Nonlinear State Dynamics, Computational Approximations and Multistage Manufacturing System Application *

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Abstract

In this paper, we treat stochastic optimal control problems that are nonlinear in the state, but otherwise are a Linear-Quadratic-Gaussian-Poisson problem in the control (LQGP/U), such that the dynamics are linear in the control with quadratic costs in control. The uncertainty in the environment is modeled by Gaussian noise for continuous background fluctuations and discrete random jumps by Poisson noise. Computational approximations in the dynamic programming approach include least squares equivalent LQGP problems in the state for a state independent approximation to solutions, modified Gauss quadrature to handle arbitrary jump distributions, and multidimensional linear interpolation for Poisson functional terms. The approximations are applied to a multistage manufacturing system (MMS) in an uncertain environment, along with implementation procedures needed to modify the formal theory for hybrid systems.

1. Introduction

Consider the following dynamical system described by the following stochastic differential equation (SDE):

$$\begin{aligned} d\mathbf{X}(t) &= [\mathbf{F}_0(\mathbf{X}(t),t) + F_1(\mathbf{X}(t),t)\mathbf{U}(t)]dt + G_0(\mathbf{X}(t),t)d\mathbf{W}(t) \\ &+ H_0(\mathbf{X}(t),t)d\mathbf{P}_0(t) + [H_1(\mathbf{X}(t),t)\mathbf{U}(t)]d\mathbf{P}_1(t), \end{aligned} \tag{1}$$

which represents the evolution of the state influenced by the control and subject to a random environment. Here, $\mathbf{X}(t)$ is the $m \times 1$ state vector in the state space $\mathcal{D}_{\mathbf{x}}$, $\mathbf{U}(t)$ is the $n \times 1$ control vector in the control space $\mathcal{D}_{\mathbf{u}}$, $d\mathbf{W}(t)$ is the $r \times 1$ Gaussian noise vector, and $d\mathbf{P}_\ell(t)$ is the $q_\ell \times 1$ space-time Poisson noise vector for $\ell = 0$ or 1. The dynamic coefficients (subscript "0" denotes control-independence and subscript "1" denotes linear in control) are the following: $\mathbf{F}_0(\mathbf{x},t)$ is the $m \times 1$ control-independent part of the drift, $F_1(\mathbf{x},t)$ is the $m \times n$ coefficient of the linear control term in the drift, $G_0(\mathbf{x},t)$ is the $m \times r$ amplitude of the Gaussian noise, and $H_0(\mathbf{x},t)$ is the $m \times q_0$ jump amplitude of the control independent Poisson noise. The linear coefficient $H_1(\mathbf{x},t)$ is the nonstandard linear algebra form, involving the *array valued right-sided dot product*,

$$[H_1(\mathbf{x}(t),t)\mathbf{u}(t)] = [\sum_{k=1}^{n} H_{1,i,j,k}(\mathbf{x},t)\mathbf{u}_k(t)]_{m\times q_1}, \tag{2}$$

which is the control dependent jump amplitude of the Poisson noise.

The dynamical system (1) is very general and allows for great realism to be modeled not only in terms of the dynamics but also the form of random disturbances to the system. These random disturbances are

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the Gaussian noise which is assumed to be a vector of independent Wiener processes and the Poisson noise terms, $d\mathbf{P}_l(t) = [dP_{l,i}(t)]_{q_l \times 1}$, which consist of q_l independent differentials of space-time Poisson processes for each $l = 0$ or 1 with z_i the Poisson jump amplitude random variable of the $dP_{l,i}(t)$ Poisson process for $l = 0$ or 1 where $i = 1$ to q_l.

