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State Dependent Jump Models in Optimal Control

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Abstract

Models of linear and nonlinear optimal control applications are considered in which random discrete jumps in the system are state dependent in both rate and amplitude. These discrete jumps are treated as a Poisson processes in continuous time. This type of random noise allows for greater realism while modeling industrial and natural phenomena in which important changes occur with jumps. Modeling concerns are described and the appropriate modifications are indicated for numerically solving the resulting optimal control problems. Applications to a multistage manufacturing system and to the management of a natural resource under stochastic price fluctuations are used to illustrate this type of dynamical formulation.

1. Introduction

Many dynamical systems that model real phenomena undergo large random fluctuations in the state. These can be rare events or catastrophes or machine failure that are responsible for important changes due to discrete state dependent jumps. These rare events or catastrophes, have a great influence on how the biological, physical or other system evolves. In order to accommodate these jumps, a state dependent Poisson process is employed. A general martingale can be used as well. Some examples of these types of systems are populations subject to bonanzas or mass moralities [9, 5, 7, 6], manufacturing systems subject to machine failure and repair [10, 12, 14], biomedical systems [11], and many other systems.

The use of stochastic differential equations to model dynamical systems has an inherent problem of insuring that the stochastic processes do not cause the state or the control of the system to become inadmissible. That is, the influence of the stochastic processes over time may cause the value for the state or the control to no longer be contained in their domain. In some applications, this can be clearly seen along boundaries of the domain. Also, the computational algorithm must contain checks and remedies to insure that the value of the state and the control remain admissible. The use of state dependent Poisson processes helps to facilitate or even force the admissibility of the state and the control.

The paper is arranged as follows. In Section 2., the formulation and moments of the state dependent Poisson process or noise is described. Examples featuring the state dependent Poisson noise are presented in Section 3. for a multistage manufacturing system and in Section 4. for natural resource and price dynamics. In Section 5., canonical models for linear (LQGP problem) and nonlinear (LQGP/U problem) optimal control problems and the resulting Hamilton-Jacobi-Bellman (HJB) equations are presented. Computational considerations for numerically solving these HJB equations is briefly described in Section 6.

2. Poisson Process

In previous work [10, 12, 13, 14], the following state independent vector valued *marked Poisson noise* is used, $d\mathbf{P}(t) = [dP_i(t)]_{q\times 1}$, which consists of q independent differentials of space-time Poisson processes that are related to the Poisson random measure, $\mathcal{P}_i(dz_i, dt)$, (see Gihman and Skorohod [3] or Hanson [4]; see Hanson and Tuckwell [7] for some biological examples):

$$dP_i(t) = \int_{\mathcal{Z}_i} z_i \mathcal{P}_i(dz_i, dt), \tag{1}$$

where z_i is the Poisson jump amplitude random variable or the mark of the $dP_i(t)$ Poisson process where $i = 1$ to q, with mean or expectation:

$$\text{Mean}[d\mathbf{P}(t)] = \Lambda\overline{\mathbf{Z}}dt, \tag{2}$$

where $\Lambda(t)$ is the diagonal matrix representation of the Poisson rates $\lambda_i(t)$ for $i = 1$ to q, $\overline{\mathbf{Z}}(t)$ is the mean of the jump amplitude mark vector and $\phi_i(z_i, t)$ is the density of the ith

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amplitude mark component. Assuming component-wise independence, $d\mathbf{P}(t)$ has covariance given by:

$$\mathrm{Covar}[d\mathbf{P}(t), d\mathbf{P}^{\top}(t)] = \Lambda(t)\sigma(t)dt \tag{3}$$

with $\sigma(t) = [\sigma_{i,j}\delta_{i,j}]_{q\times q}$ denoting the diagonalized covariance of the amplitude mark distribution for $d\mathbf{P}(t)$. Note that the mark vector is not assumed to have a zero mean, i.e., $\overline{\mathbf{Z}} \neq 0$, permitting additional modeling complexity. The marked Poisson process is such that the jump amplitudes or marks are random variables with an associated probability function that is independent of the arrival process. In other words, a marked Poisson process represents a sequence of ordered pairs

$$(T_1, M_1),\ (T_2, M_2),\ \ldots,\ (T_k, M_k), \tag{4}$$

in which T_i is the time of occurrence of the ith jump with amplitude mark M_i.

