Optimal Harvesting with Both Population and Price Dynamics

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Abstract

We consider the effects of large inflationary price fluctuations on the computed optimal harvest strategy for a randomized Schaefer model. Both prices and population sizes are assumed random with both background (Wiener) and jump (Poisson) components. Population fluctuations are assumed to be density independent, i.e., relative changes are independent of population size. Stochastic dynamic programming is employed to find the optimal harvesting effort and economic return for a realistic set of bioeconomic data for Pacific halibut. It is found that inflationary effects have a pronounced influence on the optimal return, even in a hazardous or disastrous environment. However, optimal harvesting effort levels are much less sensitive to inflationary effects.

1. Introduction

Bioeconomic resource models incorporating random fluctuations in either population size or model parameters have been the subject of much interest. Reed [1, 2] considered optimal harvest and escapement policies in the presence of general discrete-time multiplicative noise for a variety of assumptions. Gleit [3] gave an exact solution for the optimal present value, and corresponding linear

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optimal control, of an exponentially growing resource subject to Gaussian white noise fluctuations. Ludwig [4, 5] solved by perturbation methods the more general control problem for populations with Ricker type growth influenced by lognormal noise, while Ludwig and Varah studied these problems numerically in [6]. Ryan and Hanson [7] solved exactly the optimal harvest problem for constant effort and exponential growth in the presence of large fluctuations modeled by Poisson processes and in [8] numerically constructed the optimal feedback control for logistic growth with the same type of noise. See also [7, 8] for a more extensive bibliography.

In the above models per unit prices are either constant or exogenous and deterministic. However, random price fluctuations are a realistic effect and have been incorporated in standard resource models. Andersen [9] has studied continuous time optimal harvest models with logistic growth when prices follow a general probability distribution. Lewis [10, 11] has examined similar models in discrete time when both prices and population size are allowed to be random. Pindyck [12] has studied the economic consequences of uncertainty in population size as well as unit price for a variety of continuous-time harvest models, using Gaussian distributed Wiener processes to model fluctuations. Clark [13] discusses a discrete-time model with randomly varying seasonal prices. Ryan [14] considered a model in which the unit price changes suddenly at a random time. Comprehensive introductions to these problems are given by Mangel [15] and Andersen and Sutinen [16].

In the present paper, we explore the effects of price fluctuations on the computed optimal harvest strategy for a randomized Schaefer type model. In our model unperturbed prices consist of an inflation adjusted constant price term plus a supply/demand term. Random price variation is incorporated into the model through a multiplicative random process that includes both small continuous-time fluctuations and the possibility of occasional, large random changes. Since our primary interest is to model, and study numerically, the effects of randomness on supply/demand factors, we ignore both random and inflationary effects on the postulated cost function. Technically, random and inflationary effects of the type hypothesized for the supply/demand function are easy to incorporate into the model. However, random fluctuations in cost are likely to be much more complicated and require a more complicated general model. Thus, we restrict our attention to serially uncorrelated exogenous random price and population fluctuations. Our analytical emphasis is primarily numerical.

The presence of such fluctuations is well documented in fisheries. Figure 1 shows price versus year and Figure 2 shows catch versus year for the Pacific halibut (*Hippoglossus hippoglossus*) system. See the 1984 and 1985 International Pacific Halibut Commission (IPHC) Annual Reports [17, 18]. The price data show low level fluctuations followed by a precipitous decline followed by moderate fluctuations over a short time period. The price versus catch data as shown in Figure 3 reflect similar but more pronounced fluctuation with a prominent trough in the highly inflationary time around 1979. In general, random catch and recruitment fluctuations are particularly well documented. See [8] for a more detailed survey.

In the present paper we explore the effects of both random population fluctuations as well as random price fluctuations on the computed optimal harvest strategy. Our model is new in that it simultaneously incorporates the possibility of large fluctuations in both resource size and per unit prices while maintaining the general structure employed in the previously discussed work. Section 2 briefly develops the deterministic model. The stochastic model is presented in Section 3. Numerical methods and results are discussed in Section 4.

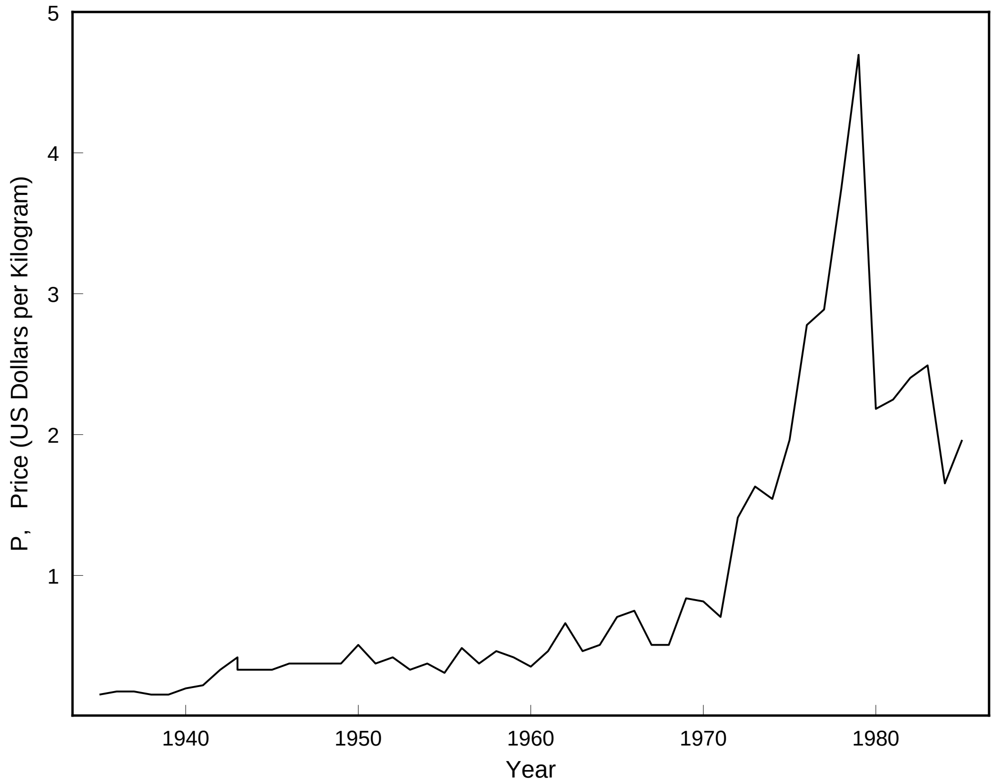

Figure 1: Pacific halibut prices in U.S. dollars per kilogram for each year from 1935 to 1985. The source of the data in the table in Appendix II in the IHPC 1984 and 1985 Annual Reports [17, 18].

