Stochastic Calculus of Heston's Stochastic–Volatility Model

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Abstract— **The Heston (1993) stochastic–volatility model is a square–root diffusion model for the stochastic–variance. It gives rise to a singular diffusion for the distribution according to Feller (1951). Due to the singular nature, the time-step must be much smaller than the lower bound of the variance. Several transformations are introduced that lead to proper diffusions including a transformation to an additive noise model with perfect-square solution, an exact, nonsingular solution special case and an alternate model. Simulation solution examples are also given.**

Index Terms— **Stochastic–volatility, square–root diffusions, transformations, stochastic calculus, diffusion approximations, nonnegative–variance, simulations.**

1. INTRODUCTION

The Heston model [19] of stochastic–volatility is a square–root diffusion model for the stochastic–variance. According to Feller [11] the model is a singular diffusion for the distribution. Unlike a regular diffusion, there is an order constraint on the relationship between the limit that the variance goes to zero and the limit that time–step goes to zero, so that any nontrivial transformation of the Heston model leads to a transformed diffusion in the Itô calculus. Several transformations are introduced that lead to proper diffusions and preservation of the nonnegativity of the variance in a perfect–square form. An exact, nonsingular solution is found for a special combination of the Heston stochastic–volatility parameters.

Due to the square–root term, the singular nature of the diffusion is intrinsic. Geometric Brownian motion is also a singular diffusion, since the diffusion vanishes when the diffusion coefficient vanishes. However, the singular nature of geometric Brownian motion is removable by the well–known logarithmic transformation, removing the state process from the right hand side and resulting in an additive Brownian motion. In general, singular diffusions can be sensitive to slight changes in the model, which may lead to significant changes in the solution, e.g., in the singular, turning point resonance problem discussed by Hanson and Wazwaz [17].

A computationally simple and practical simulation recipe of solutions to the Heston model is introduced that is consistent with the proper diffusion scaling for the time–step and the variance when both are small.

In financial markets, the log–returns differ from the geometric or linear diffusions due to several properties. Some of these are jumps and random or time–dependent statistical properties. One significant property difference is that variance, or its square root, the volatility, can be stochastically time–dependent, i.e., we have stochastic volatility. Stochastic volatility in the market has been studied and justified, mostly in options pricing, but also foreign exchange and optimal portfolios, by Andersen, Benzoni and Lund [1], Ball, and Roma [2], Ball and Torous [3], Bates [4], Duffie, Pan and Singleton [10], Hanson [16], Hanson and Yan [18], Hull and White [21], Scott [27], Wiggins [28], Yan and Hanson [29], and Zariphopoulou [31]. Andersen et al. [1], as well as others, have statistically confirmed the importance of both stochastic–volatility and jumps in equity returns. Bates [4] studied stochastic-volatility, jump-diffusion models for exchange rates.

Refined Euler discretization methods have been developed by Broadie and Kaya [5], Deelstra and Delbaen [9], Higham and Mao [20], Jäckel [22], Kahl and Jäckel [23], Lord, Koekkoek and Dijk [25], and others. In particular, Higham and Mao [20] have established strong convergence and other results for the Euler–Maruyama discretization of several versions of the mean–reverting, square–root model. Also, Lord et al. [25] carry out comparisons of a number of Euler discretization models of the more general CEV (constant elasticity of volatility) models to force nonnegativity, including many of the above mentioned discretization papers.

Glasserman [13] gives a second–order Milstein–like simulation scheme for the Heston model. Kahl and Jäckel [23] further developed fast and strong Milstein simulation schemes for stochastic–volatility models. Broadie and Kaya [5] devised an exact simulation method (ESM) for stochastic–volatility, affine–jump–diffusion models for option pricing in the sense of an unbiased Monte Carlo estimator, sampling the variance from the exact chi–squared distribution conditioned on a prior value. Smith [26] proposed an almost exact simulation method (AESM) for the Heston model that is faster and applicable to more financial derivatives.

For general overviews, the monographs of Fouque, Papanicolaou and Sircar [12], Gatheral [14], and Lewis [24] are of interest. Fouque et al. [12] cover many issues involving various models with stochastic–volatility with applications to many types of financial derivatives with techniques for estimating parameters. Lewis [24], in his interesting and useful book, presents option pricing solutions of many stochastic-volatility diffusion models, as well as many properties of stochastic volatility models. Gatheral [14] presents a well-balanced treatment of theory and practice.