As with the usual LQG problem, the quadratic (Q) part of the LQGP/U denotes the quadratic performance index with respect to control costs and permitting the costs of the state dynamics to be generally nonlinear. Here, the quadratic performance index is the cost-to-go form:

$$V(\mathbf{X},\mathbf{U},t) = \frac{1}{2}\mathbf{X}^\top(t_f)S_f\mathbf{X}(t_f) + \int_t^{t_f}\left[C_0(\mathbf{X}(\tau),\tau) + \mathbf{C}_1^\top(\mathbf{X}(\tau),\tau)\mathbf{U}(\tau) + \frac{1}{2}\mathbf{U}(\tau)^\top C_2(\mathbf{X}(\tau),\tau)\mathbf{U}(\tau)\right]d\tau, \tag{3}$$

where the time horizon is $[t, t_f)$, and S_f is the final cost matrix. This final cost is given by the quadratic form, $\mathbf{x}^\top S_f\mathbf{x} = S_f : \mathbf{x}\mathbf{x}^\top = \mathrm{Trace}[S_f\mathbf{x}\mathbf{x}^\top]$. The quadratic cost coefficient C_2 is assumed to be a symmetric, positive definite $n \times n$ array, while S_f is assumed to be symmetric, positive semi-definite $m \times m$ array in order to minimize (3). The linear cost coefficient $\mathbf{C}_1$ is a $n \times 1$ vector and C_0 is a scalar. The quadratic performance index (3) was selected to be the most general form for the LQGP/U problem.

In Section 2, an outline is given for the the numerical solution of the HJB equation. In Section 3, an application to a multistage manufacturing system (MMS) is presented in its dynamical formulation.

2. Outline for Numerical Method

The optimal, expected performance, $v(\mathbf{x}, t)$, is

$$v(\mathbf{x},t) \equiv \underset{\mathbf{u}[t,t_f)}{\mathrm{Min}}\left[\underset{\mathbf{P},\mathbf{W}[t,t_f)}{\mathrm{Mean}}\left[V(\mathbf{X},\mathbf{U},t) \mid \mathbf{X}(t)=\mathbf{x}, \mathbf{U}(t)=\mathbf{u}\right]\right]. \tag{4}$$

The state and control belong to the admissible classes for the state, $\mathcal{D}_\mathbf{x}$, and control, $\mathcal{D}_\mathbf{u}$, respectively. The final condition for the optimal, expected value is determined by

$$v(\mathbf{x},t_f) = \frac{1}{2}\mathbf{x}^\top S_f\mathbf{x} \quad \text{for} \quad \mathbf{x} \in \mathcal{D}_\mathbf{x}. \tag{5}$$

Stochastic dynamic programming is an application of the principle of optimality to the optimal expected performance value (4,3), and the general Itô chain rule for Markov stochastic processes in continuous time (see [1] for a tractable derivation). The partial differential equation of stochastic dynamic programming is known as the Hamilton-Jacobi-Bellman (HJB) equation and is subject to the final condition (5) which is given by

$$\begin{aligned} 0 \;=\;& \frac{\partial v}{\partial t}(\mathbf{x},t) + C_0(\mathbf{x},t) + \left(\mathbf{F}_0{}^\top\nabla_x[v]\right)(\mathbf{x},t) + \frac{1}{2}\left((G_0G_0^\top) : \nabla_x[\nabla_x^\top[v]]\right)(\mathbf{x},t) \\ &+ \sum_{l=1}^{q_0}\lambda_{0,l}\int_{\mathcal{Z}_{0,l}}\left[v(\mathbf{x}+\mathbf{H}_{0,l}(\mathbf{x},t)z_l,t) - v(\mathbf{x},t)\right]\phi_{0,l}(z_l,t)dz_l + \mathcal{S}^*(\mathbf{x},t), \end{aligned} \tag{6}$$

where the double dot product is defined by $A : B = \sum_i\sum_j A_{i,j}B_{j,i} = \mathrm{Trace}[AB]$ and the control minimization terms have been collected in the control switching term,

$$\begin{aligned} \mathcal{S}^*(\mathbf{x},t) \;\equiv\;& \underset{\mathbf{u}\in\mathcal{D}_\mathbf{u}}{\mathrm{Min}}\Big[\mathbf{C}_1^\top(\mathbf{x},t)\mathbf{u} + (F_1(\mathbf{x},t)\mathbf{u})^\top\nabla_x[v](\mathbf{x},t) + \frac{1}{2}\mathbf{u}^\top C_2(\mathbf{x},t)\mathbf{u} \\ &+ \sum_{l=1}^{q_1}\lambda_{1,l}\int_{\mathcal{Z}_{1,l}}\left[v(\mathbf{x}+[H_1(\mathbf{x},t)\mathbf{u}]_l z_l,t) - v(\mathbf{x},t)\right]\phi_{1,l}(z_l,t)dz_l\Big]. \end{aligned} \tag{7}$$

