Option Pricing for a Stochastic-Volatility Jump-Diffusion Model with Log-Uniform Jump-Amplitudes*

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***Abstract*— An alternative option pricing model is proposed, in which the stock prices follow a diffusion model with square root stochastic volatility and a jump model with log-uniformly distributed jump amplitudes in the stock price process. The stochastic-volatility follows a square-root and mean-reverting diffusion process. Fourier transforms are applied to solve the problem for risk-neutral European option pricing under this compound stochastic-volatility jump-diffusion (SVJD) process. Characteristic formulas and their inverses simplified by integration along better equivalent contours are given. The numerical implementation of pricing formulas is accomplished by both fast Fourier transforms (FFTs) and more highly accurate discrete Fourier transforms (DFTs) for verifying results and for different output.**

Key words: stochastic-volatility, jump-diffusion, risk neutral option pricing, uniform jump-amplitudes.

I. INTRODUCTION

Despite the great success of the Black-Scholes model [6], [22], this log-normal pure diffusion model fails to reflect the three empirical phenomena: (1) the asymmetric leptokurtic features, that is, the return distribution is skewed to the left, and has a higher peak and two heavier tails than those of the normal distribution; (2) the volatility smile, that is, the implied volatility is not a constant as assumed in the Black-Scholes model; and (3) the large random fluctuations such as crashes and rallies.

Therefore, many financial engineering studies have been undertaken to modify and improve the Black-Scholes model to explain some or all of the above three empirical phenomena. Popular models include, for example, (i) the jump-diffusion models of Merton [23] and Kou [21]; (ii) the constant-elasticity-of-variance model of Cox and Ross [8]; (iii) the stochastic-volatility models of Hull and White [17], Stein and Stein [25] and Heston [14]; (iv) the stochastic-volatility and stochastic-interest-rates models of Amin and Ng [1], Bakshi and Chen [4], and Scott [24]; (v) the stochastic-volatility jump-diffusion models of Bates [5], and Scott [24].

Jarrow and Rosenfeld [18] and Jorion [19] demonstrate that jumps are empirically important for several financial markets. Recently, some empirical research result indicate that both stochastic-volatility and discrete jump components are critical ingredients of the data-generating mechanism. Stochastic volatility is needed to calibrate the longer maturities and jumps to reflect shorter maturity option pricing. As discussed in Andersen, Benzoni and Lund [2], Bates [5] and Bakshi, Cao and Chen [3], the most reasonable model of stock prices would include both stochastic-volatility and jump-diffusion.

In this paper, an alternative stochastic-volatility jump-diffusion model is proposed, which has square-root and mean-reverting stochastic-volatility process and log-uniformly distributed jump amplitudes in Section II. In Sect. III, a formal "*closed form solution*" (according to Heston [14]) for risk-neutral pricing of European options is given by first converting the problem into characteristic functions (Fourier transforms) , then using the Fourier inversion formula for probability distribution functions to find a more numerically robust form (but, not everyone would call it closed). In Sections IV and V, the solution to the problem is computed from the Fourier inverse using both fast Fourier transforms (FFTs) for speed and more general discrete Fourier transforms (DFTs) for accuracy and validation. In Sect. VI, the conclusions are given.

II. STOCHASTIC-VOLATILITY JUMP-DIFFUSION MODEL

It is assumed that a risk-neutral probability measure $\mathcal{M}$ exists, the asset price $S(t)$, under this measure, follows a jump-diffusion process, with zero-mean at the risk neutral rate r and conditional variance $V(t)$,

$$\begin{aligned} dS(t) &= S(t)\Big((r-\lambda\bar{J})dt+\sqrt{V(t)}dW_s(t) \\ &\quad +J(Q)dN(t)\Big), \end{aligned} \tag{1}$$

It is only necessary to know that the risk-neutral measure exists [16]. Since Scott [24] finds that interest rate volatility has little impact on short-term option prices, the interest rate r will be assumed constant in this paper.

The instantaneous volatility follows a pure mean-reverting and square-root diffusion process, given as

$$dV(t)=k\left(\theta-V(t)\right)dt+\sigma\sqrt{V(t)}dW_v(t). \tag{2}$$

The variables $W_s(t)$ and $W_v(t)$ are standard Brownian motions for $S(t)$ and $V(t)$, respectively, with $\mathrm{E}[dW_i(t)]=0$, $\mathrm{Var}[dW_i(t)]=dt$, for $i=s$ or v, and correlation

$$\mathrm{Corr}[dW_s(t),dW_v(t)]=\rho. \tag{3}$$

In (1), $J(Q)$ is the Poisson jump-amplitude, Q is an underlying Poisson amplitude mark process selected [10],

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[13], [12], [26], [9] so that

$$Q = \ln(J(Q) + 1), \tag{4}$$

for convenience, $N(t)$ is the standard Poisson jump counting process with jump intensity λ, and

$$\mathrm{E}[dN(t)] = \lambda dt = \mathrm{Var}[dN(t)].$$

Also, the symbolic jump term $J(Q)dN(t)$ in (1) denotes the Poisson sum,

$$J(Q)dN(t) = \sum_{i=1}^{dN(t)} J(Q_i), \tag{5}$$

where Q_i is the ith jump-amplitude random variable taken from a set of independent, identically distributed (IID) random variables.