This formulation of of the Poisson process can be viewed as the sequence of events is inadequate for modeling jumps in the system based on the value of the state, $\mathbf{X}(t)$. The *state dependent Poisson noise* yields a sequence of events that is represented as:

$$(T_i(\mathbf{X}(T_i)), M_i(\mathbf{X}(T_i))\ \text{ for i = 1 to k}. \tag{5}$$

This representation of the Poisson process provides more realism and flexibility for a wider range of stochastic control applications since the arrival times and amplitudes may depend of the state of the system. Additionally, this formulation allows for simpler dynamical system modeling of random phenomena.

The state dependent vector valued marked Poisson noise is defined as

$$d\mathbf{P}(\mathbf{X}(t), t) = [dP_i(\mathbf{X}(t), t)]_{q\times 1}, \tag{6}$$

which consists of q independent differentials of space-time Poisson processes that are functions of the state, $\mathbf{X}(t)$, which are related to the Poisson random measure, $\mathcal{P}_i(dz_i, \mathbf{X}(t), dt)$:

$$dP_i(\mathbf{X}(t), t) = \int_{\mathcal{Z}_i} z_i \mathcal{P}_i(dz_i, \mathbf{X}(t), dt), \tag{7}$$

where z_i is the Poisson jump amplitude random variable or the mark of the $dP_i(\mathbf{X}(t), t)$ Poisson process where $i = 1$ to q, with mean or expectation:

$$\begin{aligned} \mathrm{Mean}[d\mathbf{P}(\mathbf{X}(t), t)] &= \Lambda(\mathbf{X}(t), t)dt \int_{\mathcal{Z}} \mathbf{z}\phi(\mathbf{z}, \mathbf{X}(t), t)d\mathbf{z} \\ &\equiv \Lambda(\mathbf{X}(t), t)\overline{\mathbf{Z}}(\mathbf{X}(t), t)dt, \end{aligned} \tag{8}$$

where $\Lambda(\mathbf{X}(t), t)$ is the diagonal matrix representation of the state dependent Poisson rates $\lambda_i(\mathbf{X}(t), t)$ for $i = 1$ to q, $\overline{\mathbf{Z}}(\mathbf{X}(t), t)$ is the mean of the jump amplitude mark vector and $\phi_i(z_i, \mathbf{X}(t), t)$ is the density of the ith amplitude mark component. Assuming component-wise independence, $d\mathbf{P}(\mathbf{X}(t), t)$ has covariance given by

$$\mathrm{Covar}[d\mathbf{P}(\mathbf{X}, t), d\mathbf{P}^{\top}(\mathbf{X}, t)] = \Lambda(\mathbf{X}, t)\sigma(\mathbf{X}, t)dt \tag{9}$$

with $\sigma(\mathbf{X}(t), t) = [\sigma_{i,j}\delta_{i,j}]_{q\times q}$ denoting the diagonalized covariance of the amplitude mark distribution for $d\mathbf{P}(\mathbf{X}(t), t)$. Again, the mark vector is not assumed to have a zero mean, i.e., $\overline{\mathbf{Z}} \neq 0$, permitting additional modeling complexity. Note, that for discrete distributions the above integrals need to be replaced by the appropriate sums.

3. Multistage Manufacturing System Model

In [10, 12, 14], linear and nonlinear models are applied to the control of production in a multistage manufacturing system. For each stage i in the manufacturing process, let $n_i(t)$ represents the number of operational workstations at time t which is a state variable, where $0 \leq n_i(t) \leq N_i$. In this presentation we focus only on the evolution of the number of operational workstations, which is determined by the failure and repair processes of the workstations. The number of operational workstations evolves according to a purely stochastic differential equation (SDE) given by:

$$dn_i(t) = dP_i^R(t) - dP_i^F(t), \tag{10}$$

where $dP_i^R(t)$ and $dP_i^F(t)$ are Poisson processes used to model the repair and failure processes, respectively. This formulation is lacking since repairs or failures can occur when they are not allowed, so a modification of the models in [10, 12, 14] is proposed below to force these state constraints via the use of state dependent Poisson noises.