Unlike standard PDEs, solving the HJB equation (6,7) with final condition (5) requires finding the optimal value function $v(\mathbf{x}, t)$ simultaneously with the optimal control vector $\mathbf{u}^*(\mathbf{x}$,t) and is far too complex to consider exact analytical solutions. Therefore, numerical methods for an approximation for the optimal control and optimal expected performance is necessary. The HJB equation (6) is different from standard PDEs due to its functional dependence embodied in the switching term (7) which is required to get the optimal control along with the optimal value, and the the global dependence of the Poisson jump integral term is unlike the local dependence of standard partial derivatives. This global functional dependence makes Poisson noise terms far more complex than the local second order partial derivatives due to Gaussian noise.

A predictor-corrector Crank-Nicolson method is used to solve the HJB equation (6,7,5) for the nonlinear system (1). For each time step, an initial guess for the optimal expected performance and optimal control, the predictor, is formed and then is repeatedly corrected by searching over the discretized state space until convergence of the optimal expected performance and optimal control occurs simultaneously. Each search over the discretized state space is computationally intensive, and can be reduced by a the selection of a good predictor.

The feature of this method is the effort that goes into the selection of the of the predictor. In general, the computational effort needed for assembling the predictor is less than that of a single correction, and should reduce the overall number of corrections necessary. First a Least Squares Approximation (LSA) (see [4]) is formed for the LQGP/U problem (1,3) which leads to a genuine LQGP problem in both the state and control for the current time step. This approximated system can then be solved by the method presented in Westman and Hanson [2] which requires the solution of a system of unidirectionally coupled nonlinear ordinary differential equations. The predictor is then formed by the LSA for the nonlinear system and known or calculated previous values for optimal expected value and control. For a more detailed description of the method described here see Westman and Hanson [5].

Another challenge in this method is the approximation of the integrals in (6,7) which represent jumps in the state due to the various Poisson noise terms. This is accomplished by using a modified Gauss quadrature in conjunction with a multidimensional linear interpolation to determine the values at the node points. For details on the derivation of the Gauss quadrature see Westman and Hanson [4, 5].

3. MMS Dynamical System Formulation

Consider a multistage manufacturing system (MMS) that produces a single consumable commodity, that requires a process consisting of k linear stages. For example, the manufacture or assembly of an automobile or a printed circuit board (PCB). The example presented here is from Westman and Hanson [3].

For each stage $i = 1$ to k, at time t into the manufacturing or planning horizon, there are $n_i(t)$ operational workstations which are assumed to have identical properties and produce at the same rate $c_i(t)$ with a capacity of producing M_i parts per unit time. Each workstation is subject to failure and can be repaired with the mean time between failures and the repair duration exponentially distributed. The production rate $c_i(t)$ is the utilization, i.e., the fraction of time busy.

The total number of workstations at stage i is N_i, so $n_i(t)$ can be viewed as a birth and death process on the interval $0 \leq n_i(t) \leq N_i$ with random jump sizes, determined by

$$dn_i(t) = dP_i^R(t) - dP_i^F(t), \tag{8}$$

where $dP_i^R(t)$ and $dP_i^F(t)$ are Poisson processes used to model the repair (birth) processes and the failure (death) processes, respectively. The surplus aggregate level represents the surplus (if positive) or shortfall (if negative) of the production of pieces that have successfully completed i stages of the manufacturing process, given by

