Computational Methods for Portfolio and Consumption Policy Optimization in Log-Normal Diffusion, Log-Uniform Jump Environments.

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Abstract

Computational methods for a jump-diffusion portfolio optimization application using a log-uniform jump distribution are considered. In contrast to the usual geometric Brownian motion problem based upon two parameters, mean appreciation and diffusive volatility, the jump-diffusion model will have at least five, since jump process needs at least a rate, a mean and a variance, depending on the jump-amplitude distribution. As the number number of parameters increases, the computational complexity of the problem of determining the parameter set of the underlying model becomes greater. In a companion stochastic parameter estimation paper, real market data, here a decade of log-returns for Standard and Poor's 500 index closings, is used to fit the jump-diffusion parameters, with constraints based on matching the data mean and variance to keep the unconstrained parameter space to 3 dimensions. A weighted least squares method has been used. The jump-diffusion theoretical distribution and weights has been derived. In this computational paper, the computational features of a new multidimensional, derivative-less global search method used in the companion paper are discussed. The main part of this paper is to discuss the computational solution of an optimal portfolio and consumption finance application with these more realistic parameter results. The constant relative risk aversion (CRRA) canonical model is used to reduce the high dimensionality of the PDE of stochastic dynamic programming problem to something more reasonable. Many computational issues arise due to the jump process part of the model, since several jump integrals arise which are not present in the pure diffusion with drift model. The log-uniformly distributed jumps allow a wider range of portfolio policies than does previous work with normally distributed jumps.

1 Introduction

In portfolio optimization, large scale computations enter in at least two ways. The first is the estimation of realistic financial parameters from appropriate large scale financial market data, such that the parameter estimation does not suffer from over-fitting problems. The parameter estimation is facilitated by a robust, global, multi-dimensional, optimization tool, which is under development, and does not require derivatives. This computational tool is based upon the one-dimensional golden section search method. The stochastic theory for this way is treated in the companion paper [7].

The second way large scale computations enter is in the stochastic optimal control problem using computational dynamic programming for general Markov processes in continuous time to compute the portfolio and consumption of wealth optimization. The jump process leads to higher computational complexity since it adds global dependence in the form of jump integrals to the local dependence of the partial derivatives introduced by the diffusion process. The computational complexity of PDE of dynamic programming already suffers from the curse of dimensionality caused by the discretization of the PDE state space. Much of the computational complexity and large scale computations can be reduced by employing canonical models such as Constant Relative Risk-Aversion (CRRA) power utilities, for the utilities of terminal wealth and instantaneous consumption. The jump integral computations can be systematically handled by our generalization of Gaussian quadrature for general statistical distributions.

As an application to demonstrate these computational methods, we treat a stochastic optimal control problem, constrained by the stochastic dynamics of wealth and the investment objective is to maximize the conditional, expected discounted utilities of terminal wealth and instantaneous consumption.

2 Optimal Portfolio and Consumption Problem

Let S(t) be the price of a stock or mutual stock fund at time t that satisfies the jump–diffusion stochastic differential equation (SDE),

$$dS(t) = S(t)\left[\mu_d dt + \sigma_d dZ(t) + J(Q)dP(t)\right] \ , \quad S(0) = S_0, \quad S(t) > 0, \tag{2.1}$$

where μ_d is the mean appreciation return rate, σ_d is the diffusive volatility, $dZ(t)$ is a one–dimensional mean–zero differential diffusion process with variance dt, $J(Q)$ is a jump amplitude depending on a random variable Q with log–return mean μ_j and variance σ_j^2, and $dP(t)$ is a standard differential Poisson process with jump rate λ with common mean and variance of λdt. Here, we will assume that the jump–diffusion parameters μ_d, σ_d, μ_j, σ_j and λ are constants. The stochastic processes $dZ(t)$ and $dP(t)$ are Markov and pairwise independent. The jump amplitude process $J(Q)$, given a Poisson jump in time, is also independently distributed.