Note, that if $n_i(t) = 0$ then workstation failure can not occur, similarly if $n_i(t) = N_i$ then workstation repair can not occur. The SDE (10) for the number of operational workstations can be expressed as a function of $n_i(t)$ in the following way:

$$dn_i(t) = \left\{ \begin{array}{ll} dP_i^R(t), & n_i(t) = 0 \\ dP_i^R(t) - dP_i^F(t), & 0 \leq n_i(t) \leq N_i \\ -dP_i^F(t), & n_i(t) = N_i \end{array} \right\}, \tag{11}$$

where the arrival rates, $1/\lambda_i^F$ and $1/\lambda_i^R$, and the mark probabilities $1/\Phi_i^F$ and $1/\Phi_i^R$, are for failures and repairs, respectively. This formulation is quite cumbersome to implement in numerical methods for the solution to the resulting Hamilton-Jacobi-Bellman (HJB) equation. Also, this type of formulation may transform a linear system into nonlinear system that requires a greater deal of computational effort to obtain a solution.

A more convenient way to model this phenomena would be to use state dependent Poisson noise. This leads to the following SDE for (10):

$$dn_i(t) = dP_i^R(\mathbf{X}(t), t) - dP_i^F(\mathbf{X}(t), t), \tag{12}$$

where the global state vector $\mathbf{X}(t) = [\mathbf{n}^{\top}(t), \hat{\mathbf{x}}^{\top}(t)]^{\top}$ is partitioned into the operational workstation vector $\mathbf{n}(t)$, along with additional state component vector $\hat{\mathbf{x}}(t)$ as used in [10, 12, 14]. The arrival rates for failures and repairs are given by:

$$1/\lambda_i^F(\mathbf{X}(t), t) = \left\{ \begin{array}{ll} 0 & n_i(t) = 0 \\ 1/\lambda_i^F & 1 \leq n_i(t) \leq N_i \end{array} \right\}, \tag{13}$$

and

$$1/\lambda_i^R(\mathbf{X}(t),t) = \left\{ \begin{array}{ll} 1/\lambda_i^R & 0 \le n_i(t) \le N_i - 1 \\ 0 & n_i(t) = N_i \end{array} \right\}, \quad (14)$$

with the mark probabilities given by $1/\Phi_i^F(\mathbf{X}(t),t)$ and $1/\Phi_i^R(\mathbf{X}(t),t)$, respectively. The arrival rates divide the state space for $n_i(t)$ into 3 distinct regions that are the same as that of (11). However, each value that $n_i(t)$ can assume must be considered separately since the mark distribution is different for each value. The advantage of this formulation over that of (11) is that only the *parameters* for the Poisson noise change as opposed to changes in the actual dynamical system. This formalism in (12,13,14) improves on our prior work in [10, 12, 14].

4. Natural Resource and Price Dynamics Model

In [6], the optimal control problem for a bioeconomic system undergoing both natural resource and price fluctuations is formulated and solved computationally, along with some analysis of the quasi-deterministic approximation [5] using infinitesimal moments. The motivating application comes from the Pacific halibut stock which is closely regulated by the International Pacific Halibut Commission, conveniently providing a long history of Pacific halibut data, including prices (see [6] for more information).

A simple, manageable model, in the notation used here, of halibut stock with biomass $X_1(t) = x_1$ (a random variable) at time t, grows according to a logistic law (nonlinear) in absence of disturbances with intrinsic growth rate r_1 and carrying capacity K, but is harvested at a rate $H(t) = QU(t)X_1(t)$ where $U(t) = u$ is the harvesting effort (a control) and Q is the catchability (efficiency). According to the data, the price obtained from the harvest is approximately modeled [6] by the supply-demand relationship $p(t) = (p_0/H(t) + p_1) \cdot X_2(t)$, i.e., price per unit harvest rate, where p_0 and p_1 are empirically determined constant coefficients, while $X_2(t) = x_2$ denotes a random inflationary price fluctuation factor, and grows (or declines if the rate is negative) with linear rate coefficient r_2. The factor $X_2(t)$ is a better natural stochastic modeling state variable than the price itself due to the presence of the reciprocal state-control harvesting rate $H(t)$ in the deterministic part of the supply-demand relationship.