However, here we are interested in the properties of the Heston model alone, in simple methods of revealing its nonnegativity and the consistency of the Itô diffusion approximation under transformation of the stochastic variance when

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the stochastic–variance can be small. As Jäckel [22] states regarding the Heston variance process model:

> *In an infinitesimal neighbourhood of zero, Itô's lemma cannot be applied to the variance process. The transformation of the variance process to a volatility formulation results in a structurally different process!*

Similarly, Lord et al. [25] briefly follows up on Jäckel's warning. Their comments suggest that a more thorough investigation of the problem is merited.

This *Itô dilemma* for the Heston model and transformations will be examined for several transformations and leads to questions about structural consistency of the Heston model itself and the solution consistency of related simulations. One of these consistent transformations, to a state-independent guarantees positivity of the variances. Otherwise, the usual Euler simulation of the Heston model leads to a number of negative values of the variance depending on a certain rato of Heston model parameters.

In Section 2, the stochastic–volatility, or stochastic–variance dynamics, is specified. In Section 3, the nonnegativity of the variance is verified using a proper singular form of a perfect-square form of the solution found from the variance-independent transformed form of the model. In Section 4, a consistency condition for the Itô lemma diffusion approximation is derived, when the variance is very small and positive, placing constraints on the relative smallness of the time-step; this also has implications for stochastic–variance simulations. In Section 5, a proper singular limit formulation is given for the perfect-square form of Section 3. In Section 6, an exact, nonsingular solution is given for special values of the Heston model [19] stochastic-volatility parameters. In Section 7, an alternate implicit integral form is given that incorporates the deterministic solution. In Section 8, selected simulations are given as illustrations of the theory. In Section 9, conclusions are drawn.

2. Heston's Stochastic Volatility Model.

The mean–reverting, square–root–diffusion, stochastic–volatility model of Heston [19] is frequently used. Heston's model derives from the CIR model of Cox, Ingersoll and Ross [7] for interest rates. The CIR paper also cites the Feller [11] justification for proper (Feller) boundary conditions, process nonnegativity and the distribution for the general square-root diffusions.

The stochastic–variance is modeled with the Cox–Ingersoll–Ross (CIR) [6], [7] and often used Heston [19] mean–reverting stochastic–variance $V(t)$ and square–root diffusion $\sqrt{V(t)}$, with a triplet of parameters $\{\kappa_v(t), \theta(t), \sigma_v(t)\}$:

$$dV(t)=\kappa_v(t)\left(\theta_v(t)-V(t)\right)dt+\sigma_v(t)\sqrt{V(t)}dW_v(t), \quad (1)$$

with $V(0) = V_0 > 0$, log–rate $\kappa_v(t) > 0$, reversion–level $\theta_v(t) > 0$ and *volatility of variance* $\sigma_v(t) > 0$, where $W_v(t)$ is a standard Brownian motion for $V(t)$. Equation (1) comprises the underlying stochastic–volatility (SV) model, which will be called the Heston model here, but often the term Heston model applies to the system of underlying and its stochastic–volatility.

It is necessary that the continuous variance is nonnegative, i.e., $V(t) \geq 0$, but in simulation practice the discretized variance needs to be constrained to be sufficiently positive to avoid singularities and to preserve the diffusion approximation with or without transformations. The nonnegativity for the usual range of the parameters has been shown using the distribution by Feller in his seminal singular diffusion paper [11]. However, the simple Euler simulations can generate small negative values of the variance and this is confirmed in this paper. The likely reason is the simulations yields a discrete process and not the continuous process of the theoretical model (1), which imply a reflecting boundary near zero for positive parameters.

In the next section, there are some recent, practical results for the positivity of the variance for the Heston [19] model, an implicit perfect square solution in the general parameter case and an explicit form for the case where the speed of reversion times the level of reversion is one quarter of the square of the *volatility of the variance* (often called the volatility of volatility) coefficient.

3. Verification of Nonnegativity of Stochastic Variance by Transformation to Perfect-square Form.

In some financial applications such as the Merton-type optimal portfolio problem, the optimal stock-fraction is singular as the variance goes to zero. The corresponding stock-fraction term is sometimes called the *Merton fraction* and is inversely proportional to the variance v,

$$\frac{\mu(t)-r(t)}{(1-\gamma)v},$$

where $\mu(t)$ is the asset drift coefficient, $r(t)$ is the spot rate at t and γ is the power of the risk-aversion utility. For such fractions is important to know if the model yields positive variance in calculations, beyond the theoretical nonnegative variance constraint. However, if there are finite bounds on the stock-fraction in the optimal portfolio problem, then that would provide a cutoff for these singularities. See Hanson [16] for a stochastic-volatility, jump-diffusion Merton optimal portfolio problem example.

