Stochastic Calculus of Heston's Stochastic-Volatility Model

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Abstract

The Heston stochastic-volatility model is a square-root diffusion model for the stochastic-variance. It gives rise to a singular diffusion for the distribution as noted by Feller (1951). Hence, there is an order constraint on the relationship between the limit that the variance goes to zero and the limit that time-step goes to zero, so that any non-trivial transformation of the Heston model leads to a transformed diffusion in the Itô Calculus. Several transformations are introduced that lead to proper diffusions and preservation of the nonnegativity of the variance in a perfect-square form. An exact, nonsingular solution is found for a special combination of the Heston stochastic volatility parameters.

A computationally simple and practical simulation recipe of solutions of the Heston model is introduced that is consistent with the proper diffusion scaling for the time-step and the variance when both are small.

Key words: stochastic-volatility, square-root diffusions, simulations, transformations, Itô diffusion approximation, nonnegative-variance verification.

1 Introduction.

In financial markets, the log-returns differ from the geometric or linear diffusions due to several properties. Some of these are jumps and random or time-dependent statistical properties. One significant property difference is that variance, or its square root, the volatility, can be stochastically time-dependent, i.e., we have stochastic volatility. Stochastic volatility in the market, mostly in options pricing, has been studied by Garman and Klass [12], Johnson and Shanno [21], Ball and Torous [3], Hull and White [18], Wiggins [27], Stein and Stein [26, see corrections in [2]], Ball and Roma [2], Scott [24], and Lord, Koekkoek and Dijk [23]. Andersen, Benzoni and Lund [1], as well as others, have statistically confirmed the importance of both stochastic volatility and jumps in equity returns. Bates [4] studied stochastic-volatility, jump-diffusion models for exchange rates.

The mean-reverting, square-root-diffusion, stochastic-volatility model of Heston [17] is frequently used. Heston's model derives from the CIR model of Cox, Ingersoll and Ross [7] for interest rates. The CIR paper also cites the Feller [10] justification for proper (Feller) boundary conditions, process nonnegativity and the distribution for the general square-root diffusions. In a companion paper to the CIR model paper, Cox et al. [6] present the more general theory for asset processes.

In their monograph, Fouque, Papanicolaou and Sircar [11] cover many issues involving various models with stochastic volatility with applications to many types of financial derivatives with techniques for estimating parameters. Lewis [22], in his interesting and useful book, presents option pricing solutions of many stochastic-volatility diffusion models, as well as many properties of stochastic volatility models. In their often cited paper on affine jump-diffusions, Duffie, Pan and Singleton [9] include a section on various stochastic-volatility, jump-diffusion models. Yan and Hanson [28, 29, 16] explored theoretical and computational issues for both European and American option pricing using stochastic-volatility, jump-diffusion models with log-uniform jump-amplitude distributions. Wiggins [27] considers the optimal portfolio problem for the log-utility investor with stochastic volatility and using equilibrium arguments for hedging. Zariphopoulou [30] analyzes the optimal portfolio problem with CRRA utility, a *stochastic factor*, i.e., stochastic volatility, and unhedgeable risk.

Glasserman [13] gives a second-order Milstein-like simulation scheme for the Heston model. Jäckel and Kahl [20] further developed fast and strong Milstein simulation schemes for stochastic volatility models. Broadie and Kaya [5] devised an exact simulation method (ESM) for stochastic-volatility, affine-jump-diffusion models for option pricing in the sense of an unbiased Monte Carlo estimator, sampling the variance from the exact chi-squared distribution conditioned on a prior value. Smith [25] proposed an almost exact simulation method (AESM) for the Heston model that is faster and applicable to more financial derivatives.

However, here we are interested in the properties of the Heston model alone, in simple methods of revealing its non-negativity and the consistency of the Itô diffusion approximation under transformation of the stochastic variance when the stochastic variance can be small. As Jäckel [19] states regarding the Heston variance process model (here (2.1)):

> *In an infinitesimal neighbourhood of zero, Itô's lemma cannot be applied to the variance process. The transformation of the variance process to a volatility formulation results in a structurally different process!*

This remark is also emphasized by Lord, Koekkoek and Dijk [23].