Upon extending the prior model [6] to include state dependent environmental fluctuations in the resource biomass and price, the vector state dynamics are given by

$$\begin{aligned} d\mathbf{X}(t) &= (\mathbf{F}_0(\mathbf{X}(t))+F_1(\mathbf{X}(t))U(t))dt \quad (15) \\ &+ G_0(\mathbf{X}(t))d\mathbf{W}(t)+H_0(\mathbf{X}(t))d\mathbf{P}(\mathbf{X}(t),t), \end{aligned}$$

where the state vector is $d\mathbf{X}(t) = [dX_1(t) \;\; dX_2(t)]^\top$, the Gaussian noise term is $\mathbf{W}(t) = [W_1(t) \;\; W_2(t)]^\top$, and the Poisson noise term is

$$\mathbf{P}(\mathbf{X}(t),t) = \left[\begin{array}{c} [P_{1,i}(\mathbf{X}(t),t)]_{q_1 \times 1} \\ [P_{2,i}(\mathbf{X}(t),t)]_{q_2 \times 1} \end{array} \right], \quad (16)$$

with a multitude of Poisson noise terms given for modeling a multitude of environmental effects. These multitude effects could include severe predation, epizootics (disease epidemics in the natural resource) or high variability in the market for the resource, i. e., fluctuation rates are magnified by crowding in biomass or large changes in the market price. Abiotic (nonbiological) effects, such as severe temperature changes, may be present as well, but due to their abiotic properties their state dependent influence is nearly linear in the jump amplitude rather than nonlinear in jump amplitude and rate. The state dependent array coefficients are given by

$$\mathbf{F}_0(\mathbf{x}) = [r_1 x_1(1 - x_1/K) \quad r_2 x_2]^\top,$$

$$F_1(\mathbf{x}) = [-Qx_1 \quad 0]^\top,$$

$$G_0(\mathbf{x}) = \left[\begin{array}{cc} \sigma_1 x_1 & 0 \\ 0 & \sigma_2 x_2 \end{array} \right], \text{ and}$$

$$H_0(\mathbf{x}) = \left[\begin{array}{cc} x_1 \cdot [a_i \delta_{i,j}]_{q_1 \times q_1} & 0_{q_1 \times q_2} \\ 0_{q_2 \times q_1} & x_2 \cdot [b_i \delta_{i,j}]_{q_2 \times q_2} \end{array} \right],$$

where the a_i and b_i are linear jump amplitude coefficients. Assuming quadratic costs, $c(u) = c_1 u + c_2 u^2$, the instantaneous discounted net costs or discounted costs less harvest revenue in terms of the current notation are

$$\begin{aligned} C(\mathbf{x},u,t) &= C_0(\mathbf{x},t) + C_1(\mathbf{x},t)u + C_2 u^2/2 \quad (17) \\ &= e^{-\delta t}\left((-p_0 x_2) + (c_1 - p_1 x_2)u + c_2 u^2\right), \end{aligned}$$

where δ is the nominal discount rate (inflationary effects are already included in the inflationary factor, otherwise the real discount rate would be used) and the minimization is over the time horizon $[t, t_f]$. This is another example of a LQGP/U problem, or LQGP problem in control only, allowing the dynamics to be nonlinear in the state, so that the dynamic model will be a reasonable model of the limited resource.

5. Canonical Models and Stochastic Dynamic Programming

The linear dynamical system for the LQGP problem is governed by the stochastic differential equation (SDE) subject to Gaussian and state dependent Poisson noise disturbances is given by (for details of the state independent jump case see Westman and Hanson [10]):

$$\begin{aligned} d\mathbf{X}(t) &= [A(t)\mathbf{X}(t) + B(t)\mathbf{U}(t) + \mathbf{C}(t)]dt \quad (18) \\ &+ G(t)d\mathbf{W}(t) + [H_1(t) \cdot \mathbf{X}(t)]d\mathbf{P}_1(\mathbf{X}(t),t) \\ &+ [H_2(t) \cdot \mathbf{U}(t)]d\mathbf{P}_2(\mathbf{X}(t),t) + H_3(t)d\mathbf{P}_3(\mathbf{X}(t),t), \end{aligned}$$

for general Markov processes in continuous time, with $m \times 1$ state vector $\mathbf{X}$(t), $n \times 1$ control vector $\mathbf{U}$(t), $r \times 1$ Gaussian noise vector $d\mathbf{W}$(t), and $q_\ell \times 1$ space-time Poisson noise vectors $d\mathbf{P}_\ell(\mathbf{X}(t),t)$, for $\ell = 1$ to 3. The dimensions of the respective coefficient matrices are: $A(t)$ is $m \times m$, $B(t)$ is $m \times n$, $\mathbf{C}(t)$ is $m \times 1$, $G(t)$ is $m \times r$, while the $H_\ell(t)$ are dimensioned, so that $[H_1(t) \cdot \mathbf{x}] = [\sum_k H_{1ijk}(t)x_k]_{m \times q_1}$, $[H_2(t) \cdot \mathbf{u}] = [\sum_k H_{2ijk}(t)u_k]_{m \times q_2}$