2. Deterministic Model

A frequently employed model for the harvesting of a renewable resource of size $N(s)$ at time s is the differential equation

$$dN(s) = [r_1 N(1 - N/K) - H(s)]\, ds, \qquad s > 0, \quad N(0) = x. \tag{1}$$

Here, r_1 and K are the population's intrinsic growth rate and carrying capacity, respectively. The harvest term is assumed to be given by *catch per unit effort hypothesis* [19]

$$H(s) = q \cdot E \cdot N(s), \tag{2}$$

where q is the catchability coefficient. The effort $E = E(N(s), s)$, in feedback control form here, is a measure of harvesting effort and is assumed to satisfy the condition

$$E_{\min} \leq E \leq E_{\max} < \infty.$$

The value of the harvest is given by the discounted present value of future resources

$$v(x; E) = \int_0^T e^{-\delta s}[pqEN(s) - c(E)]\, ds, \tag{3}$$

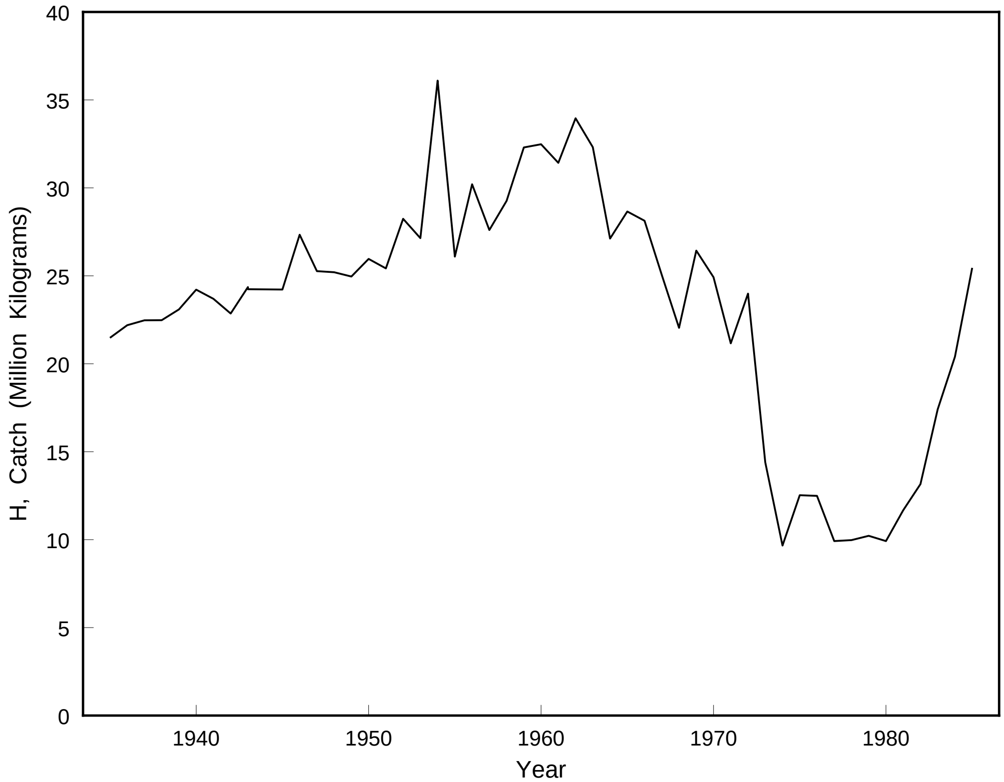

Figure 2: U.S.-Canadian catch in millions of kilograms for each year from 1935 to 1985. The source is the same as Figure 1.

with T the time horizon, δ the discount rate, p the price of a unit of harvested biomass, and $c(E)$ the cost of a unit of effort when the population size is N. The instantaneous net return or profit is given by

$$R(s) = p \cdot H(s) - c(E) = p \cdot q \cdot E \cdot N(s) - c(E)$$

at time s. If it is assumed that the goal of the harvest is to find the effort level E^* that maximizes the total profit, then we must compute

$$v^*(x) \equiv v\left(x; E^*\right) = \max_E [v(x; E)], \tag{4}$$

subject to the dynamical constraint in Eq. (1). This is a problem in optimal control theory and can be studied using Pontryagin's maximum principle (see Clark [19] on the maximum principle in the context of fishery bioeconomics). However, that method does not readily extend to the stochastic case. A more efficient form for computing optimal controls in the presence of random fluctuations is the Bellman equation of dynamic programming (see Bryson and Ho [20], for instance). Thus, we consider the current value form of Eq. (3) given by