Let the density of the jump-amplitude mark Q be uniformly distributed:

$$\phi_Q(q) = \frac{1}{b-a}\left\{\begin{array}{ll} 1, & a \le q \le b \\ 0, & \text{else} \end{array}\right\}, \tag{6}$$

where $a < 0 < b$. The mark Q has moments, such that the mean is

$$\mu_Q \equiv \mathrm{E}_Q[Q] = 0.5(b+a)$$

and variance is

$$\sigma_Q^2 \equiv \mathrm{Var}_Q[Q] = (b-a)^2/12.$$

The jump-amplitude J itself has mean

$$\bar{J} \equiv \mathrm{E}[J(Q)] = (\exp(b) - \exp(a))/(b-a) - 1. \tag{7}$$

Choosing the log-uniform distribution has several reasons: first, since the exponentially small tails of the log-normal or log-double-exponential distribution are contrary to the flat and thick tails of the long time financial market log-return data. Next, around the near-zero peak of the log-double-exponential and the log-normal, the jumps are small, so are not qualitatively different or separately detectable from the continuous diffusion fluctuations. Moreover, an infinite jump domain is unrealistic , since the jumps should be bounded in real world financial markets and infinite domain leads to unrealistic restrictions in portfolio optimization [11].

The square-root stochastic-volatility process (2) has two major advantages. First, the model can allow for systematic volatility risk. The second is that the process generates an analytically tractable method of pricing options without sacrificing accuracy of requiring undesirable restrictions on parameter values [5].

By the Itô's chain rule and under a risk-neutral probability measure $\mathcal{M}$, the log-return process $\ln(S(t))$ satisfies the SDE

$$\begin{aligned} d\ln(S(t)) &= (r - \lambda\bar{J} - V(t)/2)dt + \sqrt{V(t)}dW_s(t) \\ &\quad + QdN(t)\,, \end{aligned} \tag{8}$$

where r is the risk-free interest rate, such that $\mathrm{E}_{\mathcal{M}}[S(T)|S(t_0)] = S(t_0)\exp(r(T - t_0))$ for some initial time t_0 with risk-neutral drift $\mu_{\mathcal{M}} = r - \lambda\bar{J}$.

III. European Call Option Price

We let C denote the price at time t of a European style call option on $S(t)$ with strike price K and expiration time T. Using the fact that the terminal payoff of a European call option on the underlying stock S with strike price K is $\max(S(T) - K, 0)$ and assuming the short-term interest rate r is constant over the lifetime of the option, the price of the European call at time t equals discounted, conditional expected payoff

$$\begin{aligned} & C(S(t), V(t), t; K, T) \\ &= e^{-r(T-t)}\mathrm{E}_{\mathcal{M}}[\max[S_T - K, 0]|S(t), V(t)] \\ &= e^{-r(T-t)}\left(\int_K^\infty S_T p_{\mathcal{M}}(S_T|S(t), V(t))dS_T\right. \\ &\quad \left. -K\int_K^\infty p_{\mathcal{M}}(S_T|S(t), V(t))dS_T\right) \\ &= S(t)P_1(S(t), V(t), t; K, T) \\ &\quad -Ke^{-r(T-t)}P_2(S(t), V(t), t; K, T), \end{aligned} \tag{9}$$

where $\mathrm{E}_{\mathcal{M}}$ is the expectation with respect to the risk-neutral probability measure,

$$p_{\mathcal{M}}(S_T|S(t), V(t))$$

is the corresponding conditional density given $(S(t), V(t))$,

$$\begin{aligned} & P_1(S(t), V(t), t; K, T) \\ &= e^{-r(T-t)}\int_K^\infty S_T p_{\mathcal{M}}(S_T|S(t), V(t))dS_T/S(t) \\ &= \int_K^\infty S_T p_{\mathcal{M}}(S_T|S(t), V(t))dS_T/\mathrm{E}_{\mathcal{M}}[S(T)|S(t), V(t)], \end{aligned} \tag{10}$$

by the risk-neutral property, is the risk-neutral probability that $S(T) > K$ (since the integrand is nonnegative and the integral over $[0, \infty)$ is one) and risk-neutral in-the-money (ITM) probability is

$$P_2(S(t), V(t), t; K, T) = \mathrm{Prob}[S(T) > K|S(t), V(t)] \tag{11}$$

is the complementary risk neutral distribution function. The European option evaluation problem is to evaluate P_1 and P_2 under the distribution assumptions embedded in the risk neutral probability measure. The difficulty is that the cumulative distribution function for most distributions is infeasible [5].