On the other hand, the nonnegativity of the stochastic variance, $V(t) \geq 0$, was settled long ago for the square-root diffusion model by Feller [11], using very elaborate Laplace transform techniques on the corresponding Kolmogorov forward equation to obtain the noncentral chi-squared distribution for the distribution. He has given the boundary condition classification for the distribution of the process in terms of the parameters, which helps to determine the values that would guarantee positivity preservation in the range of nonnegativity preserving values. So, in the time-independent form notation here, positivity and uniqueness of the distribution is assured if $\kappa_v\theta_v/\sigma_v^2 > 1/2$ with zero boundary conditions in value and flux at $v = 0$, while if $0 < \kappa_v\theta_v/\sigma_v^2 < 1/2$ then only positivity can be assured for the distribution is the flux vanishes at $v = 0$. For other qualifications and information, see Cox et al. [7], Glassman [13], Jäckel [22], Broadie and Kaya [5], Kahl and Jäckel [23], Smith [26] and Lord et al. [25], in addition to Feller [11]. This includes various distribution simulation techniques, many associated with the corresponding asset option problem with stochastic volatility.

Using the general transformation techniques in Hanson [15] with $Y(t) = F(V(t), t)$, it is possible to find a general perfect square solution to (1). Using Itô's lemma for truncation to the diffusion approximation, the following transformed SDE is obtained,

$$\begin{aligned} dY(t) = {} & F_t(V(t),t)dt + F_v(V(t),t)dV(t) \\ & + \frac{1}{2}F_{vv}(V(t),t)\sigma_v^2(t)V(t)dt, \end{aligned} \tag{2}$$

to dt–precision. Then a simpler form is sought with volatility-independent noise term, i.e.,

$$\begin{aligned} dY(t) = {} & \Big(\mu_y^{(0)}(t) + \mu_y^{(1)}(t)\sqrt{V(t)} \\ & + \mu_y^{(2)}(t)\Big/\sqrt{V(t)}\Big)\, dt + \sigma_y(t)dW_v(t), \end{aligned} \tag{3}$$

with $Y(0) = F(V_0, 0)$, where $\mu_y^{(0)}(t)$, $\mu_y^{(1)}(t)$, $\mu_y^{(2)}(t)$ and $\sigma_y(t)$ are time-dependent coefficients to be determined. Equating the coefficients of $dW_v(t)$ terms between (2) and (3), given $V(t) = v \geq 0$, leads to

$$F_v(v,t) = \left(\frac{\sigma_y}{\sigma_v}\right)(t)\frac{1}{\sqrt{v}}, \tag{4}$$

and then partially integrating (4) yields

$$F(v,t) = 2\left(\frac{\sigma_y}{\sigma_v}\right)(t)\sqrt{v} + c_1(t), \tag{5}$$

which is the desired transformation with a function of integration $c_1(t)$. Additional differentiations of (4) produce

$$F_t(v,t) = 2\left(\frac{\sigma_y}{\sigma_v}\right)'(t)\sqrt{v} + c_1'(t)$$

and

$$F_{vv}(v,t) = -\frac{1}{2}\left(\frac{\sigma_y}{\sigma_v}\right)(t)v^{-3/2}.$$

Terms of order $v^0 dt = dt$ imply that $c_1'(t) = \mu_y^{(0)}(t)$, but this equates two unknown coefficients, so we set $\mu_y^{(0)}(t) = 0$ and $c_1(t) = 0$ for convenience. Equating terms of order $\sqrt{v}dt$,

$$\mu^{(1)}(t) = \left(2\left(\frac{\sigma_y}{\sigma_v}\right)' - \kappa_v\left(\frac{\sigma_y}{\sigma_v}\right)\right)(t) \tag{6}$$

and for order $dt/\sqrt{v}$,

$$\mu^{(2)}(t) = \left(\kappa_v\theta_v - \sigma_v^2/4\right)(t)\left(\frac{\sigma_y}{\sigma_v}\right)(t). \tag{7}$$

However, there are more unknown functions than equations, so $\mu^{(1)}(t) = 0$ is set in (6) since that leads to an exact differential for σ_y/σ_v with solution

$$\left(\frac{\sigma_y}{\sigma_v}\right)(t) = \left(\frac{\sigma_y}{\sigma_v}\right)(0)e^{\overline{\kappa}_v(t)/2},$$

where

$$\overline{\kappa}_v(t) \equiv \int_0^t \kappa_v(s)ds.$$

For convenience, we set $\sigma_y(0) = \sigma_v(0)$. Thus (6) becomes

$$\mu_y^{(2)}(t) = e^{\overline{\kappa}_v(t)/2}\left(\kappa_v\theta_v - \sigma_v^2/4\right)(t),$$

completing the coefficient determination.

