Stochastic Analysis of Jump-Diffusions for Financial Log-Return Processes (corrected)

Floyd B. Hanson[1] and John J. Westman[2]

[1] University of Illinois, Laboratory for Advanced Computing, Chicago IL 60607-7045, USA
[2] University of California, Department of Mathematics, Los Angeles CA 90095-1555, USA

Abstract. A jump-diffusion log-return process with log-normal jump amplitudes is presented. The probability density and other properties of the theoretical model are rigorously derived. This theoretical density is fit to empirical log-returns of Standard & Poor's 500 stock index data. The model repairs some failures found from the log-normal distribution of geometric Brownian motion to model features of realistic financial instruments: (1) No large jumps or extreme outliers, (2) Not negatively skewed such that the negative tail is thicker than the positive tail, and (3) Non-leptokurtic due to the lack of thicker tails and higher mode.

This is the corrected version of the published paper.

1 Introduction

Encouraged by the long term success of the Black-Scholes [3] option pricing model in spite of its deficiencies, many financial modelers have tried to improve on this basic model. Merton [18] applied discontinuous sample path Poisson processes, along with Brownian motion processes, i.e., jump-diffusions, to the problem of pricing options. Merton derived several extensions of the already classical diffusion theory of Black-Scholes minimizing the portfolio variance for jump-diffusion models using techniques similar to those used to derive the Black-Scholes formulae.

Before Black-Scholes, Merton [17] analyzed the optimal consumption and investment portfolio with either geometric Brownian motion or jump Poisson noise, illustrated explicit solutions for constant risk-aversion utility. In [16], Merton also examined constant risk-aversion problems. In [12], Karatzas, Lehoczky, Sethi and Shreve pointed out that it is necessary to enforce non-negativity feasibility conditions on both wealth and consumption. They formally derive explicit solutions from a consumption-investment dynamic programming models with a time-to-bankruptcy horizon, qualitatively correcting the results of Merton [17]. Sethi and Taksar [23] directly present corrections to certain formulae Merton's finite horizon consumption-investment model [17]. Merton [19] revisited the problem in the sixth chapter of his continuous-time finance book, correcting his earlier work by adding a simpler absorbing boundary condition at zero wealth and using other techniques.

Wilmott [25] presents a good discussion on difficulty of hedging with jump-diffusion models in finance, in fact, that *perfect risk-free hedging is impossible when there are jumps in the underlying*, using a single option. Lipton-Lifshitz [15] presents a good discussion of predictability and unpredictability, mainly for foreign exchange applications, but is applicable to other financial applications as well.

In the computational finance paper of Hanson and Westman [9], they solved an optimal portfolio and consumption policies model modified from a theoretical important event model proposed by Rishel [22]. The model is an optimal portfolio and consumption model for a portfolio of stocks and a bond. The stock prices are dependent on both deterministic (scheduled) and stochastic (unscheduled) jump external events in an environment of geometric jump-diffusion processes. The jumps can affect both the stock prices directly or indirectly through parameters. The deterministic jumps are quasi-deterministic, in that the timing of the scheduled events is deterministic, but the magnitude of the jumps is random. The computations were illustrated with a simple discrete jump model, such that both stochastic and quasi-deterministic jump magnitudes were heuristically estimated discretely distributed as single negative or positive jumps. A partial motivation for this quasi-deterministic are the more or less monthly announcements of the Federal Open Market Committee [7], but the response of the market to changes in Federal Funds Rate or Federal Discount Rate is difficult to predict. This quasi-deterministic process might be called the *Greenspan Process*. The current paper focuses more on the stock log-return distribution and the estimating the parameters of this log-normal jump-diffusion distribution for a more basic stock process.

The empirical distribution of daily log-returns for actual financial instruments differ in many ways from the ideal log-normal diffusion process as assumed in the Black-Scholes model and other financial models. The most notable difference is that actual log-returns suffer occasional large jumps in value. The negative large jumps are called crashes and buying frenzies lead to positive large jumps. Another difference is that the empirical log-return will typically be negatively skewed, since The negative jumps are usually larger than the positive jumps. Thus, the coefficient of skew [6] is negative,

$$\eta_3 \equiv M_3/(M_2)^{1.5} < 0 \; , \tag{1}$$

where M_2 and M_3 are the 2nd and 3rd central moments. Still another difference is that the empirical distribution is leptokurtic since the coefficient of kurtosis [6] satisfies

$$\eta_4 \equiv M_4/M_2^2 > 3 \; , \tag{2}$$

where the value 3 is the normal distribution kurtosis value and M_4 is the fourth central moment. Qualitatively, this means that the tails are fatter than a normal with the same mean and standard deviation, and consequently the distribution is also more slender about the mode (local maximum).

In Merton's discontinuous returns paper[18] (see also Chapter 9 along with background in Chapter 3 of [19]) treated the case of option pricing modeled by a jump-diffusion process where the jumps have a log-normal distribution in one important example. Andersen and Andreasen [1] treat the log-normal jump-diffusion option pricing problem in much more detail, both analytically through forward partial integral-differential equations and numerically mainly though alternating direction implicit methods.

Kou [14] has developed a Laplacian double exponential jump-diffusion model to account for the leptokurtic, negative skew and other properties in option pricing. In the model, jumps occur in time according to a Poisson process and the amplitude of the jump are distributed as a double exponential with mean κ and variance 2η, i.e., a shifted exponential depending on the absolute value of the deviation. Many probability properties are developed along with required special functions.

There are many other approaches, such as Poisson random measure related Lévy distributions (see [2], for example) and stochastic volatility (see [25], for instance).

In Section 2, the explicit form of the log-return density is shown to be a infinite sum of log-normal distributions weighted by Poisson counting discrete distribution when both the diffusion and Poisson jump amplitudes are both log-normal. In Section 3, the five log-normal jump-diffusion parameters are estimated for the empirical log-returns of the Standard & Poor's 500 (S&P500) stock index closing under the constraints that theoretical jump-diffusion distribution has the same mean and variance as the empirical distribution.