We follow the similar treatments in Bates [5], Heston [14] and Bakshi et al. [3]. We apply the Dynkin theorem (see [9, Chapter 8] for instance) to derive the partial integro-differential equation (PIDE) satisfied by the value of an option. The Dynkin theorem establishes a relationship between stochastic differential equations and partial differential equations. For the jump-diffusion in the time-inhomogeneous case and in one-dimension, given the stochastic differential equation for $X(t)$,

$$\begin{aligned} dX(t) &= f(X(t), t)dt + g(X(t), t)dW(t) \\ &\quad + h(X(t), t, Q)dP(t; X(t), t) \end{aligned}$$

the Dynkin theorem states that the expectation, where T is the terminal time,

$$u(x, t) = E[U(X(T))|X(t) = x]$$

is the solution to the following PIDE:

$$\begin{aligned} 0 &= \frac{\partial u}{\partial t} + f\frac{\partial u}{\partial x} + \frac{1}{2}g^2\frac{\partial^2 u}{\partial x^2} \\ &\quad + \lambda\int_Q (u(x + h(x, t, q), t) - u(x, t))\phi_Q(q)dq, \end{aligned}$$

subject to the final condition $u(x,T) = U(x)$, given the terminal distribution $U(x)$ (see [9, Chapter 8]for instance).

We make a change of variable from $S(t)$ to

$$L(t) = \ln(S(t))$$

with SDE given in (8) and inverse $S(t) = \exp(L(t))$. Eq. (9) is rewritten as

$$\begin{aligned} &C(S(t),V(t),t;K,T)e^{r(T-t)} \\ &\quad = \mathrm{E}_{\mathcal{M}}[\max(S_T - K,0)|S(t),V(t)], \end{aligned}$$

now $C(S(t),V(t),t;K,T)e^{r(T-t)}$ is the conditional expectation of the composite process, where $\mathcal{M}$ is an appropriate theoretical risk-neutralizing measure. Applying the two-dimensional Dynkin theorem for the price dynamics (1) and (2), we obtain that the value of a European-style option, as a function of the stock log-return $L(t)$ denoted by

$$\widehat{C}(L(t),V(t),t;\kappa,T) \equiv C(S(t),V(t),t;K,T),$$

i.e.,

$$\widehat{C}(\ell,v,t;\kappa,T) = \mathrm{E}_{\mathcal{M}}[\max(\exp(L(T)) - K,0)|L(t) = \ell, V(t) = v]$$

and $\kappa \equiv \ln(K)$, satisfies the following PIDE:

$$\begin{aligned} 0 &= \frac{\partial \widehat{C}}{\partial t} + \mathcal{A}[\widehat{C}](\ell,v,t;\kappa,T) \equiv \frac{\partial \widehat{C}}{\partial t} + \left(r - \lambda \bar{J} - \frac{1}{2}v\right)\frac{\partial \widehat{C}}{\partial \ell} \\ &\quad + k(\theta - v)\frac{\partial \widehat{C}}{\partial v} + \frac{1}{2}v\frac{\partial^2 \widehat{C}}{\partial \ell^2} + \rho\sigma v\frac{\partial^2 \widehat{C}}{\partial \ell \partial v} + \frac{1}{2}\sigma^2 v\frac{\partial^2 \widehat{C}}{\partial v^2} \\ &\quad - r\widehat{C} + \lambda\int_{-\infty}^{\infty}\left(\widehat{C}(\ell+q,v,t) - \widehat{C}(\ell,v,t)\right)\phi_Q(q)dq, \end{aligned} \quad (12)$$

From (9), in the current state variables,

$$\widehat{C}(\ell,v,t;\kappa,T) = e^{\ell}\widehat{P}_1(\ell,v,t;\kappa,T) - e^{\kappa - r(T-t)}\widehat{P}_2(\ell,v,t;\kappa,T),$$

where $\kappa = \ln(K)$, inserting this into (12) and separating assumed independent terms $\widehat{P}_1$ and $\widehat{P}_2$, produces two PIDEs for the risk neutralized probabilities $P_i(\ell,v,t;\kappa,T)$ for $i = 1,2$:

$$\begin{aligned} 0 &= \frac{\partial \widehat{P}_1}{\partial t} + \mathcal{A}_1[\widehat{P}_1](\ell,v,t;\kappa,T) \\ &\equiv \frac{\partial \widehat{P}_1}{\partial t} + \mathcal{A}[\widehat{P}_1](\ell,v,t;\kappa,T) + v\frac{\partial \widehat{P}_1}{\partial \ell} + \rho\sigma v\frac{\partial \widehat{P}_1}{\partial v} \\ &\quad + \left(r - \lambda\bar{J}\right)\widehat{P}_1 + \lambda\int_{-\infty}^{\infty}(e^q - 1)\widehat{P}_1(\ell+q,v,t)\phi_Q(q)dq; \end{aligned} \quad (13)$$

where the (κ,T) dependence has been suppressed, subject to the boundary condition at the expiration time $t = T$:

$$\widehat{P}_1(\ell,v,T;\kappa,T) = 1_{\ell > \kappa}, \quad (14)$$

where $1_{\ell>\kappa}$ is the indicator function for the set $\ell > \kappa$, and

$$\begin{aligned} 0 &= \frac{\partial \widehat{P}_2}{\partial t} + \mathcal{A}_2[\widehat{P}_2](\ell,v,t;\kappa,T) \quad (15) \\ &\equiv \frac{\partial \widehat{P}_2}{\partial t} + \mathcal{A}[\widehat{P}_2](\ell,v,t;\kappa,T) + r\widehat{P}_2; \end{aligned}$$

subject to the boundary condition at the expiration time $t = T$:

$$\widehat{P}_2(\ell,v,T;\kappa,T) = 1_{\ell > \kappa}. \quad (16)$$

A. Characteristic Function Formulation for Solution

The corresponding characteristic functions for $\widehat{P}_j(\ell,v,t;\kappa,T)$, with respect to the κ variable, for $j = 1:2$, defined by

$$f_j(\ell,v,t;y,T) \equiv -\int_{-\infty}^{\infty} e^{iy\kappa}d\widehat{P}_j(\ell,v,t;\kappa,T), \quad (17)$$

with a minus sign to account for the negativity of the measure $d\widehat{P}_j$, will also satisfy similar PIDEs:

$$\frac{\partial f_j}{\partial t} + \mathcal{A}_j[f_j](\ell,v,t;\kappa,T) = 0, \quad (18)$$

for $j = 1:2$, again suppressing PIDE parameters (y,T), with the respective boundary conditions:

$$f_j(\ell,v,T;y,T) = +e^{iy\ell}, \quad (19)$$

since from (17) and (14-16)

$$d\widehat{P}_j(\ell,v,T;\kappa,T) = d1_{\ell>\kappa} = dH(\ell - \kappa) = -\delta(\kappa - \ell)d\kappa.$$

To solve for the characteristic function explicitly, letting $\tau = T - t$ be the time-to-go we conjecture that the function is respectively given by:

$$f_j(\ell,v,t;y,t+\tau) = \exp(g_j(\tau) + h_j(\tau)v + iy\ell + \beta_j(\tau)), \quad (20)$$

for $j = 1:2$, where with $\beta_j(\tau) = r\tau\delta_{j,2}$ and the boundary conditions:

$$g_j(0) = 0 = h_j(0).$$

This conjecture exploits the linearity of the coefficient in the PIDEs (18).

By substituting (20) into (18) and cancelling the common factor of f_j, where

$$\begin{aligned} 0 &= -g_j'(\tau) - vh_j'(\tau) - r\delta_{j,2} + \left(r - \lambda\bar{J} \pm \frac{1}{2}v\right)iy \\ &\quad + (k(\theta - v) + \rho\sigma v\delta_{j,1})h_j - \frac{1}{2}vy^2 + \rho\sigma viyh_j + \frac{1}{2}\sigma^2 vh_j^2 \\ &\quad - \lambda\bar{J}\delta_{j,1} + \lambda\int_{-\infty}^{\infty}(e^{(iy+\delta_{j,1})q} - 1)\phi_Q(q)dq, \end{aligned} \quad (21)$$

where $\pm 1 = +\delta_{j,1} - \delta_{j,2}$, and by separating the order v and order one terms to reduce to two ordinary differential equations(ODEs),

$$\begin{aligned} h_j'(\tau) &= \frac{1}{2}\sigma^2 h_j^2(\tau) + (\rho\sigma(\delta_{j,1} + iy) - k)h_j(\tau) \\ &\quad \pm \frac{1}{2}iy - \frac{1}{2}y^2 \end{aligned} \quad (22)$$

and

$$\begin{aligned} g_j'(\tau) &= k\theta h_j(\tau) + (r - \lambda\bar{J})iy - \lambda\bar{J}\delta_{j,1} - r\delta_{j,2} \\ &\quad + \lambda\int_{-\infty}^{\infty}(e^{(iy+\delta_{j,1})q} - 1)\phi_Q(q)dq. \end{aligned} \quad (23)$$

To solve (22), we factor the RHS (right-hand-side) using

$$\eta_j = \rho\sigma(iy + \delta_{j,1}) - k \ \ \& \ \ \Delta_j = \sqrt{\eta_j^2 - \sigma^2 iy(iy \pm 1)},$$

so we have

$$h_j'(\tau) = \frac{1}{2}\sigma^2\left(h_j + \frac{\eta_j + \Delta_j}{\sigma^2}\right)\left(h_j + \frac{\eta_j - \Delta_j}{\sigma^2}\right).$$

By separating of variables using partial fractions,

$$\frac{1}{\Delta_j}\left(\frac{1}{h_j+\frac{\eta_j-\Delta_j}{\sigma^2}}-\frac{1}{h_j+\frac{\eta_j+\Delta_j}{\sigma^2}}\right)dh_j = d\tau,$$

and integrating both sides, we obtain separation,

$$\ln\left(\frac{h_j+\frac{\eta_j-\Delta_j}{\sigma^2}}{h_j+\frac{\eta_j+\Delta_j}{\sigma^2}}\right) = \Delta_j\tau + C.$$

Solving for h_j satisfying the boundary conditions yields the solutions

$$h_j(\tau) = \frac{(\eta_j^2-\Delta_j^2)(e^{\Delta_j\tau}-1)}{\sigma^2(\eta_j+\Delta_j-(\eta_j-\Delta_j)e^{\Delta_j\tau})}; \qquad (24)$$

and

$$\begin{aligned} g_j(\tau) &= ((r-\lambda\bar{J})iy-\lambda\bar{J}\delta_{j,1}-r\delta_{j,2})\tau \\ &\quad +\lambda\tau\int_{-\infty}^{\infty}(e^{(iy+\delta_{j,1})q}-1)\phi_Q(q)dq \qquad (25)\\ &\quad -\frac{k\theta}{\sigma^2}\left(2\ln\left(1-\frac{(\Delta_j+\eta_j)(1-e^{-\Delta_j\tau})}{2\Delta_j}\right)\right. \\ &\quad \left.+(\Delta_j+\eta_j)\tau\right), \end{aligned}$$