Assembling these results we form the solution as follows,

$$Y(t) = 2e^{\overline{\kappa}_v(t)/2}\sqrt{V(t)}$$

from (5), and

$$dY(t) = e^{\overline{\kappa}_v(t)/2}\left(\left(\frac{\kappa_v\theta_v - \sigma_v^2/4}{\sqrt{V}}\right)(t)dt + (\sigma_v dW_v)(t)\right)$$

from (3) and inverting this yields the *transparent non-negativity result*:

Theorem 1A. Nonnegativity of Variance: *Let $V(t)$ be the solution to the Heston model (1), subject to conditions on the diffusion approximation truncation (2) to be determined (Theorem 1B), then*

$$V(t) = e^{-\overline{\kappa}_v(t)}\left(\frac{Y(t)}{2}\right)^2 \geq 0, \tag{8}$$

due to the perfect square form, where

$$Y(t) = 2\sqrt{V_0} + 2I_g(t) \tag{9}$$

and

$$\begin{aligned} I_g(t) = 0.5\int_0^t e^{\overline{\kappa}_v(s)/2}\Bigg(&\left(\frac{\kappa_v\theta_v - \sigma_v^2/4}{\sqrt{V}}\right)(s)ds \\ & + (\sigma_v dW_v)(s)\Bigg). \end{aligned} \tag{10}$$

This is an implicit form that is singular unless the solution $V(t)$ is bounded away from zero, $V(t) > 0$. More generally it is desired that the solution is such that $1\Big/\sqrt{V(t)}$ is integrable in t as $V(t) \to 0^+$, so the singularity will be ignorable in theory.

4. Model Consistency for Itô Lemma Diffusion Approximation Truncation Under Transformation and Limit of Vanishing Variance.

In general, we will assume v is both positive and bounded, i.e., $0 < \varepsilon_v \leq v \leq B_v$, where B_v is a realistic rather than theoretical upper bound. It is necessary to check the consistency of the Itô lemma diffusion approximation truncation specified in (2) because of the competing time-variance limits. As the time-increment $\Delta t \to 0^+$ in the mean square limit for the Itô approximation and as the variance singularity is approached, $V(t) \to 0^+$, i.e., $\varepsilon_v \to 0^+$, difficulties arise from the limited differentiability for small values of the square root of variance. This means that it no longer make sense to assume that the state variable $V(t)$ is fixed if a uniform approximation in Δt and $V(t)$ is needed for model consistency and robustness.

The $F(v,t)$ given in (5) has the form $F(v,t)=\beta_0(t)\sqrt{v}+c_1(t)$, and the partial derivatives satisfy the power law

$$\frac{\partial^k F}{\partial v^k}(v,t)=\beta_k(t)v^{-(2k-1)/2},$$

where the coefficient $\beta_k(t)$ satisfies the recursion $\beta_{k+1}(t)=-(k-0.5)\beta_k(t)$ when $k\geq 0$ with $\beta_1(t)=(\sigma_y/\sigma_v)(t)$. Hence, the partial derivatives will be bounded as long as v is positive. For $\Delta t\ll 1$, the corresponding increment in F will be expandable as a Taylor series depending on the relative sizes of Δt and $v=V(t)$, as

$$\begin{aligned}\Delta F(V(t),t)&=F(V(t)+\Delta V(t),t+\Delta t)-F(V(t),t)\\&=\frac{\partial F}{\partial v}\Delta V(t)+\frac{\partial F}{\partial t}\Delta t+\frac{1}{2!}\frac{\partial^2 F}{\partial v^2}(\Delta V)^2(t)\\&\quad+\frac{\partial^2 F}{\partial v\partial t}\Delta V(t)\Delta t+\frac{1}{2!}\frac{\partial^2 F}{\partial t^2}(\Delta t)^2\\&\quad+\frac{1}{3!}\frac{\partial^3 F}{\partial v^3}(\Delta V)^3(t)+\frac{1}{2!}\frac{\partial^3 F}{\partial v^2\partial t}(\Delta V)^2(t)\Delta t\\&\quad+\frac{1}{2!}\frac{\partial^3 F}{\partial v\partial t^2}\Delta V(t)(\Delta t)^2+\frac{1}{3!}\frac{\partial^3 F}{\partial t^3}(\Delta t)^3\\&\quad+\cdots.\end{aligned}$$