2 Density for Log-Normal Jump-Diffusions

Let S(t) be the price of a single stock or mutual fund that satisfies the Markov, geometric, log-normal jump-diffusion stochastic differential equation (SDE),

$$dS(t) = S(t)\left[\mu_d dt + \sigma_d dZ(t) + J dP(t)\right] \ , \tag{3}$$

$S(0) = S_0$ and $S(t) > 0$, where μ_d is the mean appreciation return rate, σ_d is the diffusive volatility, $Z(t)$ is a continuous, one-dimensional Brownian motion process, J is a random jump amplitude with log-return mean μ_j and variance σ_j^2 defined below, and $P(t)$ is a discontinuous, one-dimensional, standard Poisson process with jump rate λ. Here, we will assume that the jump-diffusion parameters μ_d, σ_d, μ_j, σ_j and λ are constant. The stochastic processes $Z(t)$ and $P(t)$ are Markov and pairwise independent. The jump amplitude process J, given a Poisson jump in time, is also independently distributed.

The continuous, diffusion process $Z(t)$ is standard and is specified by the two infinitesimal moments,

$$E[dZ(t)] = 0$$

and

$$\mathrm{Var}[dZ(t)] = dt \ ,$$

since the process is normally distributed. **The discontinuous space–time jump process $JdP(t)$ is just a symbol that can be defined by a stochastic integral of Poisson random measure $\mathcal{P}(dt, dq)$ or as a sum of $dP(t)$ jumps of a compound Poisson process**,

$$JdP(t) = \int_{\mathcal{Q}} J(q)\mathcal{P}(dt, dq) = \sum_{\mathbf{i=1}}^{\mathbf{dP(t)}} \mathbf{J(Q_i)} \ , \tag{4}$$

with $E[\mathcal{P}(dt, dq)] = \lambda dt \phi_Q(q)dq$ and $\sum_{\mathbf{i=1}}^{\mathbf{0}} \mathbf{J(Q_i)} \equiv \mathbf{0}$, where q is the mark for the jump amplitude process corresponding to the underlying random variable $\mathbf{Q_i}$. The differential Poisson process is basically a counting process with the probability of the jump count given by the usual Poisson distribution,

$$p_k(\lambda dt) = \mathrm{Prob}[dP(t) = k] = \exp(-\lambda dt)(\lambda dt)^k/k!, \quad k = 0, 1, \dots \ ,$$

with parameter $u = \lambda dt > 0$. The jump in the stock price corresponding to the jump of the space-time Poisson process is

$$[S](t_j) \equiv S(t_j^+) - S(t_j^-) = J(Q)S(t_j^-)$$

at some jump time t_j. Hence, it is assumed that $-1 < J(Q) < \infty$ so that one jump does not make the underlying stock worthless. The infinitesimal moments of the jump process are

$$E[J(Q)dP(t)] = \lambda dt \int_{\mathcal{Q}} J(q)\phi_Q(q)dq = E[J(Q)]\lambda dt$$

and

$$\mathrm{Var}[J(Q)dP(t)] = \lambda dt \int_{\mathcal{Q}} J^2(q)\phi_Q(q)dq = E[J^2(Q)]\lambda dt \ ,$$

where $\phi_Q(q)$ is the Poisson mark density, providing it exists on the mark space $\mathcal{Q}$.

Before describing the jump amplitude distribution in more detail, the stock price SDE (3) is first transformed to the SDE of the instantaneous stock log-returns using the stochastic chain rule for Markov processes in continuous time [11,4],

$$d[\ln(S(t))] = \mu_{ld}dt + \sigma_d dZ(t) + \ln(1 + J(Q))dP(t) \ , \tag{5}$$

where the log-diffusion drift $\mu_{ld} \equiv \mu_d - \sigma_d^2/2$ includes the Itô calculus shift of the mean appreciation rate by the diffusion coefficient and the log-return jump amplitude is the logarithm of the relative post-jump amplitude $\ln(1+J)$.

This log-return SDE (5) will be the model that will be compared to the S&P500 log-returns, since the log-returns are the preferred financial investment metric measuring the relative changes in investment value, as opposed to the absolute change of the stock price represented by the geometric jump-diffusion SDE in (3).

Since $J > -1$, it is convenient to select the mark process to be the log-return jump amplitude

$$Q = \ln(1 + J) \ ,$$

which has the inverse

$$J(Q) = \exp(Q) - 1 \ ,$$

on the mark space $\mathcal{Q} = (-\infty, +\infty)$. On this fully infinite domain, the ideal choice for the mark density is the normal density,

$$\phi_Q(q) = \phi_n(q; \mu_j, \sigma_j^2) \equiv \frac{\exp(-(q-\mu_j)^2/(2\sigma_j^2))}{\sqrt{2\pi\sigma_j^2}} \ , \tag{6}$$

having a mean $E[Q] = \mu_j$ and variance $\mathrm{Var}[Q] = \sigma_j^2$, which define the log-return jump amplitude moments through $Q = \ln(1+J(Q))$. Hence, $J(Q)+1$ is log-normally distributed, with mean

$$E[J(Q)] = E\left[e^Q\right] - 1 = e^{\mu_j+\sigma_j^2/2} - 1 \ ,$$

and variance

$$\mathrm{Var}[J(Q)] = E^2\left[e^Q\right] \cdot \left(e^{\sigma_j^2} - 1\right) \ .$$

The basic moments of the stock log-return differential are

$$M_1^{(jd)} \equiv E[d[\ln(S(t))]] = (\mu_{ld} + \lambda\mu_j)dt \ , \tag{7}$$

$$\begin{aligned} M_2^{(jd)} &\equiv \mathrm{Var}[d[\ln(S(t))]] = \mathrm{Var}[\sigma_d dW(t)] \\ &\quad + \left(\mathrm{Var}[\mathbf{Q}] + \mathbf{E^2}[\mathbf{Q}]\right)\mathrm{Var}[\mathbf{dP(t)}] \\ &= (\sigma_{\mathbf{d}}^{\mathbf{2}} + \lambda(\sigma_{\mathbf{j}}^{\mathbf{2}} + \mu_{\mathbf{j}}^{\mathbf{2}}))\mathbf{dt} \ . \end{aligned} \tag{8}$$

The log-return is the primary model independent variable of interest in this paper, as the investor is interested in the percent or relative change in a portfolio and the log-return is the continuous limit of the relative change.