Applying the uniform distribution of jump-amplitude mark Q in our general formulas, leads to the following integral in (25):

$$\begin{aligned} \int_{-\infty}^{\infty}(e^{(iy+\delta_{j,1})q}-1)\phi_Q(q)dq &= \frac{1}{b-a}\int_a^b(e^{(iy+\delta_{j,1})q}-1)dq \\ &= \frac{e^{(iy+\delta_{j,1})b}-e^{(iy+\delta_{j,1})a}}{(b-a)(iy+\delta_{j,1})}-1, \end{aligned}$$

demonstrating the simplicity and utility of the log-uniform jump amplitude distribution.

B. Inverse Fourier Transform Solution for Tail Probabilities

The tail probabilities P_1 and P_2 can be calculated by finding the inverse Fourier transforms of the characteristic functions and are given (see Kendall et al. ([20]) by

$$\begin{aligned} P_j(S(t),V(t),t;K,T) &= \frac{1}{2}+\frac{1}{\pi}\int_{0^+}^{+\infty} \qquad (26)\\ &\cdot\mathrm{Re}\left[\frac{e^{-iy\ln(K)}f_j(\ln(S(t)),V(t),t;y,T)}{iy}\right]dy, \end{aligned}$$

for $j = 1:2$, where the leading term of $1/2$ is one half the residue at the $y = 0$ pole, $\mathrm{Re}[\ ]$ denotes the real component of a complex number which arises from the principal value limits of the combined real parts of the simplifying equivalent contour. This is what Heston [14] calls a '*closed form solution*". The singularity at $y = 0$ is only an apparent singularity in that the integrand of should be bounded as $y \to 0^+$ in most cases. Nevertheless, the infinite integrals involved by the Fourier transforms need to be evaluated by some numerical integration method.

C. Put Option by Put-Call Parity

The price of a European put on the same stock can be determined from the *put-call parity*. As both Hull [16] and Higham [15] that the put-call parity is based primarily on the properties of the maximum function, in absence of friction terms like dividends, and hence is independent of any particular process, so that

$$\begin{aligned} C(S(0),V(0),t;K,T)+Ke^{-rT} &= P(S(0),V(0),t;K,T) \\ &\quad +S(0), \end{aligned}$$

or in other words, European put option price is

$$\begin{aligned} P(S(0),V(0),t;K,T) &= C(S(0),V(0),t;K,T) \\ &\quad +Ke^{-rT}-S(0), \qquad (27) \end{aligned}$$

easily calculated once the call option price is known.

IV. Computing Inverse Fourier Integrals

The inverse Fourier integral (26) can be computed by means of standard procedures of numerical integration with some precautions. Two methods are compared: the discrete Fourier transform (DFT) with Gaussian Quadrature sub-integral refinement for accuracy and the other is the fast Fourier transform (FFT) for speed of computation.

A. Discrete Fourier Transform (DFT) Approximations

Since the integrand of (26) has a bounded limit as $y \to 0+$, is otherwise smooth and decays very fast, it is rewritten in the general approximate form for DFT,

$$\begin{aligned} I[F](\kappa) &\equiv \int_0^{\infty}F(y;\kappa)dy \\ &\simeq \sum_{j=1}^{N}I_j(\kappa)=\sum_{j=1}^{N}\int_{(j-1)h}^{jh}F(y;\kappa)dy, \qquad (28) \end{aligned}$$

for sufficiently large N and such integrals are the basis of the discrete Fourier transform, where h is a fixed gross step size depending on some integral cutoff $R_y = \max[y] \simeq N*h$. The sub-integrals on $((j-1)h, jh)$ in (28) for $j = 1\!:\!N$ are computed by means of ten-point Gauss-Legendre formula for refined accuracy need for oscillatory integrands and for the fact that it is an open quadrature formula that avoids any non-smooth behavior as $y \to 0+$. The number of steps N is not static, but ultimately determined by a local stopping criterion: the integration loop is stopped if the ratio of the contribution of the last strip to the total integration becomes smaller than 0.5e-7. By using formula (28) we also have to specify a suitable step size h. By trials, $h = 5$ is a good choice that we can get sufficiently fast convergence and good precision. We use the principal square root to evaluate the square root of complex in our pricing formula. Also, we use principal branch for the value of the complex natural logarithm.