$$V(x, t; E) = \int_t^T e^{-\delta(s-t)} [pqEN(s) - c(E)]\, ds, \tag{5}$$

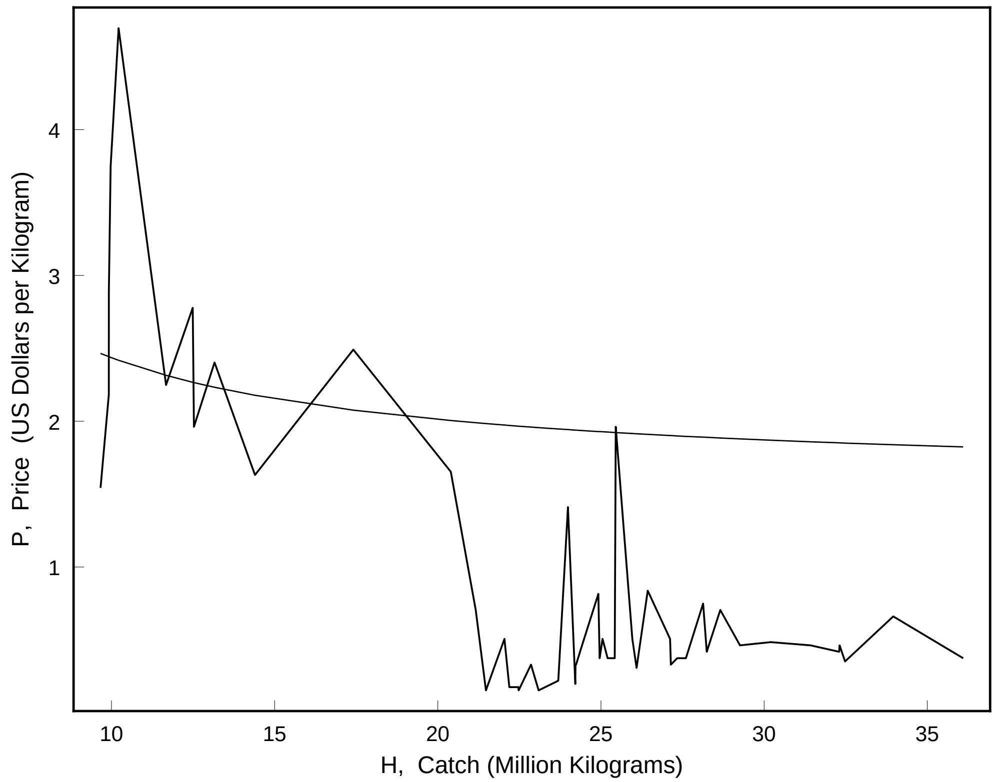

Figure 3: Pacific halibut price in U.S. dollars per kilogram versus catch in millions of kilograms for the years from 1935 to 1985. The source is the same as Figure 1. Also, the linear regression for the price times catch as a function of catch from 1980 to 1985 is displayed as the smooth hyperbolic price curve.

and apply the principle of optimality to derive an equation for

$$V^*(x,t) = \max_E [V(x,t;E)],$$

(see [8] for a simple, formal derivation),

$$V_t^*(x,t) + r_1 x(1 - x/K)V_x^*(x,t) - \delta V^*(x,t) + S^*(x,t) = 0, \tag{6}$$

where

$$S^*(x,t) = \max_E \left[(p - V_x^*(x,t))\, qEx - c(E) \right]. \tag{7}$$

Let $E_R(x,t)$ be the regular solution of $c'(E) = (p - V_x^*(x,t))qx$ corresponding to the unconstrained maximum in Eq. (7). For instance, in the case of quadratic costs with $c(E) = c_1 \cdot E + c_2 \cdot E^2$, we obtain

$$E_R(x,t) = \frac{(p - V_x^*(x,t))qx - c_1}{2 \cdot c_2}.$$

Upon imposing the constraints on the harvesting effort E, we get the *bang-regular-bang* control

$$E^*(x,t) = \begin{cases} E_{\max}, & E_{\max} \le E_R(x,t) \\ E_R(x,t), & E_{\min} \le E_R(x,t) \le E_{\max} \\ E_{\min}, & E_R(x,t) \le E_{\min} \end{cases} . \tag{8}$$

The full problem is determined by imposing the final boundary condition

$$V^*(x,T) = 0, \tag{9}$$

and natural boundary condition

$$V^*(0,t) = -\frac{(c_1 + c_2 E_{\min})E_{\min}}{\delta}\left(1 - e^{-\delta(T-t)}\right), \tag{10}$$

in the case of increasing, convex quadratic costs, i.e., $c_1 > 0$ and $c_2 > 0$, provided that the discount rate δ is positive. Note that if $x = 0$, then $N(t) = 0$ by the vanishing of the right hand side of (1) with (2) at extinction and the optimal harvesting effort must be at the minimum $E_{\min}$ due to the negativity of the cost function.

See [8] for details. This is the form of the problem most easily generalized to the stochastic model discussed in Section 3.

3. Stochastic Model

An Itô stochastic differential equation describing the growth and harvesting of a model resource population subjected to large random changes in size is the randomized Schaefer model [8]

$$\begin{aligned} dN(s) &= [r_1 N(1 - N/K) - H(s)]\, ds + \sigma_1 N\, dW_1(s) + N \sum_{j=1}^{n} a_j\, dZ_j(s, f_j), \\ N(t) &= x, \end{aligned} \tag{11}$$

where r_1, K and $H(s) = qEN(s)$ are as described in Section 2.

There are two random components in Eq. (11). Large rare finite amplitude fluctuations are characterized by the density independent, compound Poisson process

$$N \sum_{j=1}^{n} a_j\, dZ_j(s; f_j).$$

Here, the relative jump amplitude $a_j > -1$, the jump rate $f_j > 0$, the Wiener noise coefficient $\sigma_1 \ge 0$, and the jth incremental Poisson process $dZ_j(s; f_j)$ has infinitesimal mean and variance $f_j\, ds$ drawn from the set $\{dZ_1, \ldots, dZ_n\}$ of independent Poisson processes. The independent density amplitude factor N is chosen so that any variation is proportional to current population size as measured by the a_j, i.e., relative changes are independent of density N. Background fluctuations are modeled by the normalized, Gaussian distributed Wiener process dW_1 with zero mean and infinitesimal variance ds. Thus, in the model (11) the population size is known at a time $s > 0$ with future population size determined from the deterministic component, the serially uncorrelated fluctuations occurring from dW_1, and the random increases occurring at the times of events of the dZ_j with frequency f_j. Large scale effects, typically brought about by very complex and poorly understood mechanisms, are thus estimated in (11) as a lumped sum of density independent terms. Such large fluctuations are commonly observed in marine recruitment data and are well documented in the fisheries literature [21, 22, 23, 24]. An analysis of the moments to the process defined by (11) in the absence of harvesting is given in [25].