Equation (2.1) can be transformed to the more convenient log-return form by an application of the stochastic calculus chain rule to find the logarithmic differential yielding,

$$d[\ln(S(t))] \quad = \quad \mu_{ld}dt + \sigma_d dZ(t) + \ln(1 + J(Q))dP(t) \ , \tag{2.2}$$

where the log–diffusion drift $\mu_{ld} \equiv \mu_d - \sigma_d^2/2$ corrects the diffusion drift by the diffusion coefficient. Here, the random mark variable is chosen as the log-return jump amplitude, i.e., $Q = \ln(1+J(Q))$, uniformly distributed with density $\phi_Q(q) = \phi^{(u)}(q; Q_a, Q_b) = 1/(Q_b - Q_a)$ on $[Q_a, Q_b]$ where $Q_a < 0 < Q_b$. In a prior paper [5], a normal mark distribution was used with a fair amount of success, but the uniform distribution appears to be more realistic since it has finite support. Eq. (2.2) was used in our stochastic companion paper [7] to facilitate the parameter estimation.

In addition to a stock, the portfolio is hedged with a bond, which is assumed to satisfy a deterministic exponential process

$$dB(t) = rB(t)dt \ , \quad B(0) = B_0 \ . \tag{2.3}$$

with the bond price continuously compounded at a fixed rate of interest, r. Let $U_0(t)$ be the fraction of the instantaneous change in the portfolio due to changes in the bond investment and $U_1(t)$ be the fraction due to changes in the stock investment, such that $U_0(t) + U_1(t) = 1$.

The portfolio wealth process at time t changes due to changes in the portfolio fraction depending on the relative change in portfolio prices less instantaneous consumption of wealth:

$$dW(t) \;=\; W(t)\left[r dt + U_1(t)\left\{(\mu_d - r)dt \;+\; \sigma_d dZ(t) + J(Q)dP(t)\right\}\right] - C(t)dt \;, \tag{2.4}$$

where $C(t)$ is the instantaneous rate of consumption, assumed to be non–negative as well as constrained relative to wealth, i.e., $0 \le C(t) \le C^{(0)}_{\max} W(t)$ given $C^{(0)}_{\max}$, and $U_0(t) = 1 - U_1(t)$ has been eliminated by bond-stock fraction conservation.

The investor's objective is to maximize the conditional, expected current value of the discounted utility $\mathcal{U}_f(w)$ of terminal wealth at the end of the investment terminal time T and the discounted utility of instantaneous consumption, $\mathcal{U}(C(t))$, i.e.,

$$v^*(t,w) \;=\; \max_{\{u,c\}[t,T)}\left[E\left[e^{-\beta(T-t)}\mathcal{U}_f(W(T)) + \int_t^T e^{-\beta(\tau-t)}\mathcal{U}(C(\tau))d\tau \,\middle|\, \mathcal{C}(t)\right]\right] \;, \tag{2.5}$$

conditioned on the state–control set $\mathcal{C}(t) = \{W(t) = w, U_1(t) = u, C(t) = c\}$, where $0 \le t < T$, $0 \le c \le C^{(0)}_{\max} w$ for non–negative consumption feasibility with maximal relative limits $C^{(0)}_{\max}$, $w \ge 0$ for non–negative wealth feasibility, and $\beta > 0$ is a fixed discount rate. Thus, the instantaneous consumption $c = C(t)$ and stock portfolio fraction $u = U_1(t)$ serve as control variables, while the wealth $w = W(t)$ is the state variable. The objective (2.5) is subject to the terminal wealth condition $v^*(T,w) = \mathcal{U}_f(w)$ and zero wealth absorbing boundary condition to avoid the possibility of arbitrage [9],

$$v^*(t,0^+) = \mathcal{U}_f(0)e^{-\beta(T-t)} + \mathcal{U}(0)(1 - e^{-\beta(T-t)})/\beta \tag{2.6}$$

and assuming that the consumption must be zero when the wealth is zero.