If $\Delta t\ll 1$ with conditioning the current variance on v, letting $\mu_v(t)=\kappa_v(t)(\theta_v(t)-v)$ and $\Delta W_v(t)=\sqrt{\Delta t}Z_v(t)$ with standard normal $Z_v(t)\stackrel{\text{dist}}{=}\mathcal{N}(0,1)$, then

$$[\Delta V(t)|V(t)=v]\simeq\sigma_v(t)\sqrt{v\Delta t}Z_v(t)+\mu_v(t)\Delta t.$$

In terms of small Δt when $k>1$, the pure variance derivatives, i.e., those having only v-derivatives, will dominate the cross variance-time derivatives, the mixed v and t derivatives, as well as the pure time derivatives, since for $\Delta t\ll 1$ then $\Delta t\ll\sqrt{\Delta t}\ll 1$, while considering v fixed. Thus for $k>1$, only the powers of diffusion part of $[\Delta V(t)|V(t)=v]$ need be considered. The mean estimate of the absolute value of dominant diffusion power is

$$\mathrm{E}\left[|\sigma_v(t)\sqrt{v}\Delta W_v|^k\,\middle|\,V(t)=v\right]=\alpha_k(t)(v\Delta t)^{k/2},$$

where $\alpha_k(t)=\sigma_v^k(t)\mathrm{E}[Z_v^k(t)]$. The products of these terms produce an estimate of the corresponding dominant terms in the Taylor expansion,

$$\begin{aligned}\left|\frac{\partial^k F}{\partial v^k}(v,t)\right|\mathrm{E}[(\sigma_v(t)\sqrt{v}\Delta W_v)^k]&=\gamma_k(t)\frac{\Delta t}{\sqrt{v}}\left(\frac{\Delta t}{v}\right)^{(k-2)/2}\\&\leq\gamma_k(t)\frac{\Delta t}{\sqrt{\varepsilon_v}}\left(\frac{\Delta t}{\varepsilon_v}\right)^{(k-2)/2},\end{aligned}$$

for $\gamma_k(t)=\alpha_k(t)|\beta_k(t)|$, separated into the order $\Delta t/\sqrt{v}$ of the Itô diffusion approximation ($k=2$) term and the factor relative to it. Hence, to eliminate all terms of higher order than $k=2$, we need $\Delta t/v\ll 1$, i.e.,

$$\Delta t\ll\varepsilon_v\ll 1$$

to obtain a proper Itô diffusion approximation (2) for the transformation $Y(t)=F(V(t),t)$ in (5). Summarizing the results we have

Theorem 1B. Conditions for a Consistent Itô Lemma Diffusion Approximation Truncation for Transforming the Heston Model (1) to a Variance-Independent Noise Model: *Let the variance be positive and finite such that $0<\varepsilon_v\leq V(t)\leq B_v$, then the variance independent model (2) is a consistent Itô diffusion approximation to the Heston model (1) uniform in the limits $\Delta t\ll 1$ and $\varepsilon_v\ll 1$ provided*

$$\Delta t\ll\varepsilon_v\ll 1. \tag{11}$$

5. Solution Consistency for Singular Limit Formulation Suitable for Theory and Computation.

However, as $V(t)\to 0^+$, it is necessary to *verify that the solution (12) satisfies the Heston model (1) in the limit*, due to the questions involving the validity of the Itô lemma and the singular integral $I_g(t)$ in (10).[1]

First recall that from (8)–(10)

$$V(t)=e^{-\overline{\kappa}_v(t)}\left(\sqrt{V_0}+I_g(t)\right)^2. \tag{12}$$

Modifying the *method of ignoring the singularity* [8] to this implicit singular formulation, let

$$V^{(\varepsilon_v)}(t)\equiv\max(V(t),\varepsilon_v) \tag{13}$$

where $\varepsilon_v>0$ such that $\Delta t/\varepsilon_v\ll 1$ is some reference numerical increment $\Delta t\to 0^+$. This ensures that the time–step goes to zero faster than the cutoff singular denominator. The result is Itô diffusion approximation (2) consistency and the numerical consistency of the solution (12). Next (12)–(10) is reformulated as a recursion using some algebra for the next time increment Δt and the method of integration is specified for each subsequent time–step, i.e.,