B. Discrete and Fast Fourier Transform Comparisons

In addition to the general discrete Fourier transform (DFT), there is an extremely efficient way of computing the

DFT by the fast Fourier transform (FFT), corresponding to the inverse transform,

$$\mathcal{I}_N(s_k) = \sum_{j=0}^{N-1} e^{-i\frac{2\pi}{N}jk} F_N(y_j)\Delta y, \tag{29}$$

where $s_k = k * \Delta\kappa$ is the discrete stock price for $k = 0 : N-1$, $y_j = j * \Delta y$ is the discrete Fourier variable for $j = 0 : N-1$. typically N is a power of 2, $\mathcal{I}_N(s_k)$ is the kth value of the inverse $I[F](s_k)$ from (28), $F_N(y_j)$ is the jth value of its integrand and $\Delta(y)$ is integral step size. The algorithm reduces the number of matrix multiplication from $O(N^2)$ to $O(N \log_2 N)$. Unfortunately, the FFT cannot be directly applied to evaluate the integral, since the integrand is singular at the required evaluation point $y = 0$ and precision is limited to fixed step sizes Δy. Our FFT approach to this problem is similar to that of Carr and Madan [7]. They developed some techniques around the problem to acquire a better evaluation for the inverse Fourier transform to get the option price.

In our model, $f_2(\ell, v, t; \kappa, T)$ is the characteristic function of the tail probability with respect to κ. In order to remove the non-smooth behavior at the pole $y = 0$, the pole is shifted to the imaginary axis by multiplying option price at time t by an exponential, following an idea of Carr and Madan (CarrMadan99). Since $d\widehat{P}_2 = -p_{\mathcal{M}} dK$ can be shown to be

$$\widehat{C}(\ell, v, t; \kappa, T) = -e^{-r(T-t)} \int_\kappa^\infty (e^\ell - e^s) d\widehat{P}_2(\ell, v, t; s, T), \tag{30}$$

so that exponentially modified call price is

$$\widetilde{C}(\ell, v, t; \kappa, T; \alpha) = e^{\alpha\kappa} \widehat{C}(\ell, v, t; \kappa, T), \tag{31}$$

where the exponential coefficient α needs to be real and positive. The corresponding Fourier transform of $\widetilde{C}(\ell, V(t), t; \kappa, T; \alpha)$ is defined by

$$\mathcal{F}(\ell, v, t; y, T; \alpha) = \int_{-\infty}^\infty e^{iy\kappa} \widetilde{C}(\ell, v, t; \kappa, T; \alpha) d\kappa. \tag{32}$$

Hence, the call price can be expressed as the inverse Fourier transform multiplied by the reciprocal of the exponential factor,

$$\begin{aligned} \widehat{C}(\ell, v, t; \kappa, T) &= \frac{e^{-\alpha\kappa}}{2\pi} \int_{-\infty}^\infty e^{-iy\kappa} \mathcal{F}(\ell, v, t; y, T; \alpha) dy \\ &= \frac{e^{-\alpha\kappa}}{\pi} \int_0^\infty e^{-iy\kappa} \mathcal{F}(\ell, v, t; y, T; \alpha) dy. \end{aligned}$$

The expression for $\mathcal{F}(\ell, v, t; y, T; \alpha)$ is determined as follows:

$$\begin{aligned} \mathcal{F}(\ell, v, t; y, T; \alpha) &= -e^{-r(T-t)} \int_{-\infty}^\infty e^{iy\kappa} \int_\kappa^\infty e^{\alpha\kappa} \\ &\quad \cdot (e^s - e^\kappa) d\widehat{P}_2(\ell, v, t; s, T) ds d\kappa \\ &= \frac{e^{-r(T-t)} f_2(\ell, v, t; y - (\alpha+1)i, T)}{\alpha^2 + \alpha - y^2 + i(2\alpha+1)y}. \end{aligned}$$

In order to make (33) fit the application of the FFT, an approximation for $\widehat{C}(\ell, v, t; \kappa, T)$ to transfer the Fourier integral into discrete Fourier transform:

$$\begin{aligned} \widehat{C}_N(\ell, v, t; \kappa_k, T) &= \frac{e^{-\alpha\kappa_k}}{\pi} \sum_{j=1}^N e^{-i\kappa_k y_j} \\ &\quad \mathcal{F}_N(\ell, v, t; y_j, T; \alpha)\Delta y, \end{aligned} \tag{33}$$

for $k = 0 = (N-1)$, so FFT returns N values of κ_k where now $\kappa_k = -L + \ell + k\Delta L$ and $L = N\Delta L/2$, to keep the at-the-money (ATM) strike price in the middle of the range. In order to apply the FFT, we need to let $\Delta L \Delta y = 2\pi/N$. Hence there will be conflict that if we choose Δy small to obtain a fine grid for the integration, then strike price spacings will be large. Following [7], Simpson's rule is incorporated into the summation to improve the accuracy, so

$$\begin{aligned} \widehat{C}_N(\ell, v, t; \kappa_k, T) &= \frac{e^{-\alpha\kappa_k}}{\pi} \sum_{j=0}^{N-1} e^{-i\frac{2\pi}{N}jk} e^{iy_j(L-\ell)} \\ &\quad \mathcal{F}_N(\ell, v, t; y_j, T; \alpha)\widehat{\Delta} y \\ &\quad \cdot [3 - (-1)^j - \delta_{j,0}]/3, \end{aligned} \tag{34}$$

where $\delta_{j,k}$ is the Kronecker delta and the factor in the last line is the Simpson's term, preserving the fixed step size needed for FFT. The summation in (34) is an exact application of the FFT. Appropriate α and the Simpson modified $\widehat{\Delta} y$ need to be chosen. We use $\alpha = 2.0$ and $\widehat{\Delta} y = 0.25$.