Assuming the $v^*(t,w)$ is continuously differentiable in t and twice continuously differentiable in w (see [5] for more details), then the stochastic dynamic programming equation for Poisson jump versions follows from an application of the principle of optimality and the stochastic calculus chain rule to the

$$\begin{aligned} 0 \;=\;& v^*_t(t,w) - \beta v^*(t,w) + \mathcal{U}(c^*) + \left[(r + (\mu_d - r)u^*)w - c^*\right] v^*_w(t,w) \\ &+ \frac{1}{2}\sigma_d^2 (u^*)^2 w^2 v^*_{ww}(t,w) + \frac{\lambda}{Q_b - Q_a}\int_{Q_a}^{Q_b}\left[v^*(t,(1 + J(q)u^*)w) \;-\; v^*(t,w)\right] dq \;, \end{aligned} \tag{2.7}$$

where $u^* = u^*(t,w) \in [0,1]$ and $c^* = c^*(t,w) \in [0, C^{(0)}_{\max} w]$ are the optimal controls if they exist, while $v^*_w(t,w)$ and $v^*_{ww}(t,w)$ are the partial derivatives with respect to wealth w when $0 \le t < T$.

Non–negativity of wealth and the finite mark domain $[Q_a, Q_b]$ imply an additional consistency condition for the control, since $(1 + J(q)u^*)w$ is a wealth argument, $w \ge 0$ and $Q_a < 0 < Q_b$, then $1 + J(q)u \ge 0$ and consequently

$$U_{\min} \equiv -1/(\exp(Q_b) - 1) \le u \le +1/(1 - \exp(Q_a)) \equiv U_{\max}, \tag{2.8}$$

defines the u control domain for optimal objective (2.5). This result is in stark contrast to the $[0,1]$ control domain restriction found in [5] in the case of a normally distributed marks due to the infinite domain of the normal distribution. Here, recalling that the instantaneous bond fraction is $u_0 = 1 - u$, since $U_{\min} < 0$ then $U_{\min} < u < 0$ and $u_0 > 1$ mean that short-selling of stocks is permitted, while since $U_{\max} > 1$ then $1 < u < U_{\max}$ and $u_0 < 0$ mean that borrowing from bonds is permitted.

The utilities will be taken to be Constant Relative Risk–Aversion (CRRA) power utilities [9, Chapter 4-6] with the same power for wealth and consumption:

$$\mathcal{U}(x) = \mathcal{U}_f(x) = x^\gamma/\gamma \,, \quad x \ge 0, \quad 0 < \gamma < 1 \,. \tag{2.9}$$

These power utilities for this optimal consumption and portfolio problem lead to a canonical reduction in computational complexity for the stochastic dynamic programming PDE problem to a simpler ODE problem. The optimal utility value function has a solution separable in the wealth state variable and time,

$$v^*(t,w) = \mathcal{U}_f(w) v_0(t) \,, \tag{2.10}$$

where the wealth dependence is given explicitly and the time function is to be determined. Since $\mathcal{U}_f(0^+) = \mathcal{U}(0^+) = 0$ from (2.9), the absorbing boundary (2.6), i.e., $v^*(t,0^+)$, is automatically satisfied.

Further, the regular (unconstrained) consumption control is a linear function of the wealth,

$$c_{\rm reg}(t,w) \equiv w \cdot c^{(0)}_{\rm reg}(t) = w / v_0^{1/(1-\gamma)}(t) \,. \tag{2.11}$$

The regular stock fraction reduces to an implicitly defined wealth and time independent (essential for separability) control, $u_{\rm reg}(t,w) = u^{(0)}_{\rm reg}$,

$$\begin{aligned} u^{(0)}_{\rm reg} &= G(u^{(0)}_{\rm reg}) \equiv \frac{1}{(1-\gamma)\sigma_d^2}\left[\mu_d - r + \lambda I_1(u^{(0)}_{\rm reg})\right], \\ I_1(u) &\equiv \frac{1}{Q_b - Q_a}\int_{Q_a}^{Q_b} J(q)\,(1+J(q)u)^{\gamma-1}\,dq, \end{aligned} \tag{2.12}$$

where the uniform mark density on $[Q_a, Q_b]$ has been used. Since (2.12) only defines $u^{(0)}_{\rm reg}$ implicitly in fixed point form, $u^{(0)}_{\rm reg}$ must be found by iteration and a good choice is Newton's method [5], a fast and accurate fixed point method. The integrals are efficiently approximated by a 3–point Gauss–Statistics quadrature [13, 5] (a general Gaussian quadrature that, with a standard log–uniform jump density , is the Gauss–Legendre quadrature, but on $[0,1]$ with different nodes $\{(5-\sqrt{15})/10, 5/10, (5+\sqrt{15})/10\}$ and weights $\{5/18, 8/18, 5/18\}$, having fifth degree polynomial precision). The optimal controls, when there are constraints, are given in the form: $c^*(t,w)/w = c_0^*(t) = \max[\min[c^{(0)}_{\rm reg}(t), C^{(0)}_{\max}], 0]$, provided $w > 0$, and $u^* = \max[\min[u^{(0)}_{\rm reg}, U_{\max}], U_{\min}]$, independent of w and t along with $u^{(0)}_{\rm reg}$.