Next the log-normal density will be found by basic probabilistic methods and the results are summarized in the following theorem:

Theorem: The probability density for the log-normal jump-diffusion log-return differential $d[\ln(S(t))]$ specified in the SDE (5) is given by

$$\phi_{d\ln(S(t))}(z) = \sum_{k=0}^{\infty} p_k(\lambda dt)\phi_{\mathbf{n}}(\mathbf{z}; \mu_{\mathbf{ld}}\mathbf{dt} + \mu_{\mathbf{j}}\mathbf{k}, \sigma_{\mathbf{d}}^{\mathbf{2}}\mathbf{dt} + \sigma_{\mathbf{j}}^{\mathbf{2}}\mathbf{k}) \ , \tag{9}$$

$-\infty < z < +\infty$, where the Poisson distribution $p_k(\lambda dt)$ is specified in (5) and the normal density ϕ_n is specified in (6).

Proof: The basic idea of the proof follows from finding the density of a sum of independent random variables, $X + YZ$ given the densities of the components X and the symbolicYZ. Here, the process $X = \mu_{ld}dt + \sigma_d dZ(t)$ which is the diffusion plus log-drift term, normally distributed with density $\phi_n(z; \mu_{ld}dt, \sigma_d^2 dt)$, **the compound Poisson process $\mathbf{YZ} = \sum_{\mathbf{i=1}}^{\mathbf{dP(t)}} \mathbf{Q_i}$ which is the combined jump amplitude marks Q_i of the log-return, independent identically**, normally distributed with density $\phi_n(y; \mu_j, \sigma_j^2)$ and $Z = dP(t)$ which is the differential Poisson process. The discrete distribution of the Poisson process given in (5).

According to Feller [8], the density of a sum of independent random variables is given by a convolution of densities,

$$\phi_{X+YZ}(z) = (\phi_X * \phi_{YZ})(z) \equiv \int_{-\infty}^{+\infty} \phi_X(z-y)\phi_{YZ}(y)dy \; . \tag{10}$$

but before calculating the convolution the density for the compound random variable $\phi_{YZ}(x)$, the density of the *compound Poisson-Normal* process, is needed.

Since for each Poisson jump count k the compound Poisson Process XY is the sum of k independent, normally distributed random variables for $k > 0$, so by the law of total probability

$$\begin{aligned}\phi_{YZ}(x) &\equiv \text{Prob}\left[\sum_{i=1}^{dP(t)} Q_i \le x\right] = \sum_{k=0}^{\infty} p_k(\lambda dt)\text{Prob}\left[\sum_{i=1}^{k} Q_i \le x\right] \\ &\equiv \sum_{k=0}^{\infty} p_k(\lambda dt)\Phi_{\sum_i^k Q_i}(x) \; .\end{aligned}$$

Thus, the deriative of the distribution of the kth jump sum above will be a set of nested convolutions of an identically, normally distributed random variables Q_i and when combined with the normally distributed diffusion density Z, we obtain

$$\begin{aligned}\phi_{X+YZ}(z) &= \phi_Z(z) + \sum_{k=1}^{\infty} p_k(\lambda dt)\left(\left(\prod_{i=1}^{k} \phi_{Q_i} *\right)\phi_Z\right)(z) \\ &= \phi_Z(z) + \sum_{k=1}^{\infty} p_k(\lambda dt)\left((\phi_Q *)^k \phi_Z\right)(z) \; ,\end{aligned}$$

the last step being due to the identically distributed property. Finally, the fact that the convolution of two normal densities is a normal density with a mean that is the sum of the means and a variance that is the sum of the variances leads to a normal density of each k jump counts upon recursion. This result is based

upon the identity for the product of two normal distributions and is derived through the application of the *completing the square* technique combining a product of two normal densities into one,

$$\phi_{\mathbf{n}}(\mathbf{x};\mu_{\mathbf{1}},\sigma_{\mathbf{1}}^{\mathbf{2}})\cdot\phi_{\mathbf{n}}(\mathbf{x};\mu_{\mathbf{2}},\sigma_{\mathbf{2}}^{\mathbf{2}}) = \phi_{\mathbf{n}}\left(\mathbf{x};\frac{\mu_{\mathbf{1}}\sigma_{\mathbf{2}}^{\mathbf{2}}+\mu_{\mathbf{2}}\sigma_{\mathbf{1}}^{\mathbf{2}}}{\sigma_{\mathbf{1}}^{\mathbf{2}}+\sigma_{\mathbf{2}}^{\mathbf{2}}},\frac{\sigma_{\mathbf{1}}^{\mathbf{2}}\sigma_{\mathbf{2}}^{\mathbf{2}}}{\sigma_{\mathbf{1}}^{\mathbf{2}}+\sigma_{\mathbf{2}}^{\mathbf{2}}}\right) \cdot\frac{\mathbf{1}}{\sqrt{\mathbf{2}\pi(\sigma_{\mathbf{1}}^{\mathbf{2}}+\sigma_{\mathbf{2}}^{\mathbf{2}})}}\exp\left(-\frac{(\mu_{\mathbf{1}}-\mu_{\mathbf{2}})^{\mathbf{2}}}{\mathbf{2}(\sigma_{\mathbf{1}}^{\mathbf{2}}+\sigma_{\mathbf{2}}^{\mathbf{2}})}\right) \tag{11}$$

