i=1}^{n}Y_i-e^{\lambda T\bar{J}})$;

Estimate bias $\widehat{b}$ according to (15);

Get European call $\widehat{\theta}=\widehat{Z}_n-\widehat{b}$;

Get European put $\widehat{\mathcal{P}}$ by (9).

V. Numerical Results and Discussions

In this section, some numerical results and discussions are given to illustrate the Monte Carlo algorithm. First of all, the elementary Monte Carlo method and the Monte Carlo method with antithetic and control variates techniques (abbreviated as AOCV) are compared. The compound Poisson process is simulated by first using the inverse transform method given by Glasserman ([7]) for the jump counting component process N_i and then the N_i jump amplitude antithetic pairs $(\widehat{\mathcal{S}}_i,\widehat{\mathcal{S}}_i^{(a)})$ are simulated by a standard uniform random number generator to get the $U_{i,j}$. These are implemented using MATLAB 6.5 and run them on the PC with a Pentium4@1.6GHz CPU. The numerical test results for elementary Monte Carlo method are listed in Table I and the Monte carlo with AOCV's are listed in Table II.

The results in Table I and Table II show that the antithetic variates combined with control variates can reduce the standard error by a factor ranging from 2 to about 14,

TABLE I

Numerical Results of elementary Monte Carlo Method

σ	K/S_0	$\mathcal{C}$	$\mathcal{P}$	ϵ	t (sec.)	$\epsilon\sqrt{t}$
0.2	0.9	13.76	0.67	0.055	2.640	0.090
	1.0	5.26	3.28	0.035	2.578	0.056
	1.1	1.38	8.49	0.014	2.562	0.022
0.4	0.9	15.99	2.90	0.048	2.562	0.077
	1.0	8.45	6.47	0.033	2.578	0.053
	1.1	4.07	11.18	0.020	2.531	0.032
0.6	0.9	19.15	6.03	0.044	2.454	0.069
	1.0	11.79	9.81	0.033	2.500	0.052
	1.1	7.09	14.21	0.023	2.500	0.036

Option parameters: $K=100$, $r=0.1$, $T=0.2$, $\lambda=64$, $a=-0.028$, $b=0.026$. Simulation number $n=10,000$. Here, $\epsilon=\sigma_{\widehat{C}_n}=\sigma^{(BS)}/\sqrt{n}$.

TABLE II

Numerical Results of Improved Monte Carlo with AOCV

σ	K/S_0	$\mathcal{C}$	$\mathcal{P}$	ϵ	t (sec.)	$\epsilon\sqrt{t}$
0.2	0.9	13.73	0.64	0.004	6.875	0.011
	1.0	5.23	3.25	0.008	6.828	0.021
	1.1	1.38	8.49	0.006	6.781	0.016
0.4	0.9	16.03	2.94	0.004	7.031	0.011
	1.0	8.42	6.44	0.004	6.922	0.011
	1.1	4.06	11.17	0.004	7.218	0.011
0.6	0.9	19.11	6.02	0.003	6.797	0.008
	1.0	11.81	9.83	0.003	6.859	0.008
	1.1	7.12	14.23	0.003	6.812	0.008

Option parameters: $K=100$, $r=0.1$, $T=0.2$, $\lambda=64$, $a=-0.028$, $b=0.026$. Simulation number $n=10,000$. Here, $\epsilon=\sigma_{\widehat{Z}_n}=\sigma_Z/\sqrt{n}$.

but also increases the computing time by 2 to 3 times. Therefore, we use standard error multiplying square root of computng time $\epsilon\sqrt{t}$ as a benchmark for the trade-off between the estimated variance and computing time, for a detailed explanation, see Boyle, Broadie and Glasserman [5]. Seen from these results, the Monte Carlo method with AOCV is an overall the better estimate than the elementary Monte Carlo method. Also, these results show that the European call option price is an increasing function of S_0 and the European put option is a decreasing function of it. Both the call and put option prices increase as the volatility σ of stock price increase. The estimated model parameters used are $\mu=0.1626$, $\sigma=0.1074$, $\lambda=64.16$, $a=-0.028$, $b=0.026$ from our double-unform distribution paper [18] to compute the Standard & Poor 500 index option prices. Also, we compute Black-Scholes call price $\mathcal{C}^{(BS)}(S_0,T;K,\sigma^2,r)$ and the put price $\mathcal{P}^{(BS)}(S_0,T;K,\sigma^2,r)=\mathcal{C}^{(BS)}(S_0,T;K,\sigma^2,r)+K\exp(-rT)-S_0$ as a rough estimation of the true values. The numerical results are listed in Table III.

