Jump-Diffusion Stock Return Models in Finance: Stochastic Process Density with Uniform-Jump Amplitude

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Abstract

The stochastic analysis is presented for the parameter estimation problem for fitting a theoretical jump-diffusion model to the log-returns from closing data of the Standard and Poor's 500 (S&P500) stock index during the prior decade 1992-2001. The jump-diffusion model combines a the usual geometric Brownian motion for the diffusion and a space-time Poisson process for the jumps such that the jump amplitudes are uniformly distributed. The uniform jump distribution accounts for the rare large outlying log-returns, both negative and positive in magnitude. The log-normal, log-uniform jump-diffusion density is derived, leading to a jump-diffusion simulator approximation for the case the the log-return time is a small fraction of a year. There are five jump-diffusion parameters that need to be determined, the means and variances for both diffusion and jumps, as well as the jump rate, given the average log-return time. A weighted least squares is used to fit the theoretical jump-diffusion model to the S&P500 data optimizing with respect to three free parameters, with the two other parameters constrained by the mean and variance of the S&P500 data. The weight distribution derives from stochastic methods. The ideal fitted model determines the three free parameters, but the corresponding simulated results resemble the original S&P500 data better. This stochastic analysis paper is a companion to a computational methods and portfolio optimization paper at this conference.

1 Introduction

A classical model of financial market return process, such as the Black-Scholes [1, 8], is the log-normal diffusion process, such that the log-return process has a normal distribution. However, real markets exhibit several deviations from this ideal, although useful, model. The market distribution, say for stocks, should have several realistic properties not found in the ideal log-normal model: (1) the model must permit large random fluctuations such as crashes or sudden upsurges, (2) the log-return distribution should be skew since large downward outliers are larger than upward outliers, and (3) the distribution should be leptokurtic since the mode is usually higher and the tails thicker than for a normal distribution. For modeling these extra properties, a jump-diffusion process with log-uniform jump-amplitude Poisson process is used to fit the S & P 500 Index log-returns. A reasonable estimation of the parameters of the log-return process can be made using a weighted least squares approximation that is an improvement over earlier jump-diffusion model results of Merton [8] and the authors [2, 4, 5]. The computational issues are principally discussed in another paper of the authors at this conference [6].

2 Density for Jump-Diffusions

Let $S(t)$ be the price of a stock or stock fund satisfies a Markov, continuous-time, geometric, jump-diffusion stochastic differential equation (SDE),

$$dS(t) = S(t)\left[\mu_d dt + \sigma_d dZ(t) + J(Q)dP(t)\right] \ , \quad S(0) = S_0 \ , \quad S(t) > 0 \ , \tag{2.1}$$

where μ_d is the mean return rate, σ_d is the diffusive volatility, $Z(t)$ is a one-dimensional stochastic diffusion process, $J(Q)$ is a log-return mean μ_j and variance σ_j^2 random jump-amplitude and $P(t)$ is a simple Poisson jump process with jump rate λ. It is assumed that the stock price parameters μ_d, σ_d^2, μ_j, σ_j^2 and λ are constants. The differential diffusion process with drift $\mu_d dt + \sigma_d dZ(t)$ is has mean $\mu_d dt$ and $\sigma_d dt$ variance. The space-time jump process $J(Q)dP(t)$ has mean $E[J(Q)]\lambda dt$, variance $E[J^2(Q)]\lambda dt$ and $dP(t)$ has the discrete distribution

$$p_k(\lambda dt) = \text{Prob}[dP(t) = k] = \exp(-\lambda dt)(\lambda dt)^k/k!, \quad k = 0 : \infty \ . \tag{2.2}$$

The processes $Z(t)$ and $P(t)$ are pairwise independent, while $J(Q)$ is also independent except that it is conditioned on the existence of a jump in $dP(t)$.