Letting the X_i be independent normal random variables with density $\phi_{X_i}(x)$, mean μ_i and variance σ_i^2 for $i=1$ to K, then using (11),

$$\begin{aligned}\mathcal{I}_{\mathbf{2}}(\mathbf{x}) &= (\phi_{\mathbf{X_1}}*\phi_{\mathbf{X_2}})(\mathbf{x}) = \int_{-\infty}^{+\infty}\phi_{\mathbf{X_1}}(\mathbf{x}-\mathbf{y})\phi_{\mathbf{X_2}}(\mathbf{y})\mathbf{dy} \\ &= \frac{\mathbf{1}}{\sqrt{\mathbf{2}\pi(\sigma_{\mathbf{1}}^{\mathbf{2}}+\sigma_{\mathbf{2}}^{\mathbf{2}})}}\exp\left(-\frac{(\mathbf{x}-\mu_{\mathbf{1}}-\mu_{\mathbf{2}})^{\mathbf{2}}}{\mathbf{2}(\sigma_{\mathbf{1}}^{\mathbf{2}}+\sigma_{\mathbf{2}}^{\mathbf{2}})}\right) \\ &\quad\cdot\int_{-\infty}^{+\infty}\phi_{\mathbf{n}}\left(\mathbf{y};\frac{(\mathbf{x}-\mu_{\mathbf{1}})\sigma_{\mathbf{2}}^{\mathbf{2}}+\mu_{\mathbf{2}}\sigma_{\mathbf{1}}^{\mathbf{2}}}{\sigma_{\mathbf{1}}^{\mathbf{2}}+\sigma_{\mathbf{2}}^{\mathbf{2}}}\frac{\sigma_{\mathbf{1}}^{\mathbf{2}}\sigma_{\mathbf{2}}^{\mathbf{2}}}{\sigma_{\mathbf{1}}^{\mathbf{2}}+\sigma_{\mathbf{2}}^{\mathbf{2}}}\right)\mathbf{dy} \\ &= \phi_{\mathbf{n}}(\mathbf{x};\mu_{\mathbf{1}}+\mu_{\mathbf{2}},\sigma_{\mathbf{1}}^{\mathbf{2}}+\sigma_{\mathbf{2}}^{\mathbf{2}})\,.\end{aligned} \tag{12}$$

This is the induction initial condition with $K=2$, giving the techniques for a proof by induction using the induction hypothesis for general K,

$$\mathcal{I}_{\mathbf{K}}(\mathbf{x}) \equiv \left(\left(\prod_{\mathbf{i=1}}^{\mathbf{K-1}}\phi_{\mathbf{X_i}}*\right)\phi_{\mathbf{X_K}}\right)(\mathbf{x}) = \phi_{\mathbf{n}}\left(\mathbf{x};\sum_{\mathbf{i=1}}^{\mathbf{K}}\mu_{\mathbf{i}},\sum_{\mathbf{i=1}}^{\mathbf{K}}\sigma_{\mathbf{i}}^{\mathbf{2}}\right). \tag{13}$$

Reusing the induction initial condition calculation, with μ_1 replaced by $\sum_{i=1}^{K}\mu_i$ and μ_2 replaced by $\sum_{i=1}^{K}\sigma_i^2$, and adding means and variances yields the proof by induction result,

$$\mathcal{I}_{\mathbf{K+1}}(\mathbf{x}) = \left(\mathcal{I}_{\mathbf{K}}*\phi_{\mathbf{X_{K+1}}}\right)(\mathbf{x}) = \phi_{\mathbf{n}}\left(\mathbf{x};\sum_{\mathbf{i=1}}^{\mathbf{K+1}}\mu_{\mathbf{i}},\sum_{\mathbf{i=1}}^{\mathbf{K+1}}\sigma_{\mathbf{i}}^{\mathbf{2}}\right). \tag{14}$$

For the final result take $X_1=Z$, the diffusion with $\mu_1=\mu_{ld}dt$ and $\sigma_1^2=\sigma_d^2dt$, and take $X_{i+1}=Q_i$, the indentically distributed jump amplitudes with $\mu i+1=\mu_j$ and $\sigma_i^2=\sigma_j^2$ for $i=1$ to k for each k count, so that

$$\phi_{\mathbf{d}\ln(\mathbf{S(t)})}(\mathbf{x}) = \sum_{\mathbf{k=0}}^{\infty}\mathbf{p_k}(\lambda\mathbf{dt})\phi_{\mathbf{n}}\left(\mathbf{x};\mu_{\mathbf{ld}}\mathbf{dt}+\mu_{\mathbf{j}}\mathbf{k},\sigma_{\mathbf{d}}^{\mathbf{2}}\mathbf{dt}+\sigma_{\mathbf{j}}^{\mathbf{2}}\mathbf{k}\right). \tag{15}$$

This completes the proof of the Theorem. △

Note that the log-return density is the Poisson mean over a normal density since (9) can be rewritten as

$$\phi_{d\ln(S(t))}(z) = E_{dP(t)}\left[\phi_n(z;\mu_{ld}dt + \mu_j dP(t), \sigma_d^2 dt + \sigma_j^2 \mathbf{dP(t)})\right]$$

where the Poisson jump counter k has been replaced by the simple differential Poisson process $dP(t)$ and the normal density ϕ_n is specified in (6). Thus the density is the expectation over a normal distribution with jumping mean and variance, the jump in the mean scaled by the mark jump mean μ_j and the jump in the variance scaled by the mark jump variance σ_j^2, respectively.

In the 2001 thesis of Düvelmeyer [5], the probability distribution function for the log of the process $\ln(S(t))$, rather than the density of the log-returns $d\ln(S(t))$ needed here, is given, but **our results in the theorem now agree with [5] for the case of log-normally distributed jumps**. This thesis work appears to be the basis for Kluge's jump-diffusion version of the *Share Simulator* [13]. Also, the intended optimal portfolio and consumption application for the present paper is very different, not generalizations of the Black-Scholes option pricing as in the thesis [5].

Using the log-normal jump-diffusion log-return density in (9), the third and fourth central moments with finite return time dt **can be computed or the moments can be computed more simply and directly from the Poisson sum form of the SDE (5) with the sum in (4)** and verified with MapleV$^{\mathsf{TM}}$ [24] symbolic computation yielding