In the economic component of the model we assume management is neutral to risk and that prices are random and given by the supply/demand relation

$$P(s) = \left(\frac{p_0}{H(s)} + p_1\right) Y(s), \tag{12}$$

where again $H(s) = qEN(s)$ is the amount of harvested biomass, $P(s) \cdot H(s)$ is the gross return on harvest $H(s)$, p_1 is a constant price per unit harvested biomass coefficient, and p_0 is the supply/demand coefficient component of the price. In (12), $Y(s)$ is a fluctuating inflationary factor satisfying the Itô stochastic differential equation

$$dY(s) = r_2 Y\, ds + \sigma_2 Y\, dW_2(s) + Y \sum_{j=1}^{m} b_j\, dQ_j(s; g_j), \qquad Y(t) = y, \tag{13}$$

with relative jump amplitude $b_j > -1$, jump rate $g_j > 0$, and Wiener coefficient $\sigma_2 \geq 0$. The dQ_j for $j = 1, \ldots, m$, and dW_2 are, respectively, incremental Poisson processes and a Wiener process, as in (11). Equation (12) describes prices inflated at an annual rate r_2, subjected to the rapid background perturbations of dW_2 as well as the occasional random jump increases or decreases brought about by the dQ_j. Such a characterization for the price P is plausible from Figure 3, which shows price versus catch for the Pacific halibut fishery from 1935 to 1985. The estimated mean price is hyperbolic in nature with both types of fluctuations evident about the mean. In particular, in the absence of discernible correlations in the fluctuations the data suggest the multiplier Y to be independent of P.

Computation of the optimal exploitation policy in the stochastic case is much more complicated than that for the deterministic model (1)–(5). Equation (5) must be modified to account for the random terms describing fluctuations in N and P and given by (11)–(13). Corresponding to (4)–(5), we seek a policy $E^*(x, y, t)$ that maximizes the expected discounted current value

$$V(x, y, t) = \text{Mean}\left[\int_t^T e^{-\delta(s-t)} \left[(p_0 + p_1 qEN)Y(s) - c(E)\right] ds \mid N(t) = x,\ Y(t) = y\right], \tag{14}$$

where

$$\text{Mean} = \underset{\{\mathbf{dZ}, \mathbf{dQ}, \mathbf{dW}\}}{\text{Mean}}$$

denotes the mean or expectation taken over the vector processes $\mathbf{dZ} = [dZ_1, \ldots, dZ_n]$, $\mathbf{dQ} = [dQ_1, \ldots, dQ_n]$, and $\mathbf{dW} = [dW_1, dW_2]$. Here,

$$P(s) \cdot H(s) - c(E) = (p_0 + p_1 \cdot H(s)) \cdot Y(s) - c(E), \tag{15}$$

is the net return on harvested biomass $H(s)$ at time s. We further specialize effort costs to the quadratic form

$$c(E) = c_1 E + c_2 E^2. \tag{16}$$

The additional quadratic cost term c_2E^2, appropriately scaled, may be viewed as a perturbation on the more typically employed linear costs as well as a technique to avoid difficulties inherent in the computation of singular controls [19, 20]. We assume that both c_1 and c_2 are positive, so that costs are an increasing function and costs grow faster than a linear function of effort. Such a cost function is, however, relatively common and has been employed in fisheries studies by a number of authors [10, 11, 27, 28, 29].

Note that we do not include inflationary effects in the costs since our main focus in on price dynamics, but since we have modeled the price dynamics including inflationary effects, the discount rate δ in Eq. (14) must be considered the nominal discount rate rather than the real or inflation corrected discount rate. Also, we have selected a finite horizon T rather than an infinite horizon, since we believe that the finite horizon case embodies more realism, particularly when motivated by fisheries problems where the fishing season can be rather short and the environment dynamic. In this current paper, we are concerned with the dynamic problem and not the equilibrium solutions such as those associated with the infinite horizon case.

Since N and Y are stochastic processes and E is a function of the state as well as time, the easiest approach to the calculation of (E^*, V^*) is via the Bellman equation of continuous-time dynamic programming [8, 15, 30]. Since N and Y involve discontinuous processes, the Bellman equation will involve functional delay terms in both x and y as well as a second-order derivative term arising from the Wiener processes in (11) and (13). Thus, E^* and V^* satisfy

$$\begin{aligned} 0 \;=\;& V_t^* + r_1 x(1-x/K)V_x^* + \frac{\sigma_1^2 x^2}{2}V_{xx}^* - \delta V^* + \sum_j f_j\left[V^*\left((1+a_j)x,y,t\right) - V^*(x,y,t)\right] \\ &+ r_2 y V_y^* + \frac{\sigma_2^2 y^2}{2}V_{yy}^* + \sum_j g_j\left[V^*(x,(1+b_j)y,t) - V^*(x,y,t)\right] + S^*(x,y,t), \end{aligned} \tag{17}$$

where S^* is the control switching term containing the argument of the maximum in (17),

$$S^*(x,y,t) = \max_E\left[p_0\cdot y + \left(p_1\cdot y - V_x^*(x,y,t)\right)qEx - \left(c_1E + c_2E^2\right)\right], \tag{18}$$

with unconstrained, regular control given as

$$E_R(x,y,t) = \frac{\left(p_1\cdot y - V_x^*(x,y,t)\right)\cdot q\cdot x - c_1}{2\cdot c_2}, \tag{19}$$

determined from the argument of the maximum in (18), and with the constrained, optimal control given as

$$E^*(x,y,t) = \begin{cases} E_{\max}, & E_{\max} \le E_R(x,y,t) \\ E_R(x,y,t), & E_{\min} \le E_R(x,y,t) \le E_{\max} \\ E_{\min}, & E_R(x,y,t) \le E_{\min} \end{cases}, \tag{20}$$

similar to the form in Eq. (8) in the one-dimensional, deterministic case, but with a two-dimensional dependence.