The numerical results in Table III show that the estimated call $\mathcal{C}$ and put $\mathcal{P}$ values by the Monte Carlo method with AOCV are within the 95% confidence interval of the true call $\mathcal{C}^*$ and put $\mathcal{P}^*$ values, i.e., $\mathcal{C}\in[\mathcal{C}^*-1.96\epsilon,\mathcal{C}^*+1.96\epsilon]$ or $\mathcal{P}\in[\mathcal{P}^*-1.96\epsilon,\mathcal{P}^*+1.96\epsilon]$ by the central limit theorem. Also, we observe that the estimated European call and put option prices are bigger than the Black-Scholes call and put option prices, respectively. This is a fact stated in the following theorem.

Theorem 5.1: The European call and put option prices

TABLE III

NUMERICAL RESULTS FOR S&P 500 OPTION PRICES

$\frac{K}{S_0}$	$\mathcal{C}$	$\mathcal{P}$	ϵ	$\mathcal{C}^{(\mathcal{BS})}$	$\mathcal{P}^{(\mathcal{BS})}$	$\mathcal{C}^*$	$\mathcal{P}^*$
0.8	269.81	0.01	2.e-3	269.80	2.e-6	269.82	0.02
0.9	132.36	1.45	0.03	130.98	0.07	132.39	1.47
1.0	40.07	20.27	0.11	30.49	10.69	40.05	20.25
1.1	5.49	76.60	0.06	1.13	72.24	5.50	76.61
1.2	0.31	147.17	0.01	4.e-3	146.87	0.32	147.19

Option parameters: $K = 1000$, $r = 0.1$, $T = 0.2$, $\sigma = 0.1074$, $\lambda = 64$, $a = -0.028$, $b = 0.026$. Simulation number $n = 10,000$. Here, $\epsilon = \sigma_{\widehat{Z}_n} = \sigma_Z/\sqrt{n}$. The call and put values are estimated by the Monte Carlo method with AOCV. The $\mathcal{C}^*$ and $\mathcal{P}^*$ values are obtained by more simulations, say $n = 400,000$ sample points.

based on the jump-diffusion model in (1) are bigger than the Black-Scholes call and put option prices, respectively, i.e., $\mathcal{C}(S_0, T; K, \sigma^2, r) \geq \mathcal{C}^{(BS)}(S_0, T; K, \sigma^2, r)$, and $\mathcal{P}(S_0, T; K, \sigma^2, r) \geq \mathcal{P}^{(BS)}(S_0, T; K, \sigma^2, r)$.

Proof: Since the Black-Scholes call option pricing formula $\mathcal{C}^{(BS)}(S, T; K, \sigma^2, r)$ is a convex function about S. By Jensen's inequality (see [9] for instance), we have

$$\begin{aligned}\mathcal{C}(S_0, T; K, \sigma^2, r) &\overset{(10)}{=} E_{\widehat{\mathcal{S}}(T)}\left[\mathcal{C}^{(BS)}\left(S_0 e^{\widehat{\mathcal{S}}(T)-\lambda\bar{J}T}, T\right)\right] \\ &\geq \mathcal{C}^{(BS)}\left(E_{\widehat{\mathcal{S}}(T)}[S_0 e^{\widehat{\mathcal{S}}(T)-\lambda\bar{J}T}], T\right) \\ &= \mathcal{C}^{(BS)}\left(S_0, T\right).\end{aligned}$$

By put-call parity and the above proven inequality,

$$\begin{aligned}\mathcal{P}(S_0, T; K, \sigma^2, r) &= \mathcal{C}(S_0, T; K, \sigma^2, r) + Ke^{-rT} - S_0 \\ &\geq \mathcal{C}^{(BS)}(S_0, T; K, \sigma^2, r) + Ke^{-rT} - S_0 \\ &= \mathcal{P}^{(BS)}(S_0, T; K, \sigma^2, r).\end{aligned}$$

□

Remark: In the proof of the Theorem 5.1, no special distribution of Q in the Jump-Diffusion model (1) is used. Hence, this is a general result also suitable for log-normal [17], log-double-exponential [15] and log-double-uniform [18] jump amplitude models for jump-diffusions.

VI. CONCLUSION

The original SDE has been transformed to a risk-neutral SDE by setting the stock price increases at the risk-neutral interest rate. Based on this risk-neutral SDE, a reduced European call option pricing formula is derived. Then, a Monte Carlo algorithm with both antithetic and control variate variance reduction techniques are applied. This algorithm is easy to implement and the numerical results show that it is also efficient, taking less than 8 seconds per case to get the practical accuracy. Finally, we show that the European call and put option prices based on the jump-diffusion model in (1) are bigger than the Black-Scholes call and put option prices, respectively.

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