$$\begin{aligned}
M_3^{(jd)} &= E\left[\left(d[\ln(S(t))] - M_1^{(jd)}\right)^3\right] \qquad (16)\\
&= \mathbf{E}\left[\left(\sigma_\mathbf{d}\mathbf{dW(t)} + \sum_{\mathbf{i=1}}^{\mathbf{dP(t)}} \mathbf{Q_i} - \lambda\mathbf{dt}\right)^{\mathbf{3}}\right]\\
&= \sigma_\mathbf{d}^\mathbf{3}\mathbf{0} + \mathbf{3}\sigma_\mathbf{d}^\mathbf{2}\mathbf{dt}\ \mathbf{0} + \mathbf{3}\sigma_\mathbf{d}\mathbf{0} + \mathbf{E}\left[\left(\sum_{\mathbf{i=1}}^{\mathbf{dP(t)}} (\mathbf{Q_i} - \mu_\mathbf{j}) + \mu_\mathbf{j}(\mathbf{dP(t)} - \lambda\mathbf{dt})\right)^{\mathbf{3}}\right]\\
&= \mathbf{E}_{\mathbf{dP(t)}}\left[\mathbf{E}_\mathbf{Q}\left[\left(\sum_{\mathbf{i=1}}^{\mathbf{dP(t)}} (\mathbf{Q_i} - \mu_\mathbf{j})\right)^{\mathbf{3}} + \mathbf{3}\mu_\mathbf{j}(\mathbf{dP(t)} - \lambda\mathbf{dt})\left(\sum_{\mathbf{i=1}}^{\mathbf{dP(t)}} (\mathbf{Q_i} - \mu_\mathbf{j})\right)^{\mathbf{2}}\right.\right.\\
&\qquad \left.\left. + \mathbf{3}\mu_\mathbf{j}^\mathbf{2}(\mathbf{dP(t)} - \lambda\mathbf{dt})^\mathbf{2} \sum_{\mathbf{i=1}}^{\mathbf{dP(t)}} (\mathbf{Q_i} - \mu_\mathbf{j}) + \mu_\mathbf{j}^\mathbf{3}(\mathbf{dP(t)} - \lambda\mathbf{dt})^\mathbf{3} \,\middle|\, \mathbf{dP(t)}\right]\right]\\
&= (\mathbf{3}\sigma_\mathbf{j}^\mathbf{2} + \mu_\mathbf{j}^\mathbf{2})\mu_\mathbf{j}\lambda\mathbf{dt}\,;
\end{aligned}$$

$$
\begin{aligned}
M_4^{(jd)} &= E\left[\left(d[\ln(S(t))] - M_1^{(jd)}\right)^4\right] \qquad (17)\\
&= \mathbf{E}\left[\left(\sigma_\mathbf{d}\mathbf{dW(t)} + \sum_{\mathbf{i=1}}^{\mathbf{dP(t)}} \mathbf{Q_i} - \lambda\mathbf{dt}\right)^{\mathbf{4}}\right]\\
&= \mathbf{3}\sigma_\mathbf{d}^\mathbf{4}\mathbf{(dt)^2} + \mathbf{4}\sigma_\mathbf{d}^\mathbf{3}\mathbf{0} + \mathbf{6}\sigma_\mathbf{d}^\mathbf{2}\mathbf{dtE}\left[\left(\sum_{\mathbf{i=1}}^{\mathbf{dP(t)}} (\mathbf{Q_i} - \mu_\mathbf{j}) + \mu_\mathbf{j}(\mathbf{dP(t)} - \lambda\mathbf{dt})\right)^{\mathbf{2}}\right]\\
&\quad + \mathbf{4}\sigma_\mathbf{d}\mathbf{0} + \mathbf{E}\left[\left(\sum_{\mathbf{i=1}}^{\mathbf{dP(t)}} (\mathbf{Q_i} - \mu_\mathbf{j}) + \mu_\mathbf{j}(\mathbf{dP(t)} - \lambda\mathbf{dt})\right)^{\mathbf{4}}\right]\\
&= \mathbf{3}\sigma_\mathbf{d}^\mathbf{4}\mathbf{(dt)^2} + \mathbf{6}\sigma_\mathbf{d}^\mathbf{2}\mathbf{dtE_{dP(t)}}\left[\mathbf{E_Q}\left[\left(\sum_{\mathbf{i=1}}^{\mathbf{dP(t)}} (\mathbf{Q_i} - \mu_\mathbf{j})\right)^{\mathbf{2}}\right.\right.\\
&\quad \left.\left. + \mathbf{2}\mu_\mathbf{j}(\mathbf{dP(t)} - \lambda\mathbf{dt})\sum_{\mathbf{i=1}}^{\mathbf{dP(t)}} (\mathbf{Q_i} - \mu_\mathbf{j}) + \mu_\mathbf{j}^\mathbf{2}(\mathbf{dP(t)} - \lambda\mathbf{dt})^\mathbf{2}\right| \mathbf{dP(t)}\right]\\
&\quad + \mathbf{E_{dP(t)}}\left[\mathbf{E_Q}\left[\left(\sum_{\mathbf{i=1}}^{\mathbf{dP(t)}} (\mathbf{Q_i} - \mu_\mathbf{j})\right)^{\mathbf{4}}\right.\right.\\
&\quad + \mathbf{4}\mu_\mathbf{j}(\mathbf{dP(t)} - \lambda\mathbf{dt})\left(\sum_{\mathbf{i=1}}^{\mathbf{dP(t)}} (\mathbf{Q_i} - \mu_\mathbf{j})\right)^{\mathbf{3}}\\
&\quad + \mathbf{6}\mu_\mathbf{j}^\mathbf{2}(\mathbf{dP(t)} - \lambda\mathbf{dt})^\mathbf{2}\left(\sum_{\mathbf{i=1}}^{\mathbf{dP(t)}} (\mathbf{Q_i} - \mu_\mathbf{j})\right)^{\mathbf{2}}\\
&\quad \left.\left. + \mathbf{4}\mu_\mathbf{j}^\mathbf{3}(\mathbf{dP(t)} - \lambda\mathbf{dt})^\mathbf{3}\sum_{\mathbf{i=1}}^{\mathbf{dP(t)}} (\mathbf{Q_i} - \mu_\mathbf{j}) + \mu_\mathbf{j}^\mathbf{4}(\mathbf{dP(t)} - \lambda\mathbf{dt})^4\right| \mathbf{dP(t)}\right]\\
&= (\mathbf{3}\sigma_\mathbf{j}^\mathbf{4} + \mathbf{6}\mu_\mathbf{j}^\mathbf{2}\sigma_\mathbf{j}^\mathbf{2} + \mu_\mathbf{j}^\mathbf{4})\lambda\mathbf{dt}\\
&\quad + \left(\mathbf{3}\lambda^\mathbf{2}(\sigma_\mathbf{j}^\mathbf{2} + \mu_\mathbf{j}^\mathbf{2})^\mathbf{2} + \mathbf{6}\lambda\sigma_\mathbf{d}^\mathbf{2}(\sigma_\mathbf{j}^\mathbf{2} + \mu_\mathbf{j}^\mathbf{2}) + \mathbf{3}\sigma_\mathbf{d}^\mathbf{4}\right)\mathbf{(dt)^2}\,,
\end{aligned}
$$

where the binomial expansion, process independence, zero mean forms, and conditional expectation with respect to $dP(t)$ **have been used. Equations (16-17)** are used to compute the theoretical coefficients of skew (1) and kurtosis (2), respectively. **Terms of** $O((dt)^2$ **are retained in the fourth moment (17) for use in fitting empirical market data when the time increment is not infinitesimal.**

