Stochastic Calculus of Heston's Stochastic-Volatility Model

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Extended Abstract:

The Heston stochastic-volatility model is a square-root diffusion model for the stochastic-variance. It gives rise to a singular diffusion for the distribution as noted by Feller [5]. Hence, there is an order constraint on the relationship between the limit that the variance goes to zero and the limit that time-step goes to zero, so that any non-trivial transformation of the Heston model leads to a transformed diffusion in the Itô Calculus. Several transformations are introduced that lead to proper diffusions and preservation of the nonnegativity of the variance in a perfect-square form. An exact, nonsingular solution is found for a special combination of the Heston stochastic volatility parameters.

A computationally simple and practical simulation recipe of solutions of the Heston model is introduced that is consistent with the proper diffusion scaling for the time-step and the variance when both are small.

In financial markets, the log-returns differ from the geometric or linear diffusions due to several properties. Some of these are jumps and random or time-dependent statistical properties. One significant property difference is that variance, or its square root, the volatility, can be stochastically time-dependent, i.e., we have stochastic volatility. Stochastic volatility in the market, mostly in options pricing, has been studied and justified by Ball and Torous [2], Bates [3], Andersen, Benzoni and Lund [1], and Lord, Koekkoek and Dijk [8].

The mean-reverting, square-root-diffusion, stochastic-volatility model of Heston [7] is frequently used. Heston's model derives from the CIR model of Cox, Ingersoll and Ross [4] for interest rates. The CIR paper also cites the Feller [5] justification for proper (Feller) boundary conditions, process nonnegativity and the distribution for the general square-root diffusions.

The stochastic variance is modeled with the Cox-Ingersoll-Ross (CIR) [4] and Heston [7] mean-reverting stochastic variance $V(t)$ and square-root diffusion $\sqrt{V(t)}$, with a triplet of parameters $\{\kappa_v(t), \theta(t), \sigma_v(t)\}$:

$$dV(t) = \kappa_v(t)\left(\theta_v(t) - V(t)\right)dt + \sigma_v(t)\sqrt{V(t)}dW_v(t), \tag{0.1}$$

with $V(0) = V_0 > 0$, log-rate $\kappa_v(t) > 0$, reversion-level $\theta_v(t) > 0$ and *volatility of volatility (variance)* $\sigma_v(t) > 0$, where $W_v(t)$ is a standard Brownian motion $V(t)$. Equation (0.1) comprises the underlying stochastic-volatility (SV) model.

It will be assumed that the variance is nonnegative, i.e., $V(t) \geq 0$, in theory, but in practice the variance needs to be sufficiently positive to avoid singularities and to preserve the diffusion approximation in transformations. The nonnegativity for the usual range of the parameters has been shown using the distribution by Feller in his seminal singular diffusion paper [5]. However, the simple Euler simulations can generate small negative values of the variance and this is confirmed in this paper. The likely reason is the simulations yields a discrete process and not the continuous process of the theoretical model (0.1), which imply a reflecting boundary near zero for positive parameters.

Using the transformation techniques of Hanson [6], it is shown that the transformation

$$Y(t) = \beta(t)/\sqrt(V(t)) + c(t), \tag{0.2}$$

given functions, $\beta(t)$ and $c(t)$, lead to a state-independent noise term and a perfect square solution:

$$\begin{aligned} V(t) &= e^{s-\overline{\kappa}_v(t)} \left(\sqrt{V_0} + I_g(t)\right)^2, \\ I_g(t) &= 0.5\int_0^t e^{\overline{\kappa}_v(s)/2} \left(\left(\frac{\kappa_v\theta_v - \frac{1}{4}\sigma_v^2}{\sqrt{V}}\right)(s)ds + (\sigma_v dW_v)(s)\right). \end{aligned} \tag{0.3}$$

It is shown that this solution is consistent with Itô's diffusion approximation lemma provided that the time-step Δt and minimum variance $\varepsilon = \min(V(t))$ are constrained by $\Delta t \ll \varepsilon \ll 1$.

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