This *Itô dilemma* for the Heston model and transformations will be examined for several transformations and leads to questions about structural consistency of the Heston model itself and the solution consistency of related simulations. One of these consistent transformations, to a state-independent guarantees positivity of the variances. Otherwise, the usual Euler simulation of the Heston model leads to a number of negative values of the variance depending on a certain rato of Heston model parameters.

In Section 2, the stochastic-volatility, or stochastic variance dynamics, is specified. In Section 3, the nonnegativity of the variance is verified using a proper singular form of a perfect-square form of the solution found from the variance-independent transformed form of the model. In Section 4, a consistency condition for the Itô lemma diffusion approximation is derived, when the variance is very small and positive, placing constraints on the relative smallness of the time-step; this also has implications for stochastic variance simulations. In Section 5, a proper singular limit formulation is given for the perfect-square form of Section 3. In Section 6, an exact, nonsingular solution is given for special values of the Heston model [17] stochastic-volatility parameters. In Section 7, an alternate implicit integral form is given that incorporates the deterministic solution. In Section 8, selected simulations are given as illustrations of the theory. In Section 9, conclusions are drawn.

2 Heston's Stochastic Volatility Model.

The stochastic variance is modeled with the Cox-Ingersoll-Ross (CIR) [6, 7] and Heston [17] mean-reverting stochastic variance $V(t)$ and square-root diffusion $\sqrt{V(t)}$, with a triplet of parameters $\{\kappa_v(t), \theta(t), \sigma_v(t)\}$:

$$dV(t) = \kappa_v(t)\left(\theta_v(t) - V(t)\right)dt + \sigma_v(t)\sqrt{V(t)}dW_v(t), \tag{2.1}$$

with $V(0) = V_0 > 0$, log-rate $\kappa_v(t) > 0$, reversion-level $\theta_v(t) > 0$ and *volatility of volatility (variance)* $\sigma_v(t) > 0$, where $W_v(t)$ is a standard Brownian motion $V(t)$.

It will be assumed that the variance is nonnegative, i.e., $V(t) \geq 0$, but see Section 3 for important practical qualifications in theory and computation. Equation (2.1) comprises the underlying stochastic-volatility (SV) model. See also [4, 24, 11, 28, 29, 16] for other applications.

In the next section, there are some recent, practical results for the positivity of the variance for the Heston [17] model, an implicit perfect square solution in the general parameter case and an explicit form for the case where the speed of reversion times the level of reversion is one quarter of the square of the *volatility of the volatility* coefficient.

3 Verification of Nonnegativity of Stochastic Variance by Transformation to Perfect-square Form.

In some financial applications such as the Merton-type optimal portfolio problem, the optimal stock-fraction is singular as the variance goes to zero. The corresponding stock-fraction term is sometimes called the *Merton fraction* and is inversely proportional to the variance v,

$$\frac{\mu(t) - r(t)}{(1-\gamma)v},$$

where $\mu(t)$ is the asset drift coefficient, $r(t)$ is the spot rate at t and γ is the power of the risk-aversion utility. For such fractions is important to know if the model yields positive variance in calculations, beyond the theoretical nonnegative variance constraint. However, if there are finite bounds on the stock-fraction in the optimal portfolio problem, then that would provide a cutoff for these singularities. See Hanson [15] for a stochastic-volatility, jump diffusion Merton optimal portfolio problem example.

On the other hand, the nonnegativity of the stochastic variance, $V(t) \geq 0$, was settled long ago for the square-root diffusion model by Feller [10], using very elaborate Laplace transform techniques on the corresponding Kolmogorov forward equation to obtain the noncentral chi-squared distribution for the distribution. He has given the boundary condition classification for the distribution of the process in terms of the parameters, which helps to determine the values that would guarantee positivity preservation in the range of nonnegativity preserving values. So, in the time-independent form notation here, positivity and uniqueness of the distribution is assured if $\kappa_v\theta_v/\sigma_v^2 > 1/2$ with zero boundary conditions in value and flux at $v = 0$, while if $0 < \kappa_v\theta_v/\sigma_v^2 <$

$1/2$ then only positivity can be assured for the distribution is the flux vanishes at $v = 0$. For other qualifications and information, see Cox et al. [7], Glassman [13], Jäckel [19], Broadie and Kaya [5], Jäckel and Kahl [20], Smith [25] and Lord et al. [23], in addition to Feller [10]. This includes various distribution simulation techniques, many associated with the corresponding asset option problem with stochastic volatility.