Substitution of the separable power solution (2.10) and the regular controls in (2.11-2.12) into the stochastic dynamic programming equation (2.7), leads to an ODE,

$$0 = v_0'(t) + (1-\gamma)\left(g_1(u^*)v_0(t) + g_2(t)v_0^{\frac{\gamma}{\gamma-1}}(t)\right) \,, \tag{2.13}$$

$$\begin{aligned} g_1(u) &\equiv \frac{1}{1-\gamma}\left[-\beta + \gamma\left(r + u(\mu_d - r)\right) - \frac{\gamma(1-\gamma)}{2}\sigma_d^2 u^2 + \lambda(I_2(u) - 1)\right] \\ g_2(t) &\equiv \frac{1}{1-\gamma}\left[\left(\frac{c_0^*(t)}{c^{(0)}_{\rm reg}(t)}\right)^\gamma - \gamma\left(\frac{c_0^*(t)}{c^{(0)}_{\rm reg}(t)}\right)\right], \\ I_2(u) &\equiv \frac{1}{Q_b - Q_a}\int_{Q_a}^{Q_b} (1+J(q)u)^\gamma dq \,, \end{aligned} \tag{2.14}$$

for $0 \leq t < T$. The coupling of $v_0(t)$ to the time dependent part of the consumption term $c^{(0)}_{\rm reg}(t)$ in $g_2(t)$ (2.14), and the relationship of $c^{(0)}_{\rm reg}(t)$ to $v_0(t)$ in (2.11), means that the ODE (2.13) is actually highly nonlinear and thus (2.13) is only of Bernoulli type implicitly. The implicit Bernoulli equation (2.13) can be formerly transformed to a linear differential equation by using $\theta(t) = v_0^{1/(1-\gamma)}(t)$, to obtain, $0 = \theta'(t) + g_1(u^*)\theta(t) + g_2(t)$, whose general solution can be inverse transformed to the particular solution for the separated time function implicitly given by

$$v_0(t) \quad = \quad \theta^{1-\gamma}(t) = \left[e^{-g_1(u^*)(T-t)} \left(1 + \int_t^T g_2(\tau) e^{g_1(u^*)(T-\tau)} d\tau \right) \right]^{1-\gamma}, \tag{2.15}$$

using the final condition $v_0(T) = 1$. Hence, both $v_0(t)$ and $c^{(0)}_{\rm reg}(t)$ must be found by computational iteration (see [5] for more details). Assembling solution for the optimal value function is $v^*(t,w) = \mathcal{U}_f(w)v_0(t)$, requires only multiplication by the utility of wealth.

3 Computational Finance Results

In the companion stochastic parameter estimation paper [7], the authors deduced the following asymptotic result for the log-return in the form of a log-normal diffusion, log-uniform jump process in the case when the return-time Δt is not an infinitesimal:

Corollary 3.1 *As $\Delta t \to 0^+$, the log-uniform jump, log-normal diffusion density can be asymptotically approximated as*

$$\begin{aligned} \phi_{\Delta \ln(S(t))}(x) \quad &\sim \quad \phi^{(\rm jd)}(x) \\ &\equiv \quad (1-\lambda\Delta t)\phi^{(n)}(x;\mu_{ld}\Delta t, \sigma_d^2\Delta t) + \lambda\Delta t \frac{\Phi^{(n)}(x-Q_b, x-Q_a; \mu_{ld}\Delta t, \sigma_d^2 \Delta t)}{Q_b - Q_a}, \end{aligned} \tag{3.16}$$

neglecting $O((\Delta t)^2)$.