and $H_3(t) = [H_{3ij}(t)]_{m\times q_3}$. Note that the space-time Poisson terms are formulated to maintain the linear nature of the dynamics, but the first two are actually bilinear in either $\mathbf{X}$ or $\mathbf{U}$ and $d\mathbf{P}_\ell$ for $\ell = 1$ or 2, respectively.

In contrast, the LQGP/U problem denotes the LQGP in *control only* problem and permits fairly arbitrary state dependence since the state dependence is usually determined by the application, while the control is probably determined to have a simpler form by the plant manager. The nonlinear dynamical system for the LQGP/U problem is governed by the stochastic differential equation (SDE) subject to Gaussian and state dependent Poisson noise disturbances is given by (for details of the state independent jump LQGP/U case see Westman and Hanson [12, 13]):

$$\begin{aligned} d\mathbf{X}(t) &= [\mathbf{F}_0(\mathbf{X}(t),t) + F_1(\mathbf{X}(t),t)\mathbf{U}(t)]dt \quad (19)\\ &+ G_0(\mathbf{X}(t),t)d\mathbf{W}(t) + H_0(\mathbf{X}(t),t)d\mathbf{P}_0(\mathbf{X}(t),t)\\ &+ [H_1(\mathbf{X}(t),t)\cdot\mathbf{U}(t)]d\mathbf{P}_1(\mathbf{X}(t),t), \end{aligned}$$

where $\mathbf{X}(t)$ is the $m\times 1$ state vector in the state space $\mathcal{D}_\mathbf{x}$, $\mathbf{U}(t)$ is the $n\times 1$ control vector in the control space $\mathcal{D}_\mathbf{u}$, $d\mathbf{W}(t)$ is the $r\times 1$ Gaussian noise vector, and $d\mathbf{P}_\ell(\mathbf{X}(t),t)$ is the $q_\ell\times 1$ space-time Poisson noise vector for $\ell = 0$ or 1. The dynamic coefficients (subscript "0" denotes control-independence and subscript "1" denotes linear in control) are the following: $\mathbf{F}_0(\mathbf{x},t)$ is the $m\times 1$ control-independent part of the drift, $F_1(\mathbf{x},t)$ is the $m\times n$ coefficient of the linear control term in the drift, $G_0(\mathbf{x},t)$ is the $m\times r$ amplitude of the Gaussian noise, and $H_0(\mathbf{x},t)$ is the $m\times q_0$ jump amplitude of the control independent Poisson noise. The linear $m\times q_1\times n$ coefficient $H_1(\mathbf{x},t)$ is the nonstandard linear algebra form, involving the *array valued right-sided inner product*,

$$[H_1(\mathbf{x},t)\cdot\mathbf{u}] = \left[\sum_{k=1}^{n} H_{1,i,j,k}(\mathbf{x},t)u_k\right]_{m\times q_1}, \qquad (20)$$

which is the control dependent jump amplitude of the Poisson noise.

The quadratic performance index or cost functional that is employed is quadratic with respect to control costs while permitting the costs due to the state to be nonlinear, is given by the *time-to-go* or *cost-to-go* functional form:

$$\begin{aligned} V[\mathbf{X},\mathbf{U},t] &= \frac{1}{2}(\mathbf{X}^\top S\mathbf{X})(t_f) \quad (21)\\ &+ \int_t^{t_f} C(\mathbf{X}(\tau),\mathbf{U}(\tau),\tau)d\tau, \end{aligned}$$

where the time horizon is (t,t_f), with $S(t_f)\equiv S_f$ is the final cost matrix. This final cost, known as the *salvage cost*, is given by the quadratic form, $\mathbf{x}^\top S_f\mathbf{x} = S_f : \mathbf{x}\mathbf{x}^\top = \mathrm{Trace}[S_f\mathbf{x}\mathbf{x}^\top]$.