Using the general transformation techniques in Hanson [14] with $Y(t) = F(V(t), t)$, it is possible to find a general perfect square solution to (2.1). Using Itô's lemma, the following transformed SDE is obtained,

$$dY(t) = F_t(V(t), t)dt + F_v(V(t), t)dV(t) + \frac{1}{2}F_{vv}(V(t), t)\sigma_v^2(t)V(t)dt, \tag{3.1}$$

to dt-precision. Then a simpler form is sought with volatility-independent noise term, i.e.,

$$dY(t) = \left(\mu_y^{(0)}(t) + \mu_y^{(1)}(t)/\sqrt{V(t)}\right)dt + \sigma_y(t)dW_v(t) \tag{3.2}$$

with $Y(0) = F(V_0, 0)$, where $\mu_y^{(0)}(t)$, $\mu_y^{(1)}(t)$ and $\sigma_y(t)$ are time-dependent coefficients to be determined. Equating the coefficients of $dW_v(t)$ terms between (3.1) and (3.2), given $V(t) = v \geq 0$, leads to

$$F_v(v, t) = \left(\frac{\sigma_y}{\sigma_v}\right)(t)\frac{1}{\sqrt{v}}, \tag{3.3}$$

and then partially integrating (3.3) yields

$$F(v, t) = 2\left(\frac{\sigma_y}{\sigma_v}\right)(t)\sqrt{v} + c_1(t), \tag{3.4}$$

which is the desired transformation with a function of integration $c_1(t)$. Additional differentiations of (3.3) produce

$$F_t(v, t) = 2\left(\frac{\sigma_y}{\sigma_v}\right)'(t)\sqrt{v} + c_1'(t) \quad \& \quad F_{vv}(v, t) = -\frac{1}{2}\left(\frac{\sigma_y}{\sigma_v}\right)(t)v^{-3/2}.$$

Terms of order $v^0 dt$ imply that $c_1'(t) = \mu_y^{(0)}(t)$, but this equates two unknown coefficients, so we set $\mu_y^{(0)}(t) = 0$ for simplicity. Equating terms of order $\sqrt{v}dt$ and integrating imply

$$\left(2\left(\frac{\sigma_y}{\sigma_v}\right)' - \kappa_v\left(\frac{\sigma_y}{\sigma_v}\right)\right)(t) = 0 \quad \Longrightarrow \quad \left(\frac{\sigma_y}{\sigma_v}\right)(t) = \left(\frac{\sigma_y}{\sigma_v}\right)(0)e^{\overline{\kappa}_v(t)/2},$$

where

$$\overline{\kappa}_v(t) \equiv \int_0^t \kappa_v(y)dy.$$

For convenience, we set $\sigma_y(0) = \sigma_v(0)$. For order $v^{-1/2}dt$, we obtain

$$\mu_y^{(1)}(t) = e^{\overline{\kappa}_v(t)/2}\left(\kappa_v\theta_v - \frac{1}{4}\sigma_v^2\right)(t),$$

completing the coefficient determination.

Assembling these results we form the solution as follows,

$$Y(t) = 2e^{\overline{\kappa}_v(t)/2}\sqrt{V(t)},$$

and inverting this yields the **desired nonnegativity result**:

Theorem 3.1. Nonnegativity of Variance: *Let $V(t)$ be the solution to the Heston model (2.1), then*

$$V(t) = e^{-\overline{\kappa}_v(t)}\left(\frac{Y(t)}{2}\right)^2 \geq 0, \tag{3.5}$$

due to the perfect square form, where

$$Y(t) = 2\sqrt{V_0} + \int_0^t e^{\overline{\kappa}_v(s)/2}\left(\left(\frac{\kappa_v\theta_v - \frac{1}{4}\sigma_v^2}{\sqrt{V}}\right)(s)ds + (\sigma_v dW_v)(s)\right). \tag{3.6}$$

This is an implicit form that is singular unless the solution $V(t)$ is bounded away from zero, $V(t) > 0$. More generally it is desired that the solution is such that $1\Big/\sqrt{V(t)}$ is integrable in t as $V(t) \to 0^+$, so the singularity will be ignorable in theory.