$$da_i(t) = [M_i n_i(t)(c_i(t)+u_i(t))-d_i(t)]\,dt+g_i(t)dW_i(t). \tag{9}$$

The change in the surplus aggregate level, $da_i(t)$, is determined by the number of pieces that have successfully completed i stages of the manufacturing process $(M_i n_i(t)(c_i(t) + u_i(t))dt)$, that are not defective, and are not consumed by stage $i+1$, $(d_i(t)dt)$, and by the status of the workstations. The term, $g_i(t)dW_i(t)$, is used to model small random fluctuations in the number pieces produced, for example the number of defective pieces. The control $u_i(t)dt$ is used to adjust the production rate, $c_i(t)$, to compensate for workstation repair and failure and random fluctuations in the manufacturing process, given by

$$dc_i(t) = u_i(t)dt. \tag{10}$$

This equation is new to this model and allows for a dynamical determination of the production rate, which was not the case in [2].

The outputs for the MMS, for each stage i, are measures for the production rate and the estimated expected value for the surplus aggregate level, $\hat{a}_i(t,t_f)$. The degenerate case, $n_i(t)=0$ for all i and t, is not allowed and *only* the regular or unconstrained control is considered. The virtual production rate, $c_{\mathrm{vir},i}(t)$, does not anticipate workstation repair or failure or compensate for random production fluctuations, is defined as $c_{\mathrm{vir},i}(t) \equiv \int_t^{t_f} d_i(\tau)d\tau/(M_i n_i(t)t)$. The regular controlled production rate, $c_{\mathrm{reg},i}(t)$, anticipates workstation repair and failure as well as production fluctuations, is defined as $c_{\mathrm{reg},i}(t) \equiv c_i(t_f) + \int_t^{t_f} u_{\mathrm{reg},i}(\tau)d\tau$. Since it is based on the regular control, $u_{\mathrm{reg},i}(t)$, it may not be physically realizable, i.e. not admissible. The constrained controlled production rate, $c_i^*(t)$, is the restriction of $c_{\mathrm{reg},i}(t)$ to be admissible, and is defined as $c_i^*(t) \equiv \min\left[c_i^*(t), c_{\max,i}(t)\right]$, where the maximum production rate that accounts for physical (workstation) and production limitations is defined as $c_{\max,i}(t) \equiv \min[1,(c_{i-1}(t)*M_{i-1}*n_{i-1}(t))/(M_i*n_i(t))]$. The value of $\hat{a}_i(t,t_f)$ is a measure of the deviation from the target production goal projected over the remaining time in the planning horizon defined as $\hat{a}_i(t,t_f) \equiv a_i(t_f) + M_i n_i(t)(c_i^*(t) - c_{\mathrm{reg},i}(t))(t_f - t)$.

The performance index is given by (3) with $C_0(\mathbf{x},t) \equiv \mathbf{x}^T \widehat{C}_0(t)\mathbf{x}$ and $\mathbf{C}_1(\mathbf{x},t) \equiv \mathbf{0}_{k\times 1}$ where the state or plant vector is given by $\mathbf{x}(t) = [\mathbf{a}(t) \quad \mathbf{n}(t) \quad \mathbf{c}(t)]^T$. The salvage cost S_f is a symmetric, positive semi-definite $3k \times 3k$ matrix, based on the idea of *Just In Time* or *stockless production*. A penalty is imposed if there is a surplus or shortfall of production.

Modifications are necessary for the method presented since the state variables consist of discrete and continuous values. These modifications can be found in Westman and Hanson [3].

4. Conclusions

These results are of great interest because the LQGP/U problem is a *nonlinear* theoretical model and a canonical computational model. The state space representation of the LSA for the nonlinear LQGP/U problem forms a genuine LQGP problem, which should accelerate the convergence of the iterative method used to approximate the solution for the LQGP/U HJB equation. The Poisson components permit the model to account for a wide variety of large random fluctuations, whereas Gaussian noise is useful for relatively less severe background fluctuations. The stochastic differential equation formulation used here makes the construction of models analogous to that for other dynamic systems. The features of this control model allow for greater realism without having to compromise aspects of physical reality than does the corresponding linear model. This problem formulation would be useful for investigating the response of systems subject to extreme random inputs or catastrophic failure. The methods presented here are applied to a MMS.

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