Since the SDE (2.1) has a geometric or linear form it can can be transformed to the simplified log-return form using the stochastic process chain rule,

$$d[\ln(S(t))] = \mu_{ld} dt + \sigma_d dZ(t) + \ln(1 + J(Q))dP(t) \ , \tag{2.3}$$

where $\mu_{ld} dt = \mu_d - \sigma_d^2/2$ is the log-diffusion drift and $\ln(1+J(Q))$ is the log-return jump-amplitude. For finite log-return jump-amplitude and to avoid complete investment loss, $J(Q) > -1$, so the underlying random jump mark amplitude $Q = \ln(1+J(Q))$ on $(-\infty, +\infty)$ is chosen for convenience.

For this paper, we are interested in a uniformly distributed mark variable Q to account for the exceptionally long negative and positive tails in financial market distributions, as can seen in the histogram of the log-returns for S & P 500 Index [10] daily closings in the decade from 1992-2001 in Figure 1. Since large jumps in the log-returns seem to be rare events relative to the background ups and downs modeled by the diffusion process, the jump-amplitude distribution will be assumed to be uniformly distributed on $[Q_a, Q_b]$, $Q_a < 0 < Q_b$, with time-independent density

$$\phi_Q(q) \equiv \phi^{(u)}(q; Q_a, Q_b) \equiv \frac{U(q; Q_a, Q_b)}{Q_b - Q_a} \ , \tag{2.4}$$

where $U(x; a, b)$ denotes a unit step function on $[a, b]$, such that

$$\mu_j = (Q_a + Q_b)/2 \quad \text{and} \quad \sigma_j^2 = (Q_b - Q_a)^2/12. \tag{2.5}$$

Thus, the combined log-normal diffusion, log-uniform jump density derives from a triad form of random processes $\xi + \eta \cdot \zeta$, with diffusion $\xi = \mu_{ld} dt + \sigma_d dZ(t)$, jump-amplitude $\eta = Q$ and jump-time $\zeta = dP(t)$ processes. This density is proven in our time-dependent finance paper [5] and is given here in the modified form,

Theorem 2.1 *The probability density for the log–normal diffusion log–uniform jump–amplitude log–return differential* $d[\ln(S(t))]$ *specified in the SDE (2.3) is given by*

$$\begin{aligned} \phi_{d\ln(S(t))}(x) &= p_0(\lambda dt)\phi^{(n)}(x; \mu_{ld} dt, \sigma_d^2 dt) \\ &\quad + \sum_{k=1}^{\infty} p_k(\lambda dt) \frac{\Phi^{(n)}(x - kQ_b, x - kQ_a; \mu_{ld} dt, \sigma_d^2 dt)}{k(Q_b - Q_a)} \ , \end{aligned} \tag{2.6}$$

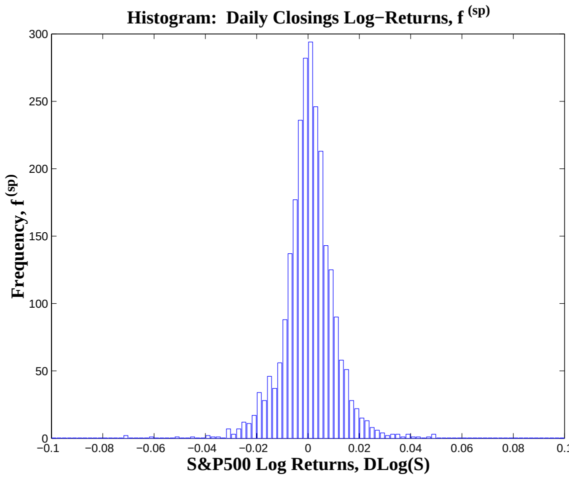

Figure 1: Histogram of log-return of daily closings in the S & P 500 Index for the decade 1992–2001, using 100 bins.