4 Model Consistency for Itô Diffusion Approximation Lemma Under Transformation and Limit of Vanishing Variance.

In general, we will assume v is both positive and bounded, i.e., $0 < \varepsilon_v \leq v \leq B_v$, where B_v is a realistic rather than theoretical upper bound. It is necessary to check the consistency of the Itô diffusion approximation in (3.1) because of the competing time-variance limits. As the time-increment $\Delta t \to 0^+$ in the mean square limit for the Itô approximation and as the variance singularity is approached, $V(t) \to 0^+$, i.e., $\varepsilon_v \to 0^+$, difficulties arise from the limited differentiability for small values of the square root of variance. This means that it no longer make sense to assume that the state variable $V(t)$ is fixed if a uniform approximation in Δt and $V(t)$ is needed for model consistency and robustness.

The partial derivatives of the $F(v,t)$ given in (3.4), in the form $F(v,t) = \beta_0(t)\sqrt{v} + c_1(t)$, and following equations imply that they satisfy the power law

$$\frac{\partial^k F}{\partial v^k}(v,t) = \beta_k(t)v^{-(2k-1)/2}$$

for coefficient $\beta_k(t)$ such that $\beta_{k+1}(t) = -(k-0.5)\beta_k(t)$ when $k \geq 0$ with $\beta_1(t) = (\sigma_y/\sigma_v)(t)$, so the partial derivative will be bounded as long as v is positive. For $\Delta t \ll 1$, the corresponding increment in F will be expandable as a Taylor series depending on the relative sizes of Δt and v, as

$$\begin{aligned}\Delta F(V(t),t) = {} & F(V(t)+\Delta V(t), t+\Delta t) - F(V(t),t) \\ = {} & \frac{\partial F}{\partial v}\Delta V(t) + \frac{\partial F}{\partial t}\Delta t + \frac{1}{2!}\frac{\partial^2 F}{\partial v^2}(\Delta V)^2(t) + \frac{\partial^2 F}{\partial v \partial t}\Delta V(t)\Delta t \\ & + \frac{1}{2!}\frac{\partial^2 F}{\partial t^2}(\Delta t)^2 + \frac{1}{3!}\frac{\partial^3 F}{\partial v^3}(\Delta V)^3(t) + \frac{1}{2!}\frac{\partial^3 F}{\partial v^2 \partial t}(\Delta V)^2(t)\Delta t + \cdots\end{aligned}$$

If $\Delta t \ll 1$, conditioning the current variance on v, letting $\mu_v(t) = \kappa_v(t)(\theta_v(t) - v)$ and $\Delta W_v(t) = \sqrt{\Delta t} Z_v(t)$ with standard normal $Z_v(t) \stackrel{\text{dist}}{=} \mathcal{N}(0,1)$, then

$$[\Delta V(t)|V(t) = v] \simeq \sigma_v(t)\sqrt{v\Delta t} Z_v(t) + \mu_v(t)\Delta t.$$

Just in terms of small Δt when $k > 1$, the pure diffusion terms, those having only v-derivatives will dominate the cross diffusion-drift terms with mixed v and t derivatives, as well as the pure drift terms with only t derivatives, since for $\Delta t \ll 1$ then $\Delta t \ll \sqrt{\Delta t} \ll 1$, also assuming v is fixed. Thus for $k > 1$, only the powers of diffusion part of $[\Delta V(t)|V(t) = v]$ need be considered. The mean estimate of the absolute value of dominant diffusion power is

$$\mathrm{E}\left[|\sigma_v(t)\sqrt{v}\Delta W_v|^k \,\middle|\, V(t) = v \right] = \alpha_k(t)(v\Delta t)^{k/2},$$