$$\begin{aligned}V^{(\varepsilon_v)}(t+\Delta t)=\max\Big(&e^{-\Delta\overline{\kappa}_v(t)}\Big(\sqrt{V^{(\varepsilon_v)}(t)}\\&+e^{-\overline{\kappa}_v(t)/2}\Delta I_g^{(\varepsilon_v)}(t)\Big)^2,\varepsilon_v\Big),\end{aligned} \tag{14}$$

where

$$\Delta\overline{\kappa}_v(t)\equiv\int_t^{t+\Delta t}\kappa_v(s)ds\to\kappa_v(t)\Delta t$$

as $\Delta t\to 0^+$. Similarly, a scaled increment of an integral is defined by

$$\begin{aligned}e^{-\overline{\kappa}_v(t)/2}\Delta I_g^{(\varepsilon_v)}(t)&\equiv 0.5\int_t^{t+\Delta t}e^{(\overline{\kappa}_v(s)-\overline{\kappa}_v(t))/2}\\&\qquad\left(\left(\frac{\kappa_v\theta_v-\sigma_v^2/4}{\sqrt{V^{(\varepsilon_v)}}}\right)(s)ds\right.\\&\qquad\left.+(\sigma_v dW_v)(s)\right)\\&\to 0.5\left(\left(\frac{\kappa_v\theta_v-\sigma_v^2/4}{\sqrt{V}}\right)(t)\Delta t\right.\\&\qquad\left.+(\sigma_v\Delta W_v)(t)\right),\end{aligned} \tag{15}$$

[1] Recall that Zabusky and Kruskal [30] showed that the well-known discretization of the Fermi-Pasta-Ulam problem numerically solved the Korteweg-deVries problem instead.

such that

$$\Delta t/\varepsilon_v \to 0^+ \quad \text{as} \quad \Delta t \to 0^+ \;\;\&\;\; \varepsilon_v \to 0^+. \tag{16}$$

An Itô–Taylor expansion to precision dt or small Δt confirms that (14)–(15) yields the Heston [19] model, *proving solution consistency*.

Thus, the square in (14) formally justifies the nonnegativity of the variance and the volatility of the Heston [19] model, for a *proper computational nonnegativity–preserving procedure*.

6. Nonsingular, Explicit, Exact Solution.

In any event, the singular term in (12)–(10) vanishes in the special parameter case, such that

$$\kappa_v(t)\theta_v(t) = \sigma_v^2(t)/4, \quad \forall\ t. \tag{17}$$

Hence, we obtain a nonnegative, nonsingular exact solution

$$V(t) = e^{-\overline{\kappa}_v(t)}\left(\sqrt{V_0} + 0.5\int_0^t e^{\overline{\kappa}_v(s)/2}(\sigma_v dW_v)(s)\right)^2, \tag{18}$$

with the recursive numerical form corresponding to the ε_v-truncated forms (14)–(15),

$$\begin{aligned}V^{(\varepsilon_v)}(t+\Delta t) = \max\Bigg(& e^{-\Delta\overline{\kappa}_v(t)}\Bigg(\sqrt{V^{(\varepsilon_v)}(t)} \\ &+\frac{1}{2}\int_t^{t+\Delta t} e^{(\overline{\kappa}_v(s)-\overline{\kappa}_v(t))/2} \\ &\cdot(\sigma_v dW_v)(s)\Bigg)^2, \varepsilon_v\Bigg),\end{aligned} \tag{19}$$

which is useful for testing simulation algorithms.

7. Alternate Solution Relative to Deterministic Solution Using Integrating Factor Transformation.

Similarly, the chain rule for the integrating factor form

$$X(t) = \exp(\overline{\kappa}_v(t))V(t) \tag{20}$$

for the general stochastic–volatility (1) leads to a somewhat simpler integrated form,

$$V(t) = V^{(\det)}(t) + e^{-\overline{\kappa}_v(t)}\int_0^t e^{\overline{\kappa}_v(s)}\left(\sigma_v\sqrt{V}dW_v\right)(s), \tag{21}$$

suppressing the maximum with respect to zero to remove spurious numerical simulations of the corresponding discretized model for the time being. In (21),

$$V^{(\det)}(t) = e^{-\overline{\kappa}_v(t)}\left(V_0 + \int_0^t \theta_v(s)d\left(e^{\overline{\kappa}_v(s)}\right)\right) \tag{22}$$

is the deterministic part of $V(t)$.