$-\infty < x < +\infty$, *where the Poisson distribution* $p_k(\lambda dt)$ *is specified in (2.2) and the normal distribution on* $[x, y]$ *is*

$$\Phi^{(n)}(x, y; \mu_{ld}dt, \sigma_d^2 dt) \equiv \int_x^y \phi^{(n)}(z; \mu_{ld}dt, \sigma_d^2 dt)dz \equiv \int_x^y \frac{\exp(-(z-\mu_{ld}dt)^2/(2\sigma_d^2 dt))}{\sqrt{2\pi\sigma_d^2 dt}}dz\ , \tag{2.7}$$

where the integrand is the normal density of the diffusion process $\xi = \mu_{ld}dt + \sigma_d dZ(t)$ *in (2.3).*

In the theorem there is no mention that dt is the infinitesimal of time, since it can be used for small but non-infinitesimal time increments Δt as needed in the financial markets. In the S & P 500 Index the average time between closings is $\Delta t = 0.003967$ years, so $(\Delta t)^2 = 0.00001574$ is negligible in comparison to Δt, if that would be sufficiently accurate. Hence, the two-term asymptotic form of (2.6) will be used:

Corollary 2.1 *As* $\Delta t \to 0^+$, *(2.6) can be asymptotically approximated as*

$$\begin{aligned} \phi_{\Delta\ln(S(t))}(x) &\sim \phi^{(\mathrm{jd})}(x) \qquad (2.8)\\ &\equiv (1-\lambda\Delta t)\phi^{(n)}(x; \mu_{ld}\Delta t, \sigma_d^2\Delta t) + \lambda\Delta t\frac{\Phi^{(n)}(x-Q_b, x-Q_a; \mu_{ld}\Delta t, \sigma_d^2\Delta t)}{Q_b - Q_a}\ , \end{aligned}$$

neglecting $O((\Delta t)^2)$.

Eq. (2.8) is consistent with the usual zero-or-one jump definition of the infinitesimal Poisson distribution given in full form by (2.2), such that there are zero jumps with probability $(1 - \lambda\Delta t)$ and one jump with probability $\lambda\Delta t$. Note that in (2.8) the zero-jump density is just the diffusion density, while the one-jump density can be called the *secant-normal* density since it is the ratio of the difference in normal distributions divided by the difference in arguments. Eq. (2.8) is also consistent with the small time form of the log-return in (2.3), such that

$$\Delta\ln(S(t)) = \int_t^{t+\Delta\tau} d\ln(S(\tau)) \sim \mu_{ld}\Delta t + \sigma_d\Delta Z(t) + Q\Delta P(t)\ , \tag{2.9}$$

provided the parameters are constant and higher order jumps are neglected, with $\Delta P(t)$ playing the role of an indicator function for either zero or one jump. Eq. (2.9) can also for the jump-diffusion simulations using $\sqrt{\Delta t}$ times a normal random number generator for $\Delta Z(t)$, a standard uniform generator on $[0,1]$ partitioned into $[0,\lambda\Delta t]$ for one-jump and $(\lambda\Delta t,1]$ for no-jump in $\Delta P(t)$, and a uniform generator on $[Q_a,Q_b]$ for simulating Q provided a one jump is selected by the simulation of $\Delta P(t)$.

3 Jump–Diffusion Parameter Estimation

For financial market modeling purposes, it is necessary to have an estimate of the parameters of the market distribution. For the log-normal diffusion, log-uniform jump-amplitude jump-diffusion theoretical model, there is a set of five parameters, $\{\mu_d,\sigma_d^2,\mu_d,\sigma_d^2,\lambda\}$, assuming the time-step Δt is known. The object of this paper is to estimate these parameters by fitting the theoretical model to the decade worth of log-returns of the S & P 500 Index from 1992 to 2001 portrayed in $N^{(\mathrm{bin})}=100$ histogram of Figure 1, subject to some constraints to keep the parameter estimation computationally reasonable. There are a total of 2522 daily closings $S_i^{(\mathrm{sp})}$, so that there are $N^{(\mathrm{sp})}=2521$ log-returns, $\Delta(\ln(S_i^{(\mathrm{sp})}))\equiv\ln(S_{i+1}^{(\mathrm{sp})})-\ln(S_i^{(\mathrm{sp})})$. The constraints used are matching the decade mean $M_1^{(\mathrm{sp})}\simeq 4.015\times10^{-4}$ and variance $M_2^{(\mathrm{sp})}\simeq 9.874\times10^{-5}$. Relative to the normal distribution, the higher order moment coefficients are $\eta_3^{(\mathrm{sp})}\equiv M_3^{(\mathrm{sp})}/(M_2^{(\mathrm{sp})})^{1.5}\simeq-0.2913$ for skewness and $\widehat{\eta}_4^{(\mathrm{sp})}\equiv M_4^{(\mathrm{sp})}/(M_2^{(\mathrm{sp})})^2-3\simeq 4.804$ for kurtosis, subtracting three for the unshifted normal kurtosis coefficient.