where $\alpha_k(t) = \sigma_v^k(t)\mathrm{E}[Z_v^k(t)]$. The products of these terms are an estimate of the corresponding dominant terms in the Taylor expansion,

$$\left| \frac{\partial^k F}{\partial v^k}(v,t) \right| \mathrm{E}[(\sigma_v(t)\sqrt{v}\Delta W_v)^k] = \gamma_k(t)\frac{\Delta t}{\sqrt{v}}\left(\frac{\Delta t}{v}\right)^{(k-2)/2} \le \gamma_k(t)\frac{\Delta t}{\sqrt{\varepsilon_v}}\left(\frac{\Delta t}{\varepsilon_v}\right)^{(k-2)/2},$$

for $\gamma_k(t) = \alpha_k(t)|\beta_k(t)|$, separated into the order $\Delta t/\sqrt{v}$ of the Itô diffusion approximation ($k = 2$) term and the factor relative to it. Hence, to eliminate all terms of higher order than $k = 2$, we need $\Delta t/v \ll 1$, i.e., $\Delta t \ll \varepsilon_v \ll 1$ to obtain a proper Itô diffusion approximation (3.1) for the transformation $Y(t) = F(V(t), t)$ in (3.4). Summarizing the results we have

Theorem 3.2. Conditions for a Consistent Itô Diffusion Approximation for Transforming the Heston Model to a Variance-Independent Noise Model: *Let the variance be positive and finite such that $0 < \varepsilon_v \le V(t) \le B_v$, then the variance independent model (3.1) is a consistent Itô diffusion approximation to the Heston model (2.1) uniform in the limits $\Delta t \ll 1$ and $\varepsilon_v \ll 1$ provided $\Delta t \ll \varepsilon_v$.*

5 Solution Consistency for Singular Limit Formulation Suitable for Theory and Computation.

However, as $V(t) \to 0^+$, the validity of neglecting higher order terms in the Taylor expansion underlying Itô's lemma is questionable, unless the integral is treated as a singular integral and the method of integration steps is properly specified.

First (3.5)-(3.6) are simply reformulated as

$$V(t) = e^{-\overline{\kappa}_v(t)} \left(\sqrt{V_0} + I_g(t) \right)^2, \tag{5.7}$$

where

$$I_g(t) = 0.5 \int_0^t e^{\overline{\kappa}_v(s)/2} \left(\left(\frac{\kappa_v\theta_v - \frac{1}{4}\sigma_v^2}{\sqrt{V}} \right)(s)ds + (\sigma_v dW_v)(s) \right). \tag{5.8}$$

Modifying the *method of ignoring the singularity* [8] to this implicit singular formulation, let

$$V^{(\varepsilon_v)}(t) = \max(V(t), \varepsilon_v)$$

where $\varepsilon_v > 0$ such that $\Delta t/\varepsilon_v \ll 1$ is some reference numerical increment $\Delta t \to 0^+$. This ensures that the time-step goes to zero faster than the cutoff singular denominator. The result is Itô diffusion approximation (3.1) consistency and the numerical consistency of the solution (5.7). Next (5.7)-(5.8) is reformulated as a recursion using some algebra for the next time increment Δt and the method of integration is specified for each subsequent time-step, i.e.,

$$V^{(\varepsilon_v)}(t+\Delta t) = \max\left(e^{-\Delta\overline{\kappa}_v(t)}\left(\sqrt{V^{(\varepsilon_v)}(t)} + e^{-\overline{\kappa}_v(t)/2}\Delta I_g^{(\varepsilon_v)}(t)\right)^2, \varepsilon_v\right), \tag{5.9}$$

where

$$\Delta\overline{\kappa}_v(t) \equiv \int_t^{t+\Delta t} \kappa_v(s)ds \to \kappa_v(t)\Delta t$$

as $\Delta t \to 0^+$. Similarly, a scaled increment of an integral is defined by