3 Log-Normal Jump-Diffusion Model Parameter Estimation

Now the log-normal jump-diffusion density (9) is available for fitting to realistic data to obtain some of the parameters of the log-normal diffusion model (5) for $d[\ln(S(t))]$. For realistic data, the daily closings of the Standard and Poor's 500 (S&P500) stock index from 1995 to July 2001 will be used from data available on-line [26]. The data consists of $nsp = 1657$ points. The S&P500 data is an example of one large mutual fund rather than a single stock but has the advantage of not being biased severely to any one stock. The data has been transformed into the discrete analog of the continuous log-return, i.e., into changes in the natural logarithm of the index closings, $\Delta[\ln(SP_i)] \equiv \ln(SP_{i+1}) - \ln(SP_i)$ for $i = 1, \ldots, nsp - 1$ points. The scatter for the $nsp - 1 = 1656$ points of $\Delta[\ln(SP_i)]$ is shown in Figure 1 versus time in years, along with confidence intervals for one, two and three standard deviations. A slight, but noticeable, time dependence of the local mean and volatility is seen, but the time-dependent behavior is the topic of another paper and the constant coefficient case needs to be treated here. In different view, the histogram of the data using 50 equally spaced data bins is given in Figure 2. The mean is $M_1^{(sp)} \simeq 5.754 \times 10^{-4}$ and the variance is $M_2^{(sp)} \simeq 1.241 \times 10^{-4}$, the coefficient of skew is

$$\eta_3^{(sp)} \equiv M_3^{(sp)}/(M_2^{(sp)})^{1.5} \simeq -0.2867$$

which is negative and the coefficient of kurtosis or normalized fourth central moment is

$$\eta_4^{(sp)} \equiv M_4^{(sp)}/(M_2^{(sp)})^{2} \simeq 6.862\ .$$

Compared to the normal distribution, the empirical distribution has negative skew while the normal distribution has zero skew. Also, the empirical kurtosis is 2.3 times the normal distribution kurtosis of 3. The S&P500 log-return skew and kurtosis are characteristic of log-returns of many market instruments as noted in the Introduction.

Using MATLAB™ [20], the theoretical log-normal jump-diffusion density $\phi_{d\ln(S(t))}$ in (9) is compared to the 50 bin histogram shown in Figure 2 by discretizing the theoretical density using the same 50 bin data structure as for the histogram. However, the five parameter set $\{\mu_d, \sigma_d^2, \mu_j, \sigma_j^2, \lambda dt\}$ had to be reduced to a more manageable set to avoid large fitting errors and to preserve the *Principle of Modeling Parsimony* (striving for economies or simplicity of the model). The empirical return time is taken as the reciprocal average number of trading days per year or $1/252.3 \simeq 0.003964 = dt$ for the data used (250 trading days seems to be a standard value [14]), so the empirical dt is small, but not infinitesimal. Two parameters, μ_d and σ_d^2, were eliminated by forcing the theoretical means and variances to be the same as

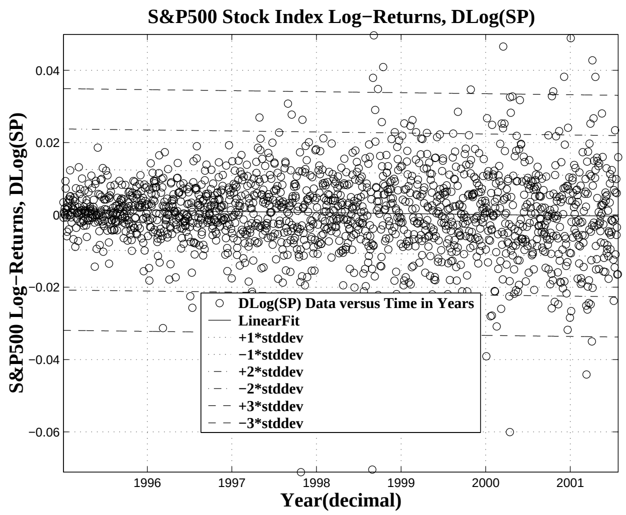

Fig. 1. Daily changes in the logarithm of the S&P500 stock index. Linear fit (light solid line) is nearly zero and horizontal. The confidence intervals for one (68%), two (95%) and three (99%) standard deviations are presented (light dashed lines)

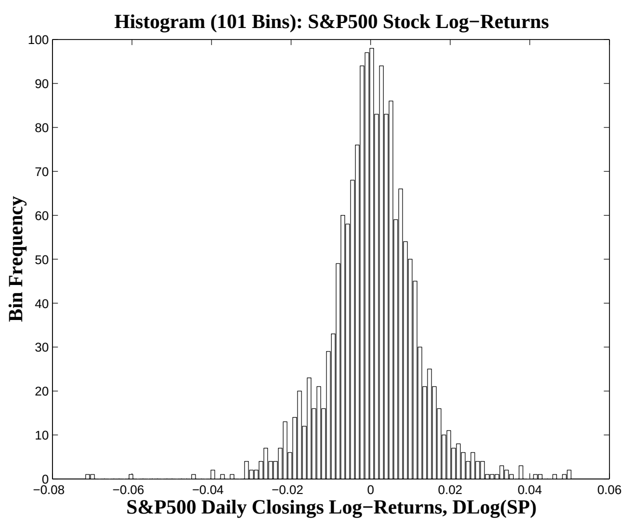

Fig. 2. Histogram of daily changes in the logarithm of the S&P500 stock index

the mean and variance of the empirical data, respectively.