The distinguishing feature of real markets are the thicker tails that are longer on the negative side compared to normal distributions, leading to negative skew and larger kurtosis coefficients. Hence, it is important that the fitting of the distributions be sufficiently weighted so that the tails are sufficiently detectable. In our papers [4, 5], an unweighted least squares was used which resulted in the negative tails over-dominating the positive tails. Here, we use a weighted least squares or χ^2 fit (see for instance the summaries in [9]),

$$\chi^2=\sum_{i=1}^{N^{(\mathrm{bin})}}\omega_i\cdot\left(f_i^{(\mathrm{jd})}-f_i^{(\mathrm{sp})}\right)^2, \tag{3.10}$$

where ω_i is the weight of the ith bin, $f_i^{(\mathrm{sp})}$ is the ith empirical S & P 500 bin frequency data and $f_i^{(\mathrm{jd})}$ is the ith theoretical jump-diffusion bin frequency corresponding to the same sample size $N^{(\mathrm{sp})}=2521$. An estimate of the weights corresponding to a errors in measurements is not easy to get, but we will use the following theoretical result to be proved elsewhere:

Theorem 3.1 *If $f_i^{(\mathrm{jdsim})}=\sum_{j=1}^N U(\Delta S_j^{(\mathrm{jdsim})};x_i,x_{i+1}^-)$ for $i=1:N^{(\mathrm{bin})}$ are the frequencies of the ith bin $[x_i,x_{i+1})$ and $\Delta S_j^{(\mathrm{jdsim})}$ is the jth jump-diffusion simulation, using N samples, as prescribed for (2.9), then the bin frequency expectation and variance are*

$$\mu_{f_i^{(\mathrm{jdsim})}}=\mathrm{E}\left[f_i^{(\mathrm{jdsim})}\right]=f_i^{(\mathrm{jd})}\quad\text{and}\quad\sigma^2_{f_i^{(\mathrm{jdsim})}}=\mathrm{Var}\left[f_i^{(\mathrm{jdsim})}\right]=N\cdot\left(1-f_i^{(\mathrm{jd})}\Big/N\right)^2 f_i^{(\mathrm{jd})}, \tag{3.11}$$

respectively, where the ith expected bin frequency after N simulations is

$$f_i^{(\mathrm{jd})}=N\cdot\int_{x_i}^{x_{i+1}}\phi_i^{(\mathrm{jd})}(x)dx.$$

The bin weights are chosen as the theoretical values,

$$\omega_i = \left(1/\sigma^2_{f_i^{(\mathrm{jd})}}\right) \bigg/ \sum_{j=1}^{N^{(\mathrm{bin})}} \left(1/\sigma^2_{f_j^{(\mathrm{jd})}}\right) \;, \tag{3.12}$$

for $i = 1 : N^{(\mathrm{bin})}$ bins, normalized to a unit sum for convenience of small minima. The problem is reduced to a 3-dimensional global minimization for the transformed parameter set $\{Q_a, Q_b, \lambda\Delta t\}$ subject to constraints,

$$M_1^{(\mathrm{jd})} = \mu_{ld}\Delta t + \mu_j\lambda\Delta t = M_1^{(\mathrm{sp})} \quad \text{and} \quad M_2^{(\mathrm{jd})} = \sigma_d^2\Delta t + (\sigma_j^2 + \mu_j^2)\lambda\Delta t = M_2^{(\mathrm{sp})} \;, \tag{3.13}$$