$$\begin{aligned} e^{-\overline{\kappa}_v(t)/2}\Delta I_g^{(\varepsilon_v)}(t) \equiv\ & 0.5\int_t^{t+\Delta t} e^{(\overline{\kappa}_v(s)-\overline{\kappa}_v(t))/2}\left(\left(\frac{\kappa_v\theta_v - \frac{1}{4}\sigma_v^2}{\sqrt{V^{(\varepsilon_v)}}}\right)(s)ds + (\sigma_v dW_v)(s)\right) \\ \to\ & 0.5\left(\left(\frac{\kappa_v\theta_v - \frac{1}{4}\sigma_v^2}{\sqrt{V}}\right)(t)\Delta t + (\sigma_v\Delta W_v)(t)\right), \end{aligned} \tag{5.10}$$

such that

$$\Delta t/\varepsilon_v \to 0^+ \text{ as } \Delta t \to 0^+ \ \&\ \varepsilon_v \to 0^+. \tag{5.11}$$

An Itô-Taylor expansion to precision dt or small Δt confirms that (5.9)-(5.10) yields the Heston [17] model, proving solution consistency. Thus, the square in (5.9) formally justifies the non-negativity of the variance and the volatility of the Heston [17] model, for a proper computational nonnegativity-preserving procedure.

However, for the general validity of applications of the chain rule and simulations, the Δt-variance limit (5.11) required for (5.7)-(5.8) implies that the non-negative variance condition is not needed in both theory and simulation using this perfect square form.

Further, the logarithmic transformation used for the geometric Brownian motion leads to singular derivatives of all orders, but the singularities are exactly cancelled out by the linear property of the underlying SDE, i.e., the singularity is removable with a logarithmic transformation.

6 Nonsingular, Explicit, Exact Solution.

In any event, the singular term in (5.7)-(5.8) vanishes in the special parameter case, such that

$$\kappa_v(t)\theta_v(t) = \frac{1}{4}\sigma_v^2(t), \quad \forall\, t. \tag{6.12}$$

Hence, we obtain a nonnegative, nonsingular exact solution

$$V(t) = e^{-\overline{\kappa}_v(t)}\left(\sqrt{V_0} + 0.5\int_0^t e^{\overline{\kappa}_v(s)/2}(\sigma_v dW_v)(s)\right)^2, \tag{6.13}$$

with the numerical form corresponding to (5.9)-(5.10),

$$V^{(\varepsilon_v)}(t+\Delta t)=\max\left(e^{-\Delta\overline{\kappa}_v(t)}\left(\sqrt{V^{(\varepsilon_v)}(t)}+\frac{1}{2}\int_t^{t+\Delta t} e^{(\overline{\kappa}_v(s)\,-\,\overline{\kappa}_v(t))/2}(\sigma_v dW_v)(s)\right)^2, \varepsilon_v\right). \quad (6.14)$$

7 Alternate Solution Relative to Deterministic Solution Using Integrating Factor Transformation.

Similarly, the chain rule for the integrating factor form $X(t) = \exp(\overline{\kappa}_v(t))V(t)$ for the general stochastic volatility (2.1) leads to a somewhat simpler integrated form,

$$V = V^{(\mathrm{det})}(t) + e^{-\overline{\kappa}_v(t)} \int_0^t e^{\overline{\kappa}_v(s)} \left(\sigma_v \sqrt{V} dW_v\right)(s), \quad (7.15)$$

using the maximum with respect to zero to remove spurious numerical simulations in absence of a perfect square form. In (7.15),

$$V^{(\mathrm{det})}(t) = e^{-\overline{\kappa}_v(t)} \left(V_0 + \int_0^t \theta_v(s) d\left(e^{\overline{\kappa}_v(s)}\right)\right) \quad (7.16)$$

is the deterministic part of $V(t)$. Note that there is only a linear change of dependent variable according to the stochastic chain rule [14] using the transformation $Y(t) = \exp(\overline{\kappa}_v(t))V(t)$. So the deterministic part is easily separated out from the square-root dependence and replaces the mean-reverting drift term. The $V^{(\mathrm{det})}(t)$ will be positive for positive parameters. For simulation purposes the incremental recursions are useful:

$$V^{(\mathrm{det})}(t + \Delta t) = e^{-\Delta\overline{\kappa}_v(t)}\left(V^{(\mathrm{det})}(t) + e^{-\overline{\kappa}_v(t)}\int_t^{t+\Delta t}\left(\theta_v d\left(e^{\overline{\kappa}_v}\right)\right)(s)\right) \quad (7.17)$$

and

$$V(t + \Delta t) = e^{-\Delta\overline{\kappa}_v(t)}\left(V(t) + e^{-\overline{\kappa}_v(t)}\int_t^{t+\Delta t} e^{\overline{\kappa}_v(s)}\left(\kappa_v\theta_v + \sigma_v\sqrt{V}dW_v\right)(s)\right), \quad (7.18)$$

Note that with constant coefficients, $\theta_v(t)=\theta_0$, $\kappa_v(t)=\kappa_0$ and $\sigma_v(t)=\sigma_0$, then (7.16-7.18) become

$$V^{(\mathrm{det})}(t) = V_0 e^{-\kappa_0 t} + \theta_0 \left(1 - e^{-\kappa_0 t}\right), \quad (7.19)$$

$$V^{(\mathrm{det})}(t + \Delta t) = \theta_0 + e^{-\kappa_0 \Delta t} \left(V^{(\mathrm{det})}(t) - \theta_0\right), \quad (7.20)$$

and

$$V(t + \Delta t) = \theta_0 + e^{-\kappa_0 \Delta t} \left(V(t) - \theta_0 + \sigma_0 \sqrt{V(t)} \Delta W(t)\right). \quad (7.21)$$

However, as Lord et al. [23] point out, a sufficiently accurate simulation scheme and a large number of simulation nodes are required so that the right-hand side of (2.1) generates nonnegative values. Nonnegative values using the stochastic Euler simulation have been verified for Heston's [17] constant risk-neutralized parameter values $\{\kappa_v = 2.00,\ \theta_v = 0.01,\ \sigma_v = 0.10\}$ as long as the scaled number of nodes per unit time $N/(\kappa_v t_f) > 100$.

Hence, since the variance by definition for real processes cannot be negative, practical considerations suggest replacing occurrences of $V(t)$ by $\max(V(t), \epsilon)$, where ϵ is some numerically small, positive quantity for numerical purposes to account for the appearances of negative variance values.

8 Selected Numerical Simulations.

In Figure 1, the simulations for the Euler approximation to Heston's stochastic variance equation (2.1), truncating any negative values to zero, compared to the perfect square solution simulations in (5.9) using $\varepsilon = 0$ since is not needed for nonpositivity. Also, shown is the deterministic solution $V^{(\mathrm{det})}(t + \Delta t)$ simulation from (7.17). The negative of the difference between the the truncated Heston Euler simulations and the perfect square form is shown at the bottom of the figure in red. The maximum absolute value of this difference is 2.46e-3. Otherwise, the Heston-Euler and perfect square simulation trajectories are barely distinguishable in Fig. 1. The parameter values used are $\{\kappa_v = 2.00,\ \theta_v = 0.01,\ \sigma_v = 0.25\}$, which coincidentally have the Heston model parameter ratio $\kappa_v\theta_v/\sigma_v^2 = 0.25$ from the exact solution using (6.12) .