$$\sigma_{\mathbf{d}}^{\mathbf{2}} = (\mathbf{M_2} - \lambda \mathbf{dt}(\sigma_{\mathbf{j}}^{\mathbf{2}} + \mu_{\mathbf{j}}^{\mathbf{2}}))/\mathbf{dt}\ , \tag{18}$$

$$\mu_d = (M_1 - \lambda dt \mu_j)/dt\ , \tag{19}$$

using the log-return moment formulas (7,8). The objective is to select the reduced set $\{\mu_j, \sigma_j^2, \lambda dt\}$ then to minimize the variance (i.e., histogram least squares) of the difference between the empirical and theoretical model distribution. The search stopping criteria was that the final maximum relative change in successive values of the parameters μ_j, σ_j^2 and λdt plus the relative change in the variance of the histogram difference was less than a 0.5e-3 tolerance. Due to the complexity of the jump-diffusion density and the need to keep finance methods simple, a multi-dimensional modification of *Golden Section Search* that needs no derivatives and searches beyond the current range when a local minimum is not found in the current search hypercube [10]. In addition, hypercube constraints were implemented so that the free model parameters $\{-\mu_j, \sigma_j^2, \lambda dt\}$ would remain non-negative and be bounded. The final parameter results are

$$
\begin{aligned}
&\mu_d \simeq 0.2712\ , \quad \sigma_d^2 \simeq 0.01048\ , \\
&\mu_j \simeq -0.0007474\ , \quad \sigma_j^2 \simeq 0.00007812\ , \quad \lambda \simeq 161.7\ ,
\end{aligned} \tag{20}
$$

with a final variance of the deviation of 11.45 and an mean deviation of 2.68×10^{-2}, with a total frequency count of 1656 over all bins. Also, note that the diffusion mean and variance have dimension per year, while the jump mean and variance are dimensionless so the jump values should be weighted by the jump intensity λ for a jump to diffusion comparison (i.e., the jump values are not as relatively small as they seem). For a better empirical to theoretical comparison, the moments of the deviation of the empirical S&P500 histograms from the normal density histogram, where the normal density matches only the empirical mean and variance since there are only two parameters to match given dt, with a mean deviation of -6.015×10^{-5}, but a deviation variance of 56.32, almost five times deviation variance of the fit jump-diffusion. The corresponding jump-diffusion normalized higher moments are the coefficients

$$\eta_3^{(jd)} \equiv M_3^{(jd)} / \left(M_2^{(jd)}\right)^{1.5} \simeq -0.2114$$

for skew and

$$\eta_4^{(jd)} \equiv M_4^{(jd)} / \left(M_2^{(jd)}\right)^{2} \simeq 8.082$$

for kurtosis, whose values are qualitatively similar to the empirical values with somewhat larger (18%) leptokurtosis and thinner (-26%) negative tails, but much more realistic than the normal density model with zero skew and kurtosis of three.

Note that the single log-normal jump amplitude distribution and the minimum variance comparison yields only a very slender distribution (very small σ_j^2) around a negative mean (μ_j), so only the dominant negative tail is represented and not the subdominant positive tail. The quality of the fit may be due to the simple minimum variance technique used, the single log-normal jump amplitude distribution and the fully infinite theoretical domain. A better model would use a log-bi-normal distribution similar to the log-bi-discrete distribution used in [9] with a positive as well as a negative discrete jump, but enhanced with a slight spread. Such a log-bi-normal distribution would add three more unknown parameters to search for, i.e., additional mean, variance and the probability of positive jump relative to a negative jump. The multidimensional Golden Section Search could be used, although it has the slowness of a general method, one that needs no derivatives. If more speed and accuracy were required, a nonlinear least squares method such as that of Levenberg-Marquardt [21] hybrid method can be used, but the parameter gradient of the log-normal jump-diffusion density would be needed, which could be facilitated by symbolic computation like MapleV™ [24]. The histogram of the final discretized theoretical density is displayed in Figure 3 and the deviation of the empirical S&P500 from the theoretical jump-diffusion histogram data is displayed in Figure 4. The discrepancy between the empirical and theoretical is best seen in the difference histogram in Figure 4, considering that the frequency scale being ten times finer for the positive deviations than in full histogram Figure 3, with significant frequency deviations of $(-15, +10)$ around the mode and adjacent shoulders fairly well-distributed within the deviation range of $(-0.03, +0.02)$.

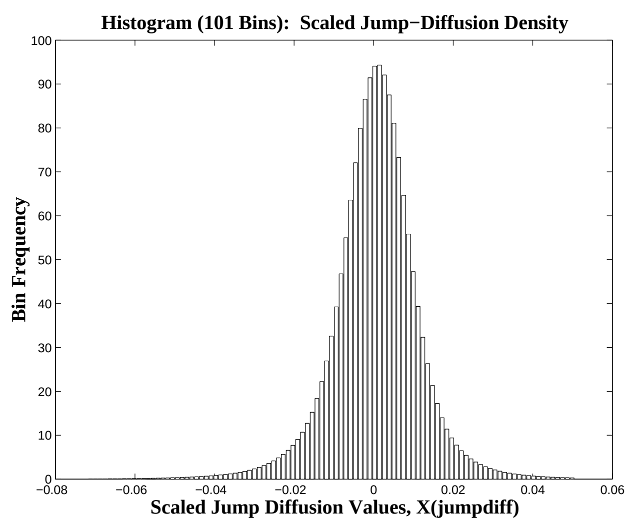

Fig. 3. Histogram of log-normal jump-diffusion as in Figure 2 with same bin structure