The instantaneous quadratic cost function in control for the LQGP/U problem is

$$C(\mathbf{x},\mathbf{u},t) = C_0(\mathbf{x},t) + \mathbf{C}_1^\top(\mathbf{x},t)\mathbf{u} + \frac{1}{2}\mathbf{u}^\top C_2(\mathbf{x},t)\mathbf{u}. \qquad (22)$$

In order to minimize (21, 22) requires that the quadratic cost coefficient $C_2(\mathbf{x},t)$ is assumed to be a positive definite $n\times n$ array, while S_f is assumed to be a positive semi-definite $m\times m$ array. The linear cost coefficient $\mathbf{C}_1(\mathbf{x},t)$ is a $n\times 1$ vector and $C_0(\mathbf{x},t)$ is a scalar.

Returning to the genuine LQGP problem, the standard instantaneous cost function is given by

$$C(\mathbf{x},\mathbf{u},t) = \frac{1}{2}\left[\mathbf{x}^\top Q(t)\mathbf{x}, +\mathbf{u}^\top R(t)\mathbf{u}\right]. \qquad (23)$$

In order to minimize (21, 23) requires that the quadratic control cost coefficient $R(t)$ is assumed to be a positive definite $n\times n$ array, while the quadratic state control coefficient $Q(t)$ is assumed to be a positive semi-definite $m\times m$ array. The coefficients $R(t)$ and $Q(t)$ are assumed to be symmetric for simplicity. A form similar to (22) can be used, however *a formal closed form analytical solution for the control problem requires that the form of the instantaneous cost function be quadratic in both state and control.*

The LQGP problem is defined by (18, 21, 23) and the LQGP/U problem is defined by (19, 21, 22). The quadratic performance index (21 with 23 or 22) is selected to be the most general form for the LQGP or LQGP/U problems, respectively.

The stochastic dynamic programming approach is used to solve the control problems. So, let a functional, the *optimal, expected cost*, be defined as:

$$v(\mathbf{x},t) \equiv \underset{\mathbf{u}[t,t_f)}{\mathrm{Min}}\left[\underset{\mathbf{P},\mathbf{W}[t,t_f)}{\mathrm{Mean}}\left[V\left|\begin{array}{c}\mathbf{X}(t)=\mathbf{x}\\ \mathbf{U}(t)=\mathbf{u}\end{array}\right.\right]\right], \qquad (24)$$

where the restrictions on the state and control are that they belong to the admissible classes for the state, $\mathcal{D}_\mathbf{x}$, and control, $\mathcal{D}_\mathbf{u}$, respectively. A final condition on the optimal, expected value is determined from the final or *salvage* cost using (24) with $V[\mathbf{X},\mathbf{U},t_f]$ in (21):

$$v(\mathbf{x},t_f) = \frac{1}{2}\mathbf{x}^\top S_f\mathbf{x}, \qquad (25)$$

for $\mathbf{x}$ in $\mathcal{D}_\mathbf{x}$.

Upon applying the principle of optimality to the optimal, expected performance index, (24, 21, 23), and the chain rule for Markov stochastic processes in continuous time for the LQGP problem yields

$$\begin{aligned} 0 &= \frac{\partial v}{\partial t} + \underset{\mathbf{u}}{\mathrm{Min}}\left[(A\mathbf{x}+B\mathbf{u}+\mathbf{C})^T\nabla_x[v]\right. \quad (26)\\ &+ \frac{1}{2}\left(GG^T\right):\nabla_x[\nabla_x^\top[v]] + \frac{1}{2}\mathbf{x}^\top Q\mathbf{x} + \frac{1}{2}\mathbf{u}^\top R\mathbf{u}\\ &+ \sum_{k=1}^{q_1}\lambda_{1,k}\int_{\mathcal{Z}_{1,k}}\left[v(\mathbf{x}+[H_1(t)\cdot\mathbf{x}]_k z_k,t)-v\right]\phi_{1,k}dz_k\\ &+ \sum_{k=1}^{q_2}\lambda_{2,k}\int_{\mathcal{Z}_{2,k}}\left[v(\mathbf{x}+[H_2(t)\cdot\mathbf{u}]_k z_k,t)-v\right]\phi_{2,k}dz_k\\ &+ \left.\sum_{k=1}^{q_3}\lambda_{3,k}\int_{\mathcal{Z}_{3,k}}\left[v(\mathbf{x}+\mathbf{H}_{3,k}(t)z_k,t)-v\right]\phi_{3,k}dz_k\right], \end{aligned}$$