serving as eliminants of $\mu_{ld}\Delta t$ and $\sigma_d^2\Delta t$, with the jump-moments definition (2.5) of μ_j and σ_j^2 relating them to Q_a and Q_b (in rare case, non-negativity must be enforced on the variances). The global minimizer *Golden Super Finder (GSF)* [7], developed for financial problems in [4, 5], was used to estimate the fit (3.10). This method is an extensive modification of the method of golden section search (see [9]) and is described more in [6]. The final parameter results are

$$\mu_d \;\simeq\; 0.06386 \;, \quad \sigma_d^2 \simeq 0.005513 \;, \mu_j \simeq 0.0007624 \;, \quad \sigma_j^2 \simeq 0.0003679 \;, \quad \lambda \simeq 55.46 \;, \tag{3.14}$$

with minimum $\chi^2_{\min} \simeq 2.621 \times 10^{-5}$ with a relative value-location hybrid stopping criterion of 5×10^{-3} in a total of 16 GSF-iterations.

The final successful minimum weighted least squares iteration results are illustrated in Figure 2, with both theoretical and simulation histograms. The histogram on the right for the simulations more closely resembles the S & P 500 data histogram, the S & P 500 being a large realistic simulation.

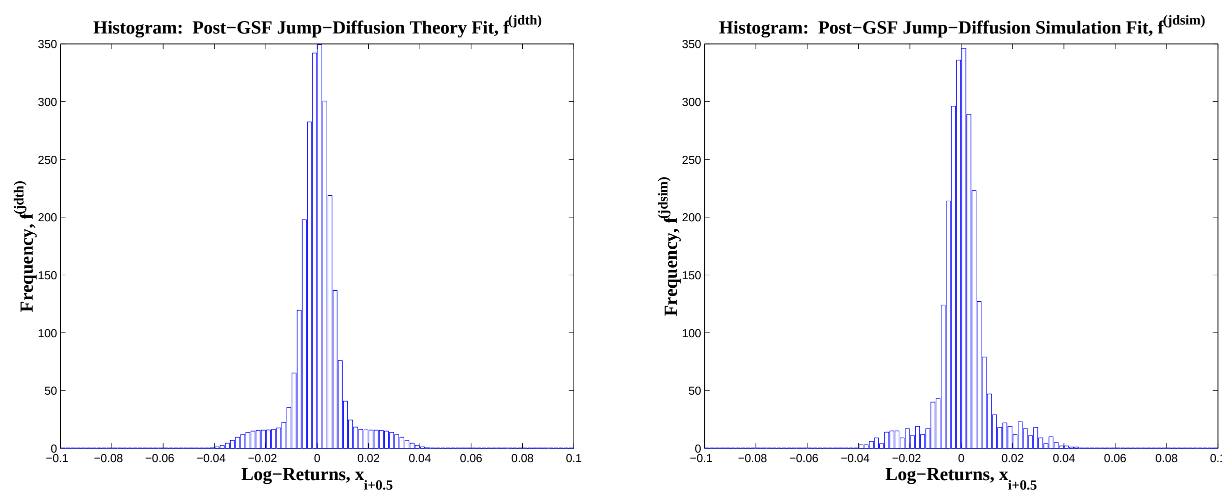

Figure 2: Histogram of log-returns from the log-normal diffusion, log-uniform jump-diffusion model fitted to the S & P 500 Index log-returns for the decade 1992–2001 shown in Fig. 1, using 100 bins. The figure on the left is the fitted theoretical jump-diffusion histogram, while the figure on the right is the corresponding simulated jump-diffusion histogram using the same final parameter results and the same number of samples as the S & P 500 .

Conclusions

In this paper, significant progress has been made toward fitting the theoretical log-normal diffusion, log-uniform jump-diffusion model to realistic financial market data, here the 1992-2001 log-returns of the S & P 500 Index . The log-uniform jump distribution is a big improvement over the log-normal jump distribution used in [4]. The crucial advance was to use a least squares method with weights and to establishing a method for computing the least square weights from the theoretical bin frequencies. In essence, the S & P 500 Index data is treated as a large scale jump-diffusion simulation.

The resulting estimated jump-diffusion parameter set can add more realism to financial market applications, such as the optimal portfolio and consumption policy problem treated in a computational companion paper [6] of the authors at this conference.

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