V. Numerical Results and Discussion

Both methods are implemented and validated against with each other. The two methods give the similar results within double precision accuracy. The FFT method can compute the different levels strike price near at-the-money (ATM) in about 5 seconds, which has advantage when one want a full view about the option price. However, because the regular spacings on log strike price are required, the method is not convenient to give out the results for one specific strike price as can be implemented with the DFT. Using standard integration methods with the DFT has the advantage of producing results for a given strike price in about 0.5 seconds.

The option prices from the stochastic-volatility jump-diffusion (SVJD) model are compared with those of Black-Scholes (BS) model. As expected, call and put option prices of SVJD model are higher than those of Black-Scholes model with respect to the strike price. The reason is the stochastic-volatility and jump increase the risk premium. For longer the maturity time, the difference found to be are bigger. However, for same maturity time, the near-ATM option prices from two models have largest difference for strike price.

See Figures 1-3 comparing the DFT results call option prices for the SVJD model compared with the corresponding Black-Scholes (pure diffusion) call option prices for the maturity times T =0.1, 0.25 and 1.0 years, i.e., 36 days, one quarter and one year, respectively. For the corresponding DFT put option price results, see Figure 4-6 comparing the results put option prices for the SVJD model compared with the corresponding Black-Scholes option prices for the maturity times T =0.1, 0.25 and 1.0 years, i.e., 36days, one quarter and one year, respectively.

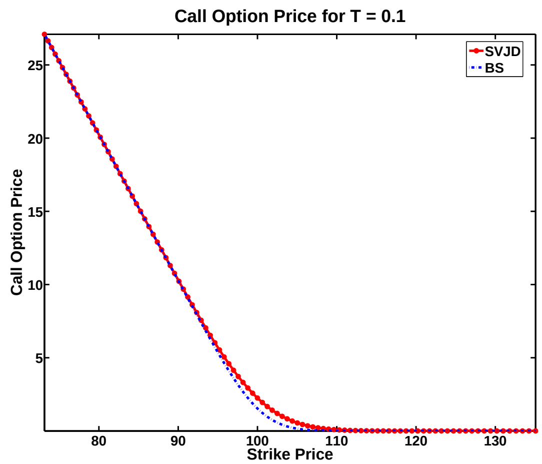

Fig. 1. DFT call option prices for the SVJD model compared to the corresponding pure diffusion Black-Scholes values for parameter values $r = 3\%$, $S_0 = \$100$ and $T = 0.1$ years for the options; while for the stochastic volatility they are $\sigma = 7\%$, $V = 0.012$, $\rho = -0.622$, $\theta = 0.53$ and $k = 0.012$; and for the uniform jump model they are $a = -0.028$, $b = 0.026$ and $\lambda = 64$.

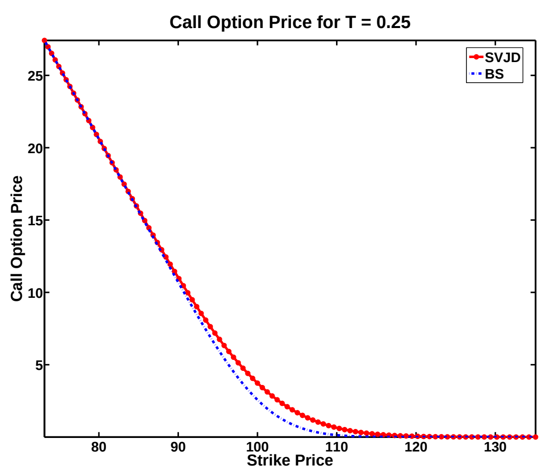

Fig. 2. DFT call option prices for the SVJD model compared to the corresponding pure diffusion Black-Scholes values for parameter values $r = 3\%$, $S_0 = \$100$ and $T = 1/4$ years (i.e., one quarter) for the options; while for the stochastic volatility they are $\sigma = 7\%$, $V = 0.012$, $\rho = -0.622$, $\theta = 0.53$ and $k = 0.012$; and for the uniform jump model they are $a = -0.028$, $b = 0.026$ and $\lambda = 64$.

VI. Summary and Conclusion

An alternative stochastic-volatility jump-diffusion model is proposed with square root mean reverting for stochastic-volatility combined with log-uniform jump amplitudes. Characteristic functions of the log of the terminal stock price and the conditional risk neutral option prices are derived analytically. The option prices can be expressed in terms of

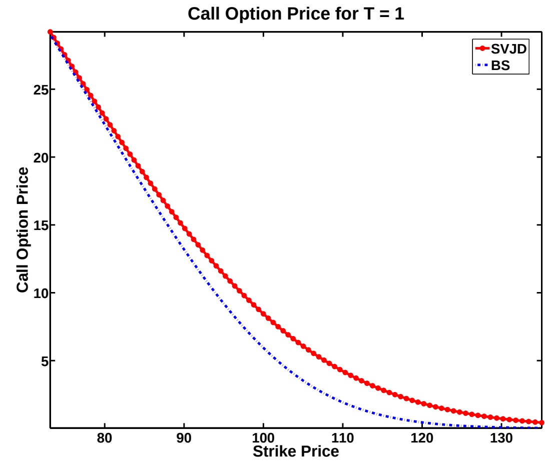

Fig. 3. DFT call option prices for the SVJD model compared to the corresponding pure diffusion Black-Scholes values for parameter values $r = 3\%$, $S_0 = \$100$ and $T = 1$ year for the options; while for the stochastic volatility they are $\sigma = 7\%$, $V = 0.012$, $\rho = -0.622$, $\theta = 0.53$ and $k = 0.012$; and for the uniform jump model they are $a = -0.028$, $b = 0.026$ and $\lambda = 64$.