Equation (17) is augmented by the side conditions

$$V^*(x,y,T) \;=\; 0, \tag{21}$$

$$V^*(0,0,t) \;=\; -\frac{(c_1 + c_2E_{\min})E_{\min}}{\delta}\left(1 - e^{-\delta(T-t)}\right), \tag{22}$$

with the same reasoning given for (9) and (10) in the deterministic case, but here with $Y(s) \equiv 0$ when $y = Y(t) = 0$. The most appropriate method of solution for (17)–(22) appears to be numerical. Perturbative techniques might be suggested for σ_1 and σ_2 small, but these are not likely to be effective because of the $O(1)$ nature of the functional terms in (17). The numerical procedures are discussed more fully in Section 4.

4. Numerical Results

In this section we examine the results of the numerical solution of Eq. (17) for certain values of the parameters. The numerical solution of (17) is also outlined.

Our parameter values are based on Pacific halibut data over a number of years and come from a variety of sources. We use estimates of $r_1 = 0.71$/year and $K = 80.5 \times 10^6\,\mathrm{Kg}$ (see Clark [19]). The price and cost data were taken from the 1984 and 1985 IPHC Annual Reports [17, 18] for the period 1980 to 1985 to allow some temporal perspective and avoid the anomalous inflationary period of the late 1970's. Linear regression was used to fit value versus catch, $\widetilde{V} = p_0 + p_1 \cdot H$, for 1980 to 1985. Although we use only six data points, $p_0 = \$8.46$/year and $p_1 = \$1.59/$Kg with 78% of the variance explained. The results of the linear regression for the value is displayed in Figure 3 as the smooth hyperbolic curve of the price versus catch

$$\widetilde{P} = \widetilde{V}/H = p_0/H + p_1,$$

fitting only the higher price fluctuations in the original IPHC price versus catch data from 1935 to 1985.

Other parameter values are taken to be: $r_2 = 0.01$/year, $T = 10$ years, $\delta = 0.06$/year, $c_1 = \$96 \times 10^{-6}/$(skate-year)$/\mathrm{year}$ (a standard skate is a 550 meter ground line with 100 hooks. Note that in the IPHC data [17, 18] the annual effort is given in units of skates with year dimensions implicit, thus year dimensions have been explicitly added here to effort, cost and catchability to preserve dimensional correctness), $c_2 = \$0.10 \times 10^{-6}/$(skate-year)$^2/\mathrm{year}$, $q = 3.30 \times 10^{-6}/$(skate-year)/year, $E_{\min} = 0$, and $E_{\max} = r_1/q = 0.2152 \times 10^6$ skate-years. Since our primary focus is on discontinuous effects, we take $\sigma_1 = 0$ and $\sigma_2 = 0$, removing the continuous background noise. We further lump the additive effects of the jumps by taking $f_1 = 0.5$ with $f_j = 0$ for $j \geq 2$, $a_1 = -0.5$ with $a_j = 0$ for $j \geq 2$, $g_1 = 0.5$ with $g_j = 0$ for $j \geq 2$, and $b_1 = 0.5$ with $b_j = 0$ for $j \geq 2$.

The numerical solution of (17) has been obtained by employing a hybrid extrapolated predictor-corrector and Crank-Nicolson finite difference method modified to account for

1. functional terms that appear due to the Poisson processes used to characterize the large fluctuations in both population and inflation rate, and
2. the maximization embodied in the switching term (18).

We discretize using $x_i = (i-1)\Delta x$, $i = 1, \cdots, N_x$ for the population, $y_j = (j-1)\Delta y$, $j = 1, \cdots, N_y$ for the inflationary factor, and $t_k = T - (k-1)\Delta t$, $k = 1, \cdots, N_t$ for the time, where $\Delta x = K/(N_x - 1)$, $\Delta y = e^{r_2 T}/(N_y - 1)$, and $\Delta t = T/(N_t - 1)$. The dependent variable $V^*(x_i, y_j, t_k)$ is represented by the discrete variable $V_{i,j,k}$. Second order central finite differences are used for spatial derivatives such that $V_x^*(x_i, y_j, t_k)$ is approximated by $DVX_{i,j,k} = \frac{1}{2}(V_{i+1,j,k} -$

$V_{i-1,j,k})/\Delta x$ and $V_y^*(x_i, y_j, t_k)$ is approximated by $DVY_{i,j,k} = \frac{1}{2}(V_{i,j+1,k} - V_{i,j-1,k})/\Delta y$, with appropriate forms for the boundaries. The second derivatives terms $V_{xx}^*(x_i, y_j, t_k)$ and $V_{yy}^*(x_i, y_j, t_k)$ are discretized by the central difference formulas $DDVX_{i,j,k} = (V_{i+1,j,k} - 2V_{i,j,k} + V_{i-1,j,k})/(\Delta x)^2$ and $DDVY_{i,j,k} = (V_{i,j+1,k} - 2V_{i,j,k} + V_{i,j-1,k})/(\Delta y)^2$, respectively. The backward time derivative $V_t^*(x_i, y_j, t_{k+0.5})$ is approximated by $DVT_{i,j,k} = -(V_{i,j,k+1} - V_{i,j,k})/\Delta t$, which is also a second order central finite difference, but about the half time steps of the Crank-Nicolson method. The functional terms $V^*((1+a_l)x_i, y_j, t_k)$ and $V^*(x_i, (1+b_l)y_j, t_k)$ are approximated, respectively, by linear interpolation between the two nearest nodal values $V_{\hat{i},\hat{j},k}$ consistent with order of the errors in the second order central finite differences used for the spatial derivatives. We denote the linear interpolation of $V^*((1+a_l)x_i, y_j, t_k)$ by $ZV_{i,j,k,l}$ and $V^*(x_i, (1+b_l)y_j, t_k)$ by $QV_{i,j,k,l}$.