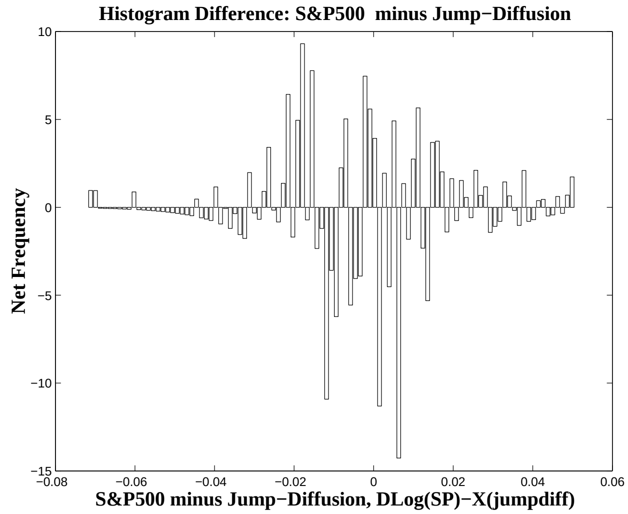

Fig. 4. Difference of histograms with daily changes in the logarithm of the S&P500 stock index minus the log-normal jump-diffusion bin values (Note: scale in this figure is one tenth the scale of the previous figure)

Conclusions

The probability density for a jump-diffusion whose jump amplitudes are distributed log-normally has been found and rigorously justified using basic probabilistic theory. This density is a discrete weighted sum of normal densities whose parameters depend on the $\{\mu_d, \sigma_d^2\}$ mean-variance parameters of the continuous drift-diffusion subsystem, the $\{\mu_j, \sigma_j^2\}$ mean-variance parameters of the log-normal jump mark distribution and the Poisson jump rate λ, with the weights being the Poisson discrete distribution of the jump counts. The log-normal jump-diffusion should be useful for fitting data for real investment markets with a distribution of random jumps, negative skew and leptokurtic properties that are not present in the standard log-normal or geometric diffusion model alone.

Acknowledgements

The authors thank Professor Bozenna Pasik-Duncan for the invitation to this splendid Kansas Workshop on Stochastic Theory in honor of Professor Tyrone Duncan's 60th birthday, providing local support via the University of Kansas and the National Science Foundation. The first author (FBH) acknowledges that this work was supported in part by the National Science Foundation Grant DMS-99-73231. **Additional thanks are due to Zongwu Zhu of**

the University of Illinois for pointing out some errors in the original paper that are here corrected.

References

1. Andersen L. and Andreasen J. (2000) Jump-Diffusion Processes: Volatility Smile Fitting and Numerical Methods of Option Pricing. Review of Derivatives Research 4:231-262
2. Barndorff-Neilsen O. E. and Shephard N. (2001) Non-Gaussian OU Based Models and Some of Their Uses in Financial Economics. J Roy Stat Soc Ser B 63:167-241
3. Black F. and Scholes M. (1973) The Pricing of Options and Corporate Liabilities. J Political Economy 81:637-659
4. Gihman I. I. and Skorohod A. V. (1979) Controlled Stochastic Processes. Springer-Verlag, New York
5. Düvelmeyer D. (2001) Untersuchungen zu Chancen und Risiken von Anti-Trend-Strategien am Beispiel des DAX-Futures. Thesis, Facultät für Mathematik, Technische Universität Chemnitz, Chemnitz, *URL: http://www-usercgi.tu-chemnitz.de/∼dana/diplom_pdf_dd.zip*
6. Evans M., Hastings N. and Peacock B. (2000) Statistical Distributions, 3rd edn. John Wiley, New York
7. Federal Open Market Committee (2001) Press Releases. Federal Reserve System, *URL: http://www.federalreserve.gov/FOMC/PreviousCalendars.htm*
8. Feller W. (1971) An Introduction to Probability Theory and Its Application, vol 2, 2nd edn. John Wiley, New York
9. Hanson F. B. and Westman J. J. (2001) Optimal Consumption and Portfolio Policies for Important Jump Events: Modeling and Computational Considerations. In: Proceedings of 2001 American Control Conference, 4456-4661
10. Hanson F. B. and Westman J. J. (2002) Golden Super Search: Multidimensional Modification of Golden Section Search Unrestricted by Initial Domain. in preparation
11. Kushner H. J. (1967) Stochastic Stability and Control. Academic Press, New York
12. Karatzas I., Lehoczky J. P., Sethi S. P. and Shreve S. E. (1986) Explicit Solution of a General Consumption/Investment Problem. Math Oper Res 11:261-294
13. Kluge T. (2000) Share Simulator: Black-Scholes, Heston and Jump-Diffusion Models. *URL: http://www.mathfinance.de/TinoKluge/*
14. Kou S. G. (2000) A Jump Diffusion Model for Option Pricing with Three Properties: Leptokurtic Feature, Volatility Smile, and Analytical Tractability. *URL: http://www.ieor.columbia.edu/∼kou/expo.pdf*
15. A. Lipton-Lifshitz (1999) Predictability and Unpredictability in Financial Markets. Physica D 133:321-347
16. Merton R. C. (1969) Lifetime Portfolio Selection under Uncertainty: The Continuous-Time Case. Rev Economics and Statistics 51:247-257
17. Merton R. C. (1971) Optimum Consumption and Portfolio Rules in a Continuous-Time Model. J Economic Theory 3:373-413
18. Merton R. C. (1976) Option Pricing When Underlying Stock Returns are Discontinuous. J Financial Economics 3:125-144

19. Merton R. C. (1990) Continuous-Time Finance. Basil Blackwell, Cambridge, MA
20. Moler C. et al. (2000) Using MATLAB, vers 6. Mathworks, Natick, MA
21. Press W. H. et al. (1992) Numerical Recipes in C: The Art of Scientific Computing. Cambridge University Press, Cambridge, UK
22. Rishel R. (1999) Modeling and Portfolio Optimization for Stock Prices Dependent on External Events. In: Proceedings of 38th IEEE Conference on Decision and Control 2788-2793
23. Sethi S. P. and Taksar M. (1988) A Note on Merton's Optimum Consumption and Portfolio Rules in a Continuous-Time Model. J Economic Theory, 46:395-401
24. Waterloo Maple (1998) Maple V. Waterloo Maple Inc, Waterloo, CA
25. Wilmott P. (2000) Paul Wilmott on Quantitative Finance, vol 1. John Wiley, New York
26. Yahoo! Finance (August 2001) Historical Quotes, S&P 500, Symbol ^SPC. *URL: http://chart.yahoo.com/*