Note that there is only a linear change of dependent variable according to the stochastic chain rule (Hanson, 2007) using the transformation (20). So the deterministic part is easily separated out from the square-root dependence and replaces the mean-reverting drift term. The $V^{(\det)}(t)$ will be positive for positive parameters. For simulation purposes the incremental recursions are useful:

$$\begin{aligned}V^{(\det)}(t+\Delta t) = e^{-\Delta\overline{\kappa}_v(t)}\Bigg(& V^{(\det)}(t) \\ &+e^{-\overline{\kappa}_v(t)}\int_t^{t+\Delta t}\left(\theta_v d\left(e^{\overline{\kappa}_v}\right)\right)(s)\Bigg)\end{aligned} \tag{23}$$

and

$$\begin{aligned}V(t+\Delta t) = e^{-\Delta\overline{\kappa}_v(t)}\Bigg(& V(t) + e^{-\overline{\kappa}_v(t)}\int_t^{t+\Delta t} e^{\overline{\kappa}_v(s)} \\ &\cdot\left(\kappa_v\theta_v + \sigma_v\sqrt{V}dW_v\right)(s)\Bigg).\end{aligned} \tag{24}$$

Note that with constant coefficients, $\theta_v(t) = \theta_0$, $\kappa_v(t) = \kappa_0$ and $\sigma_v(t) = \sigma_0$, then (22–24) become

$$V^{(\det)}(t) = V_0 e^{-\kappa_0 t} + \theta_0\left(1 - e^{-\kappa_0 t}\right), \tag{25}$$

$$V^{(\det)}(t+\Delta t) = \theta_0 + e^{-\kappa_0\Delta t}\left(V^{(\det)}(t) - \theta_0\right), \tag{26}$$

and

$$V(t+\Delta t) = \theta_0 + e^{-\kappa_0\Delta t}\left(V(t) - \theta_0 + \sigma_0\sqrt{V(t)}\Delta W(t)\right). \tag{27}$$

However, as Lord et al. [25] point out, a sufficiently accurate simulation scheme and a large number of simulation nodes are required so that the right-hand side of (1) generates nonnegative values. Nonnegative values using the stochastic Euler simulation have been verified for Heston's [19] constant risk-neutralized parameter values $\{\kappa_v = 2.00, \theta_v = 0.01, \sigma_v = 0.10\}$ as long as the scaled number of nodes per unit time $N/(\kappa_v t_f) > 100$.

Hence, since the variance by definition for real processes cannot be negative, practical considerations suggest replacing occurrences of $V(t)$ by $\max(V(t), \varepsilon_v)$, where ε_v is some numerically small, positive quantity for numerical purposes to account for the appearances of negative variance values.

8. Selected Numerical Simulations.

In Figure 1, the simulations for the Euler–Maruyama approximation to Heston's stochastic–variance equation (1), truncating any negative values to zero, compared to the perfect square solution simulations in (14) using $\varepsilon_v = 0$ since is not needed for nonpositivity. Also, shown is the deterministic solution $V^{(\det)}(t+\Delta t)$ simulation from (23). The negative of the difference between the the truncated Heston Euler simulations and the perfect square form is shown at the bottom of the figure straddling zero except for one spike. The maximum absolute value of this difference is 2.46e-3. Otherwise, the Heston-Euler and perfect square simulation trajectories are barely distinguishable in Fig. 1. The parameter values used are $\{\kappa_v = 2.00, \theta_v = 0.01, \sigma_v = 0.25\}$, which coincidentally have the Heston model parameter ratio $\kappa_v\theta_v/\sigma_v^2 = 0.32$ from the exact solution using (17) .

In the simulation of Fig. 1, the Heston-Euler simulation before truncation produces $K_{neq} = 76$ improper nonpositive values over one million sample points, while the perfect