The Crank-Nicolson average for the midpoint time-step $V_{i,j,k+0.5}$ is denoted by $VM_{i,j,k} \equiv 0.5(V_{i,j,k} + V_{i,j,k+1})$, and the accelerating extrapolated starting value by $VE_{i,j,k} \equiv 0.5(3V_{i,j,k} - V_{i,j,k-1})$ provided $k \leq 2$, with corresponding notations for the spatial derivatives,

Thus, the discrete extrapolated, predictor approximation corresponding to the Bellman equation (17) is

$$
\begin{aligned}
V_{i,j,k+1}^{(p)} &= V_{i,j,k}^{(c,*)} + \Delta t \Big[r_1 x_i (1 - x_i/K) DVXE_{i,j,k} + \frac{1}{2}\sigma_1^2 x_i^2 DDVXE_{i,j,k} - \delta VE_{i,j,k} \\
&+ \Sigma_l f_l (ZVE_{i,j,k,l} - VE_{i,j,k}) + r_2 y_j DVYE_{i,j,k} \\
&+ \frac{1}{2}\sigma_2^2 y_j^2 DDVYE_{i,j,k} + \Sigma_l g_l (QVE_{i,j,k,l} - VE_{i,j,k}) + SE_{i,j,k} \Big],
\end{aligned}
\tag{23}
$$

where $V_{i,j,k}^{(c,*)}$ is the final correction from the kth backward time step, $DVXE_{i,j,k} = 0.5(VE_{i+1,j,k} - VE_{i-1,j,k})/\Delta x$, for example, and $VE_{i,j,k} \equiv 0.5(3V_{i,j,k}^{(c,*)} - V_{i,j,k-1}^{(c,*)})$ provided $k \leq 2$ so that corrections are available on at least two starting time steps. In the predictor evaluation step, *DVXM*, *DVYM*, *DDVXM*, *DDVYM*, *ZVM*, and *QVM* are evaluated using the discrete values

$$VM_{i,j,k}^{(p)} = 0.5(V_{i,j,k}^{(c,*)} + V_{i,j,k+1}^{(p)}).$$

From (19), it follows that the regular control $E_R(x_i, y_j, t_{k+0.5})$ at the predictor step is given approximately by

$$ERM_{i,j,k}^{(p)} = (p_1 Y_j - DVXM_{i,j,k}^{(p)} \cdot q \cdot x_i - c_1)/(2c_2). \tag{24}$$

The predicted, constrained, optimal control $EM_{i,j,k}^{(p)}$ is computed using composite formula (20) with $ERM_{i,j,k}^{(p)}$ substituted for $E_R(x, y, t)$ on the right hand side and the maximized control switching term is computed from the argument of the maximum in (18) by substituting the optimal $EM_{i,j,k}^{(p)}$ for E.

Consequently, the $(L+1)$th correction to the discretized Bellman equation is given by

$$
\begin{aligned}
V_{i,j,k+1}^{(c,L+1)} &= V_{i,j,k}^{(c,*)} + \Delta t \Big[r_1 x_i (1 - x_i/K) DVXM_{i,j,k}^{(c,L)} + \frac{1}{2}\sigma_1^2 x_i^2 DDVXM_{i,j,k}^{(c,L)} - \delta VM_{i,j,k}^{(c,L)} \\
&+ \Sigma_l f_l (ZVM_{i,j,k,l}^{(c,L)} - VM_{i,j,k}^{(c,L)}) + r_2 y_j DVYM_{i,j,k}^{(c,L)} \\
&+ \frac{1}{2}\sigma_2^2 y_j^2 DDVYM_{i,j,k}^{(c,L)} + \Sigma_l g_l (QVM_{i,j,k,l}^{(c,L)} - VM_{i,j,k}^{(c,L)}) + SM_{i,j,k}^{(c,L)} \Big],
\end{aligned}
\tag{25}
$$

for $L = 0$ to L^*, where $\mathit{VM}_{i,j,k}^{(c,0)} = \mathit{VM}_{i,j,k}^{(p)}$, i.e., the prediction is the 0th correction. The subsequent correction evaluation step is again the Crank-Nicolson average

$$\mathit{VM}_{i,j,k}^{(c,L)} = 0.5(V_{i,j,k}^{(c,*)} + V_{i,j,k+1}^{(c,L)}),$$

which is used to calculate $(L+1)$th corrections for all differenced derivatives and functions terms as well as $\mathit{ERM}_{i,j,k}^{(c,L+1)}$, $\mathit{EM}_{i,j,k}^{(c,L+1)}$ and $\mathit{SM}_{i,j,k}^{(c,L+1)}$.

Corrections are continued until a relative stopping criterion,

$$|V_{i,j,k+1}^{(c,L+1)} - V_{i,j,k+1}^{(c,L)}| < \varepsilon |V_{i,j,k+1}^{(c,L)}|$$

is satisfied for all $\{i, j\}$ at fixed discrete time $k+1$ and some relative tolerance $\varepsilon > 0$ with the stopped correction counter denoted by $L_k^* = L + 1$. The final correction value that is used in the next time step is more concisely denoted by $V_{i,j,k}^{(c,*)} = V_{i,j,k}^{(c,L_k^*)}$. Typically, only a few corrections are needed for reasonable accuracy, beyond the starting, final value at $k = 1$.

The convergence of the corrections is not a simple matter and convergence difficulties increase with the dimension of the state space, since the convergence of the discretized stochastic dynamic programming procedure critically depends on the mesh ratio of Δt compared to some metric of Δx and Δy. For more information on the approximate quasi-deterministic convergence criteria used, comparison to other methods, and additional references the reader is referred to the survey chapter of Hanson [26].