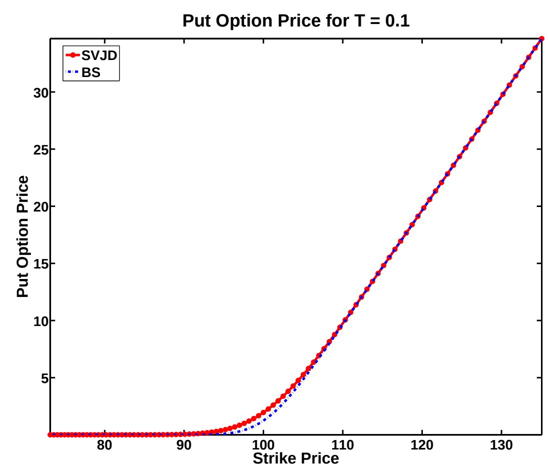

Fig. 4. DFT put option prices for the SVJD model compared to the corresponding pure diffusion Black-Scholes values for parameter values $r = 3\%$, $S_0 = \$100$ and $T = 0.1$ years for the options; while for the stochastic volatility they are $\sigma = 7\%$, $V = 0.012$, $\rho = -0.622$, $\theta = 0.53$ and $k = 0.012$; and for the uniform jump model they are $a = -0.028$, $b = 0.026$ and $\lambda = 64$.

characteristic functions in a formally closed form in terms of a Fourier inverse transform on a reduced equivalent contour. Two numerical computing algorithms are implemented. The first is a general integral for the DFT accurately approximated by 10-point Gauss-Legendre quadrature formula and second applies the FFT algorithm using an evenly-spaced Simpson's rule enhancement technique due to Carr and Madan [7]. Also an exponential modification technique of theirs was used to move a pole at the origin in the inverse Fourier transform to a less numerically sensitive position on the imaginary axis. Two methods give the similar results and have different advantages depending on the desired output. The option prices from this alternative model are compared

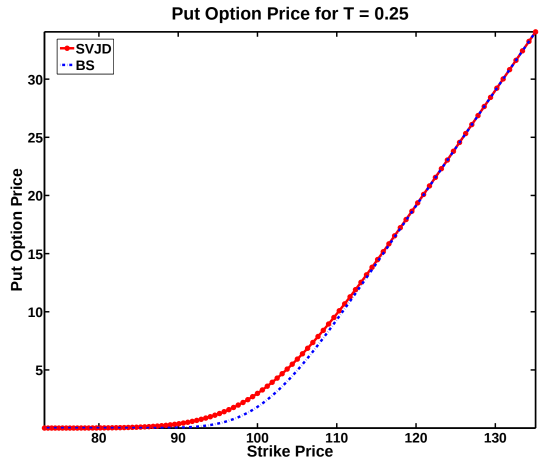

Fig. 5. DFT put option prices for the SVJD model compared to the corresponding pure diffusion Black-Scholes values for parameter values $r = 3\%$, $S_0 = \$100$ and $T = 1/4$ years for the options; while for the stochastic volatility they are $\sigma = 7\%$, $V = 0.012$, $\rho = -0.622$, $\theta = 0.53$ and $k = 0.012$; and for the uniform jump model they are $a = -0.028$, $b = 0.026$ and $\lambda = 64$.

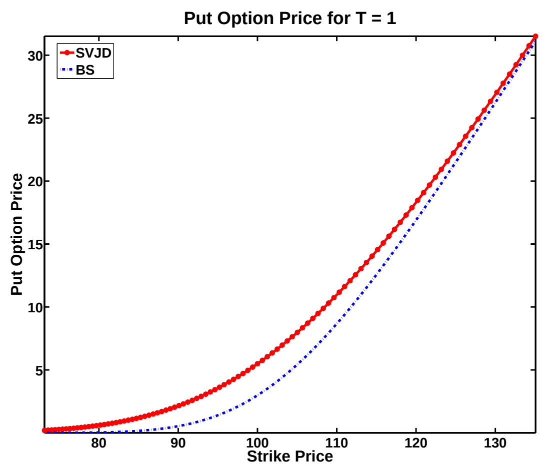

Fig. 6. DFT put option prices for the SVJD model compared to the corresponding pure diffusion Black-Scholes values for parameter values $r = 3\%$, $S_0 = \$100$ and $T = 1$ year for the options; while for the stochastic volatility they are $\sigma = 7\%$, $V = 0.012$, $\rho = -0.622$, $\theta = 0.53$ and $k = 0.012$; and for the uniform jump model they are $a = -0.028$, $b = 0.026$ and $\lambda = 64$.

with those from Black-Scholes model. The SVJD uniform jump model has higher option prices, especially for longer maturity and near at-the-money strike price.

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