with $\lambda_{i,k} = \lambda_{i,k}(\mathbf{x},t)$ and $\phi_{i,k} = \phi_{i,k}(z_k,\mathbf{x},t)$ for $i = 1, 2, 3$ where the column arrays used in the Poisson terms are defined by the notation,

$$[H_1(t)\cdot\mathbf{x}]_k \equiv \left[\sum_{j=1}^{m} H_{1,i,k,j}(t)x_j\right]_{m\times 1} \quad (27)$$

$$[H_2(t)\cdot\mathbf{u}]_k \equiv \left[\sum_{j=1}^{n} H_{2,i,k,j}(t)u_j\right]_{m\times 1} \quad (28)$$

and

$$\mathbf{H}_{3,k}(t) \equiv [H_{3,i,k}(t)]_{m\times 1}, \quad (29)$$

and the double dot product is defined by $A : B = \sum_i\sum_j A_{i,j}B_{i,j} = \mathrm{Trace}[AB^\top]$. The backward partial differential equation (PDE) (26) is known as the Hamilton-Jacobi-Bellman (HJB) equation and is subject to the final condition (25).

Similarly, for the LQGP/U problem the HJB equation can be written as

$$\begin{aligned} 0 &= \frac{\partial v}{\partial t} + C_0 + \left(\mathbf{F}_0{}^\top \nabla_x[v]\right) \quad (30)\\ &+ \frac{1}{2}\left(\left(G_0G_0^\top\right) : \nabla_x\left[\nabla_x^\top[v]\right]\right)\\ &+ \sum_{k=1}^{q_0}\lambda_{0,k}\int_{\mathcal{Z}_{0,k}}[v(\mathbf{x}+\mathbf{H}_{0,k}(\mathbf{x},t)z_k,t)\\ &- v]\,\phi_{0,k}dz_k + \mathcal{S}^*(\mathbf{x},t), \end{aligned}$$

where the control minimization terms have been collected in the control switching term,

$$\begin{aligned} \mathcal{S}^*(\mathbf{x},t) &\equiv \operatorname*{Min}_{\mathbf{u}\in\mathcal{D}_\mathbf{u}}\left[\mathbf{C}_1^\top\mathbf{u} + (F_1\mathbf{u})^\top\nabla_x[v] + \frac{1}{2}\mathbf{u}^\top C_2\mathbf{u}\right. \quad (31)\\ &+ \sum_{k=1}^{q_1}\lambda_{1,k}\int_{\mathcal{Z}_{1,k}}\left[v(\mathbf{x}+[H_1(\mathbf{x},t)\cdot\mathbf{u}]_k\,z_k,t)\right.\\ &- \left.v]\,\phi_{1,k}dz_k\right], \end{aligned}$$

with $\lambda_{i,k} = \lambda_{i,k}(\mathbf{x},t)$ and $\phi_{i,k} = \phi_{i,k}(z_k,\mathbf{x},t)$ for $i = 1$ and 2 where $\mathbf{H}_{0,k}(\mathbf{x},t) \equiv [H_{0,i,k}(\mathbf{x},t)]_{m\times 1}$ and $[H_1(\mathbf{x},t)\cdot\mathbf{u}]_k \equiv [\sum_{j=1}^{n} H_{1,i,k,j}(\mathbf{x},t)u_j]_{m\times 1}$ for $k = 1$ to q_l with $l = 0$ or 1, respectively, and is subject to the final condition equation (25). The argument of the minimum is the optimal control, $\mathbf{u}^*(\mathbf{x},t)$; if there are no control constraints the optimal control is known as the regular control, $\mathbf{u}_{\mathrm{reg}}(\mathbf{x},t)$, which can not be determined *explicitly* by standard calculus optimization in this case, due to the control dependent Poisson noise term.

6. Computational Considerations

Although the primary purpose of this paper is to discuss modeling considerations, the computational considerations are also important and will be briefly discussed. From the state dependent multistage manufacturing system and the natural resource with price dynamics applications, it is clear that a partition of the state space for the usual LQGP problem is necessary in order to represent the influence of the state dependent Poisson noise. In the case of continuous or discrete functions to represent the mark distribution, a partition of the state space needs to be formed in which not only the arrival rates but also the mark distributions are the same. It will be this partition that is essential in determining the numerical solution for the optimal control problems. The computational methods need to be extended to accommodate for this local partitioning of the state space for the genuine LQGP problem [10], solving the temporal Riccati-like system by marching in time.