Figure 4 shows the optimal current value $V^*(K, y, t)$ in million dollar units using optimal effort $qE^*(K, y, t)/r_1$ versus a scaled price factor $y \cdot \exp(-r_2 T)$, i.e., with the deterministic inflationary part $\exp(+r_2 T)$ at the final time scaled out. The figure is intended to show the effects on the optimal current value due to the inflationary factor. The curves indexed by time-to-go, $T - t = 0$, 2, 4, 6, 8, 10 starting at the bottom along the abscissa at $T - t = 0$ (i.e., the final time $t = T$) to the uppermost curve at $T - t = 10$ (i.e., initial time $t = 0$). As expected, the optimal current value increases as a function of increasing scaled inflation factor with a nearly constant slope for fixed $T - t$, except for the zero final current value at $t = T$. From the curves $T - t = 2$ to 4, 4 to 6, 6 to 8 and 8 to 10 the current value shows a substantial increase of about 2.9, 2.2, 2.0 and 1.9 times, respectively. The optimal current value as a function of population size (not pictured) is relatively flat but shows similar large increases when indexed over the indicated intervals. The displayed curves are essentially linear with slope approximately $p_1 qE(K, y, t)N(K, t)$ for the relatively short horizon $T = 10$. Thus, even when influenced by density independent disasters, the optimal current value is extremely sensitive to the stochastic inflationary factor with the rate of increase increasing for longer times. As a word of caution in interpreting Fig. 4, we note that the scaled inflationary factor $y \cdot \exp(-r_2 T)$ just indicates a rough, average scaling of y for this model and does not indicate an exponential growth in the price of halibut since we took only price parameters from the halibut fishery and not inflation parameters.

Since the price P is time dependent, the instantaneous return or value will be altered by both changes in the stock level $N(t)$ and the price level $P(t)$. In order to motivate this and our more general numerical results, we examine the simplifying quasi-deterministic approximation [25], bearing in mind that the results displayed in Figs. 4 through 6 are for the stochastic problem with random Poisson jumps in price and population. The rate of change in the inflationary factor can be

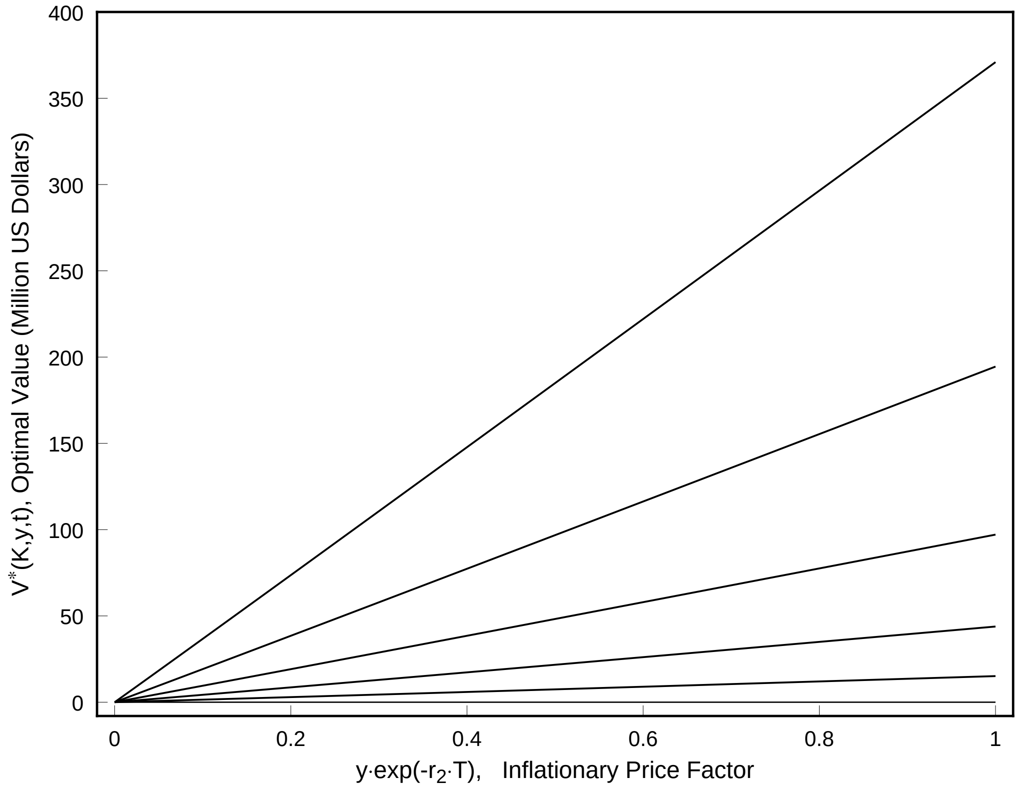

Figure 4: Optimal current value, $V^*(K, y, t)$, in millions of U.S. dollars versus the scaled price factor, $y \cdot \exp(-r_2 \cdot T)$, with time parameter $t = 0.0$, 2.0, 4.0, 6.0, 8.0, 10.0 for each curve ordered from top to bottom, respectively, and with population size fixed at carrying capacity $x = K$.

approximated by the quasi-deterministic approximation to Eq. (13), where Y_{QD} satisfies

$$dY_{QD}(t) = \mathrm{Mean}\left[dY(t) \mid Y(t) = Y_{QD}(t)\right] = \left(r_2 + \sum_j b_j g_j\right) Y_{QD}(t)\, dt, \tag{26}$$

i.e., approximated by the exponential growth:

$$Y_{QD}(t) = y(0) \exp\left(\left(r_2 + \sum_j b_j g_j\right) t\right). \tag{27}$$

Similarly, the stock level has the quasi-deterministic approximation, $N_{QD}(t)$, assuming in (11) that the effort E is constant and that

$$H_{QD}(t) = q \cdot E \cdot N_{QD}(t),$$

for simplicity,

$$dN_{QD}(t) \quad = \quad \mathrm{Mean}\left[dN(t) \mid N(t) = N_{QD}(t)\right]$$

$$= \left(r_1 \cdot (1 - N_{QD}(t)/K) - q \cdot E + \sum_j a_j f_j \right) N_{QD}(t)\, dt, \tag{28}$$

i.e., a "Schaefer" model modified by the linear, mean jump contribution. The quasi-deterministic price, from Eq. (12), is then

$$P_{QD}(t) = (p_0/H_{QD}(t) + p_1) \cdot Y_{QD}(t).$$

Consequently, the approximate instantaneous return,

$$R_{QD}(t) \equiv P_{QD}(t) \cdot H_{QD}(t) - c(E),$$

has a marginal rate of increase that decomposes into

$$\frac{dR_{QD}}{dt}(t) = p_1 \cdot q \cdot E \cdot \frac{dN_{QD}}{dt}(t) + (p_0 + p_1 \cdot q \cdot E \cdot N_{QD}(t)) \cdot \frac{dY_{QD}}{dt}(t). \tag{29}$$

Thus the approximate immediate return changes with the changes in stock level, but also with the average approximate inflationary jump rate, which will be more rapid for longer times from (27). Note that this approximate result ignores changes in the harvesting effort.