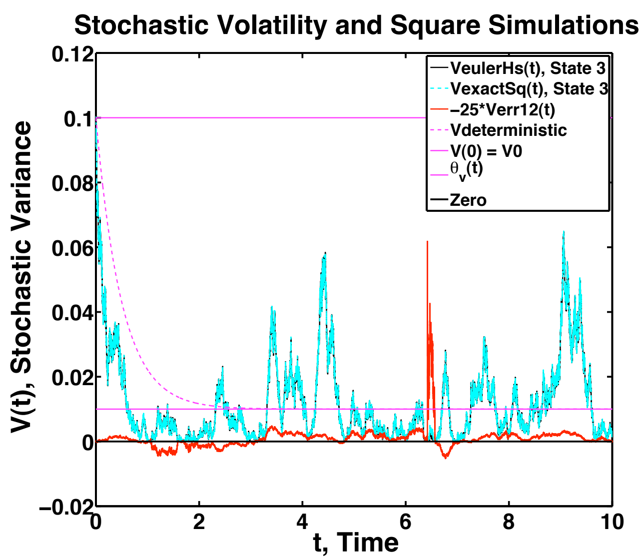

Figure 1: Comparison of the Heston-Euler simulation of (2.1) with negative values truncated to zero and the perfect square solution(5.9). Also shown are the deterministic solution (7.17) and the negative of the error magnified 25 times. The Heston model parameter ratio is $\kappa_v\theta_v/\sigma_v^2 = 0.25$. There are 10^6 sample points over 10 time units.

In the simulation of Fig. 1, the Heston-Euler simulation before truncation produces $K_{neq} = 76$ improper nonpositive values over one million sample points, while the perfect square solution does not even produce zero values. The most negative value of the Heston-Euler simulation is V=-1.01e-6. The number of negative values K_{neq} before truncation are plotted in Figure 2 against the Heston model parameter ratio $\kappa_v\theta_v/\sigma_v^2$. The number of negative values begin at a ratio value of 0.22 ± 0.001 and become extreme as the ratio approaches the Feller boundary classification separation point at 0.5. The results for the simulations use the same random number generator seed in MATLAB. Note the negative value count for the alternate implicit, integrated solution including, in part the deterministic solution of (7.18).

In fact, the alternate and Heston-Euler simulations are virtually the same, with the same maximum absolute difference of 2.46e-03 from the perfect square simulation, suggesting that the problem is a numerical one with the square root function of the variance. Also, there is appears to be a discrepancy between the variance violations of the simulations for the process model and Feller's description of the positivity for the distribution solution. The likely reason is that the Euler simulations of the diffusion process are discrete, while the theoretical process is contintinuous, in which case the trajectory must be at zero for an instant before becoming negative. However, when $V(t)=0$ then $dV(t)=\kappa_v\theta_v dt>0$, so the trajectory will reflected back to positive values. Thus, the simulated negative values, though very small must be a discretization flaw. This is a good reason for setting any negative value at least to zero.

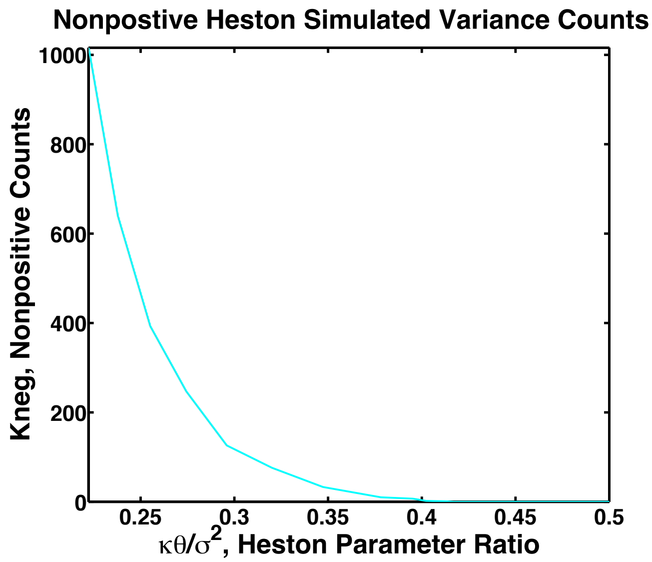

Figure 2: Nonpositive variance counts for the Heston Euler and Alternate solution simulations, counted prior to truncation to zero. The coordinate axis is the Heston model parameter ratio $\kappa_v\theta_v/\sigma_v^2$, where $\kappa_v = 2.00$ and $\theta_v = 0.01$, while $\sigma_v \in [0.20, 0.30]$.

9 Conclusions.

Some recent, practical results for the positivity of the variance are given for the Heston [17] model. There are an implicit perfect square solution in the general parameter case and an explicit form for the case where the speed of reversion times the level of reversion is one quarter of the square of the *volatility of the volatility* coefficient. Also, effort has been made to examine the consistency of the Heston model under transformations considering small variance and also the spurious simulation of small negative values.

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