However, the LQGP/U problem [12, 13, 14] is assumed to be genuinely nonlinear in the state, so partitioning is unnecessary, since the state dependence of the Poisson rates only introduces additional coefficients variable, possibly nonlinear, in the state. The discrete computational PDE-type methods of [12, 13, 14] carry over to this case with modifications for state dependent jump rates, as well as amplitude mean and covariance integrals.

6.1. LQGP Problem with Local State Independence

To solve (26) subject to the final condition (25), for the LQGP problem (for further details see Westman and Hanson [10]) a modification of the formal state decomposition of the solution for the usual LQG problem (for the usual LQG, see Bryson and Ho [1], Dorato et al. [2], or Lewis [8]) is assumed:

$$\begin{aligned} v(\mathbf{x},t) &= \frac{1}{2}\mathbf{x}^T S(t)\mathbf{x} + \mathbf{D}^T(t)\mathbf{x} + E(t) \quad (32)\\ &+ \frac{1}{2}\int_t^{t_f}\left(GG^T\right)(\tau) : S(\tau)d\tau. \end{aligned}$$

The final condition (25) is satisfied, provided that

$$S(t_f) = S_f, \quad \mathbf{D}(t_f) = \mathbf{0}, \quad \text{and} \quad E(t_f) = 0. \quad (33)$$

The ansatz (32) would not, in general, be true for the state dependent case, but would be applicable if the Poisson noise is locally state independent, while globally state dependent. That is, the state domain is decomposed into subdomains, $\mathcal{D}_\mathbf{x} = \bigcup_i \mathcal{D}_{\mathbf{x}_i}$, where the arrival rates and moments for all the Poisson processes are constant in the region $\mathcal{D}_{\mathbf{x}_i}$ and can be expressed as:

$$\left\{\begin{array}{l} \Lambda(\mathbf{X}(t),t) = \Lambda_i(t)\\ \overline{\mathbf{Z}}(\mathbf{X}(t),t) = \overline{\mathbf{Z}}_i(t)\\ \sigma(\mathbf{X}(t),t) = \sigma_i(t) \end{array}\right\}, \quad \text{for } \mathbf{X}(t)\in\mathcal{D}_{\mathbf{x}_i}, \quad (34)$$

for all subdomains i. This was the case in the multistage manufacturing system, where the value of $\mathbf{X}(t)$ is used to decompose the state domain. If (34) has any explicit dependence on $\mathbf{X}(t)$ then the resulting system would then form a LQGP/U problem and the analysis and the extended computational algorithm of [10] would no longer apply.

6.2. LQGP/U Problem

To solve (30, 31) subject to the final condition (25), in the case of the LQGP/U problem, the methods of Westman and Hanson [12, 13, 14] can be applied with modifications for the state dependent jump rates, mean jump amplitudes and their covariance,

$$\left\{ \begin{array}{c} \Lambda(\mathbf{X}(t), t) \\ \overline{\mathbf{Z}}(\mathbf{X}(t), t) \\ \sigma(\mathbf{X}(t), t) \end{array} \right\}, \tag{35}$$

as given in the integrals in (8, 9). The algorithm presented in Westman and Hanson [13] is used to solve the LQGP/U problem. and would require additional modifications in quadrature procedures to account for the added state dependence in those jump integrals. We omit further details here since related numerical considerations in computing state independent Poisson jump integrals are found in Westman and Hanson [13].

The applications to the multistage manufacturing system and to resource management with price dynamics are examples of the state dependent Poisson noise in LQGP/U problems, in absence of state domain decomposition.

7. Conclusions

These state dependent models are of great interest because they allow for modeling of industrial and natural phenomena, where important changes can occur due to state dependent jumps in the system in terms of both rates and amplitudes. This allows for greater realism to be incorporated into the dynamical system since accurate ways of representing jumps in the value of the state are included in the dynamical model. The models of the multistage manufacturing system and the management of a natural resource with stochastic price dynamics are used to illustrate the usefulness of the state dependent Poisson noise in modeling stochastic dynamical systems. To solve the resulting optimal control problems for this dynamical system formulation requires only modifications to existing computational methods. This formulation can be used in many applications which have not previously considered the use of jumps in the value of the system, for example, biomedical control, manufacturing systems with workstation maintenance, population control, flexible space structures, as well as many others.

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