Here $\phi^{(n)}(x;\mu_{ld}\Delta t, \sigma_d^2\Delta t)$ is the normal density with mean $\mu_{ld}\Delta t$ and variance $\sigma_d^2\Delta t$, while $\Phi^{(n)}(x,y;\mu_{ld}\Delta t, \sigma_d^2\Delta t)$ is the corresponding normal distribution on $[x,y]$. In [7], the histogram of the theoretical density (3.16) was fit to the histogram of empirical market data, namely the log returns of the daily closings of the S & P 500 Index , $\Delta[\ln(SP_i)] \equiv \ln(SP_{i+1}) - \ln(SP_i)$ for $i = 1 : 2521$ values from 1992 to 2001. This fitting was by the weighted least squares method in which two of five jump-diffusion parameters were eliminated by matching the theoretical and empirical mean $M_1 = \mathrm{E}[\Delta \ln(S(t))]$ and variance $M_2 = \mathrm{Var}[\Delta \ln(S(t))]$.

The minima was determined by our general multi-dimensional search method *Golden Super Finder (GSF)* [8], that is a generalization of the one-dimensional Golden Section Search (GSS) method. GSF is obviously slow due to the computational intensity, but convenient for global optimization of complicated functions on powerful workstations. The GSF method has many modification over the usual Golden Section Search method: (1) it is multi-dimensional, (2) for N dimensions or variables there are 4^N nodes using four nodes per dimension (two golden interior nodes plus two endpoints), (3) all nodes are tested for the current minimum, (4) if the current minimum occurs at a purely golden interior point GSF proceeds with a golden contraction like GSS, but if current minimum is at an end point of any dimension then GSF shifts the hypercube golden template by two nodes in that dimension in search of a better minimum, and (5) a user can specify a bounding hypercube domain in which the GSF hypercube search cannot leave, e.g., preserving non-negativity of a variance parameter.

The final results for the jump-diffusion coefficients are

$$\mu_d \simeq 0.06386\,, \quad \sigma_d^2 \simeq 0.005513\,, \quad \mu_j \simeq 0.0007624\,, \quad \sigma_j^2 \simeq 0.0003679\,, \quad \lambda \simeq 55.46\,, \quad (3.17)$$

Here, the average time between trading days $\Delta t \simeq 0.003967$ was used since it was consistent with the assumption of small $O((\Delta t)^2)$ assumed in (3.16). Additional economic rate parameters that will be used are the average rate for Moody AAA bonds of $r \simeq 7.384\%$ for data in the period 1999-2001 [2], and a corresponding discount rate $\beta \simeq 6.884\%$, 50 basis point smaller than the bond rate as is typical with the Federal Market Rates. Other parameters are $\gamma = 0.50$ common terminal wealth and instant consumption CRRA utility powers, $C_{\max}^{(0)} = 0.75$ upper bound on consumption relative to wealth, and $T = 1$ trading year terminal time.

Fast and accurate approximations are very important in financial engineering computations, so the computations were coded in MATLAB™ [10] due to its facility for developing rapid prototype solutions.

In Figure 1, the numerical approximation to the optimal, expected utility $v^*(t,w)$ is shown versus wealth w in dollars and t in trading years. When viewed for fixed time t, $v^*(t,w)$ follows the CRRA power utility template in wealth w, whereas for fixed wealth w, $v^*(t,w)$ exhibits the dependence on the separated time function $v_0(t)$ in time t. In the finite difference representation, the wealth w-intervals have been transformed into constant intervals in the utility power w^γ since as a function of w the utility is not differentiable as $w \to 0^+$. Typically, the investor, given the terminal value $v*(T,w) = \mathcal{U}_f(w)$, is interested in the starting value $v^*(0,w)$ as as function of wealth, but since the problem here is autonomous dynamic programming also generates answers for lesser investment periods $T_0 < T$ for which $v^*(T-T_0,w)$ would be the starting value. The numerical result for the constant optimal stock fraction control is $u^*(t,w) \simeq 3.271$, the same as the regular stock fraction control $u_{\text{reg}}(t,w) \simeq 3.271$ which is well within the control domain $[U_{\min}, U_{\max}] \simeq [-28.93, +31.31]$ given the estimated bounds on the marks, $Q_a \le q \le Q_b$.