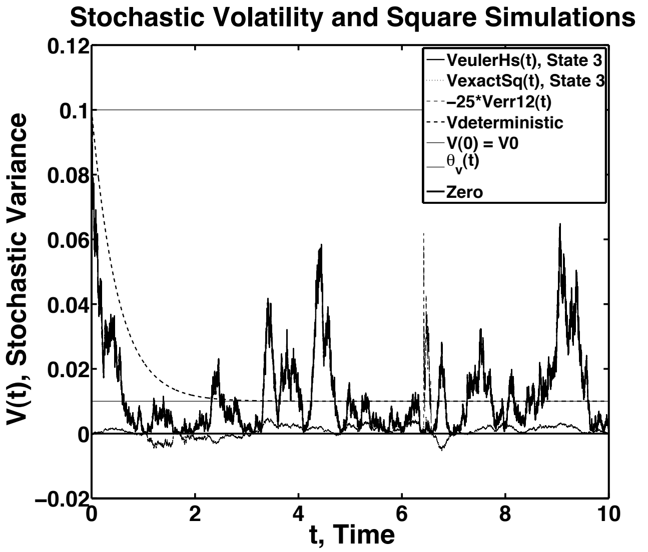

Fig. 1. Comparison of the Heston-Euler simulation of (1) with negative values truncated to zero and the perfect square solution(14). Also shown are the deterministic solution (23) and the negative of the error **Verr12(t)** *magnified 25 times*. The Heston model parameter ratio is $\kappa_v\theta_v/\sigma_v^2 = 0.32$. There are 10^6 sample points over 10 time units.

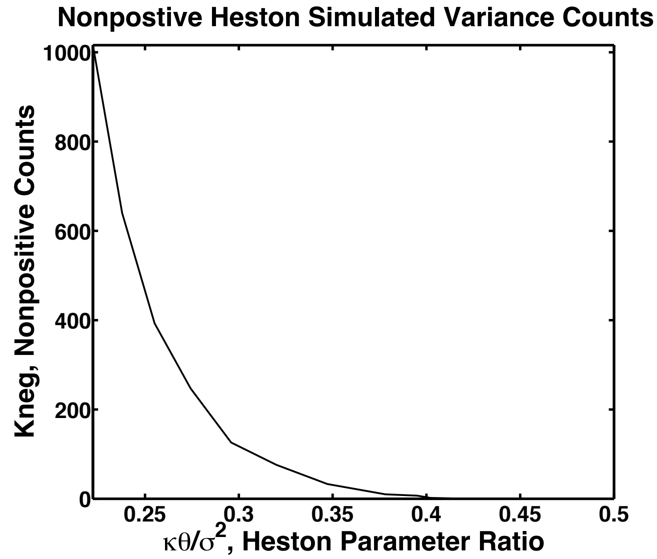

Fig. 2. Nonpositive variance counts for the Heston Euler and Alternate solution simulations, counted prior to truncation to zero. The coordinate axis is the Heston model parameter ratio $\kappa_v\theta_v/\sigma_v^2$, where $\kappa_v = 2.00$ and $\theta_v = 0.01$, while $\sigma_v \in [0.20, 0.30]$.

square solution does not even produce zero values. The most negative value of the Heston-Euler simulation is $V =$– 1.01e–6, so not very significant. The number of negative values K_{neq} before truncation are plotted in Figure 2 against the Heston model parameter ratio $\kappa_v\theta_v/\sigma_v^2$. The number of negative values begin as ratio approaches the Feller boundary classification separation point at $\kappa_v\theta_v/\sigma_v^2 = 0.5$ and are extreme at ratio value of 0.22. The results for the simulations use the same random number generator seed in MATLAB. Note the negative value count is the same for the alternate implicit, integrated solution including, in part the deterministic solution of (24).

In fact, the alternate and Heston-Euler simulations are virtually the same, with the same maximum absolute difference of 2.46e–03 from the perfect square simulation, suggesting that the problem is a numerical one with the square root function of the variance. Also, there is appears to be a discrepancy between the variance violations of the simulations for the process model and Feller's description of the positivity for the distribution solution. The likely reason is that the Euler simulations of the diffusion process are discrete, while the theoretical process is continuous, in which case the trajectory must be at zero for an instant before becoming negative. However, when $V(t) = 0$ then $dV(t) = \kappa_v\theta_v dt > 0$, so the trajectory will reflected back to positive values. Thus, the simulated negative values, though very small must be a discretization flaw. This is a good reason for setting any negative value at least to zero. See also the comments in Higham and Mao [20] or Lord et al. [25] on discretization treatments.

9. Conclusions.

The consistency of the Heston stochastic–volatility model under transformations with the Itô lemma with respect to the diffusion approximation is shown by considering the relation between the time-step and variance as they both become small using basic calculus principles.

Some practical results are given for the positivity of the variance are given for the Heston [19] stochastic–volatility model as a result of a transformation to the model diffusion approximation with purely additive noise. The solution is shown to have an implicit perfect square form in the general parameter case. For solution consistency, it is also confirmed that the transformed, truncated model formal solution *reduces back to the Heston model* in the joint small Δt and ε_v limit.

An explicit solution is given for the case where the speed of reversion times the level of reversion is one quarter of the square of the *volatility of the variance* coefficient.

The spurious simulation of small negative values from the corresponding discretized Heston model is studied.

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