In Figure 2, the computational approximation of the optimal consumption policy or control $c^*(t,w)$ is displayed versus the time t in trading years and the wealth w in dollars using the CRRA power utility model. Recall that $c^*(t,w)$ is linear in the wealth w, but inversely proportional to the square of the separated optimal value time function $v_0(t)$ to the power $1/(1-\gamma) = 2.00$ here when $\gamma = 0.5$. Hence, lines constant in time are straight lines, while the dependence in time t for fixed wealth w in $[0,100]$ is proportional to the reciprocal square of $v_0(t)$, i.e., $v_0^{-2}(t)$.

4 Conclusions

The log–normal diffusion, log–uniform jump distribution has been demonstrated on the canonical optimal portfolio and consumption control problem. The log–uniform jump distribution has significant benefits over the log–normal jump distribution used in our prior paper [5] in that the stock fraction is not severely constrained on $[0,1]$ due to the finite domain of the uniform distribution, allowing for borrowing and short-selling, thus more realism. This uniform distribution is demonstrated on the optimal portfolio and consumption policy application, yielding optimal stock

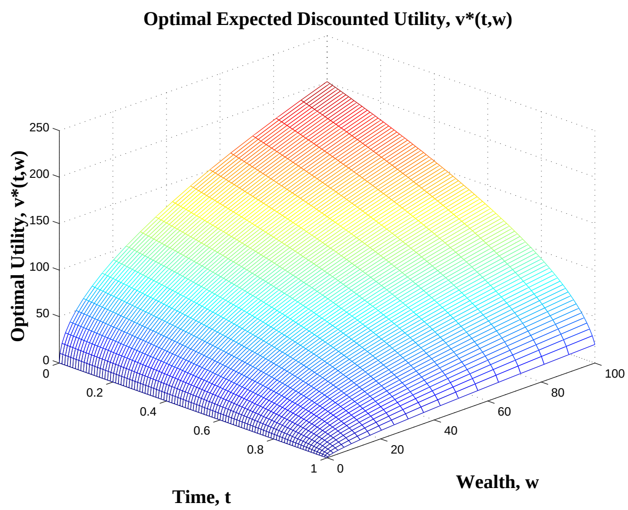

Figure 1: Optimal, expected utility numerical results $v^*(t, w)$ versus time t and wealth w for the CRRA power utility model.

fraction, consumption and expected discounted utility value.

Computational techniques are presented for handling the iterations for implicitly defined solutions such as the optimal stock fraction policy u^* and the coupled optimal value separated time function $v_0(t)$ and the optimal consumption policy c^*. Also, the Gauss–Statistics quadrature for handling the log–uniform jump amplitude integral has been used, but this technique is also useful for other jump distribution by using the appropriate standardized distribution. The features multi-dimensional optimizer Golden Super Finder [8] was used in a companion parameter estimation paper [7] have also been discussed.

Acknowledgement: Work supported in part by the National Science Foundation Computational Program Mathematics Grant DMS–99–73231.

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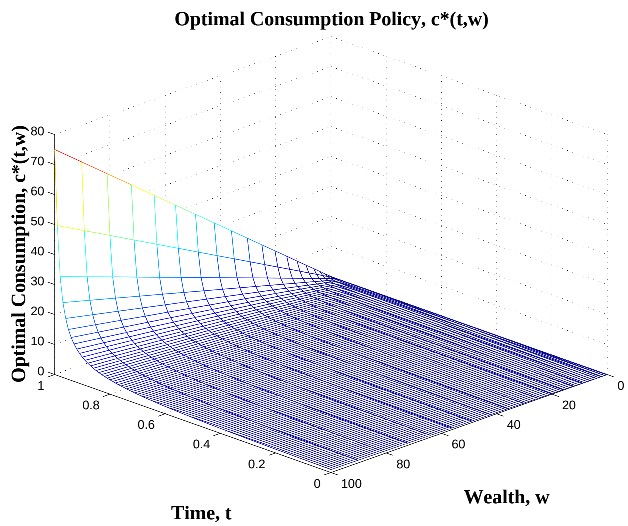

Figure 2: Optimal consumption policy numerical results $c^*(t, w)$ versus time t and wealth w for the CRRA power utility model.

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