The wide separation in the curves may be accounted for by noting that starting the discounting at $s = t$ makes the Bellman equation autonomous. Since the Wiener and Poisson processes are stationary each separate curve represents the expected addition to V^* starting from the previous time. In other words y can be thought of as a restarted inflation rate at any time. Following a jump in price this increment is positive and augments the marginal increase in the expected value of current yield revenues.

Figure 5 shows the scaled optimal feedback effort $qE^*(K, y, t)/r_1$ versus $y \cdot \exp(-r_2 T)$ for time-to-go $T - t = 0$, 2, 4, 6, 8, 10. This shows that optimal effort is not very sensitive to the lumped effects of the inflation factor and the dQ_j. While the optimal current values differ by over 380% between $T - t = 2$ and $T - t = 0$ at $x = K$, the optimal effort needed to obtain these levels shows a relative difference of about -0.13% for the same values, i.e., a slight decrease. This is consistent over the full range of time-to-go values. Thus stochastic inflationary effects dramatically change optimal return while leaving effort levels relatively untouched. This is reasonable since for x near K for increasing times the rapid inflationary increase in P is offset by a rapid increase in the shadow price V_x^*. Away from the constraints this tends to keep effort levels slowly changing.

Figure 6 displays the sensitivity of the optimal current value, $V^*(K, y, 0)$, to the inflation price factor rate r_2. The curves are parameterized by the scaled inflation price factor $y \cdot \exp(-r_2 T)$ ranging from 1.0 for the topmost curve to 0.2 at the bottom in steps of 0.2. The convexity of the curves is most upward for r_2 near 0.2 and the scaled price factor near 1.0 suggesting that return is sensitive to the starting inflation rate. For y near zero the expected return will stay positive as long as the shadow price, V_x^*, is less than $(p_0 + p_1 qEx)y$, in which case we see the expected present value approximately proportional to y. For the parameter values chosen, the combined effects of the random jumps dQ_j in the price factor and the inflationary drift $r_2 \cdot Y$ overcome the effect of the occasional disaster according to the random population jumps dZ_j that might drive the population to near zero. These combined effects are more pronounced for higher inflation levels with higher variability as the inflation rate increases.

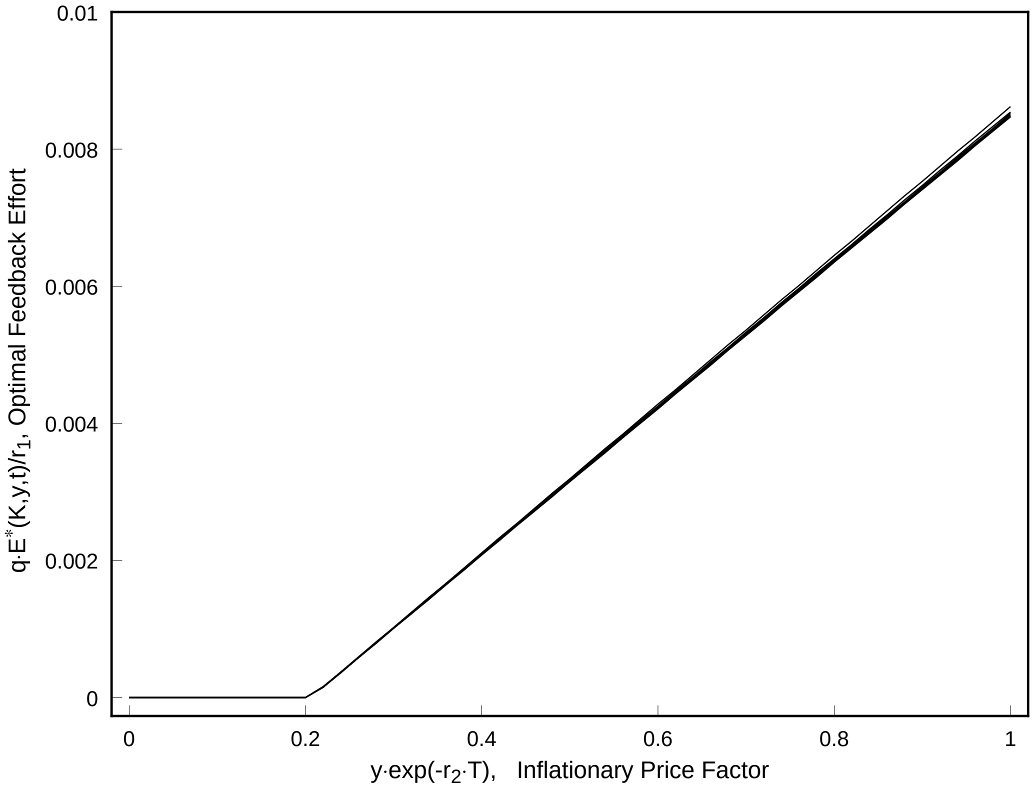

Figure 5: Optimal feedback effort, $q \cdot E^{*}/r_1(K, y, t)$, in dimensionless form versus the scaled price inflation factor, $y \cdot \exp(-r_2 \cdot T)$, with time parameter covering $t = 0.0$, 2.0, 4.0, 6.0, 8.0, 10.0 for each curve closely spaced from the bottom to top, respectively, and with population size fixed at carrying capacity $x = K$.

5. Summary

We have examined the effects of random price fluctuations on the computed optimal harvest strategy and return for a randomized Schaefer model with density independent disasters. Model population and economic parameters were taken from [19] and the 1984 and 1985 IPHC Annual Reports [17, 18].

We have found that random price fluctuations that include large inflationary increases against a background of continuous inflationary growth strongly affect optimal return but have a much less significant impact on optimal effort. Random inflationary effects, even in the presence of a hazardous environment, are therefore much more likely to play a role in determining optimal return than in scheduling effort.

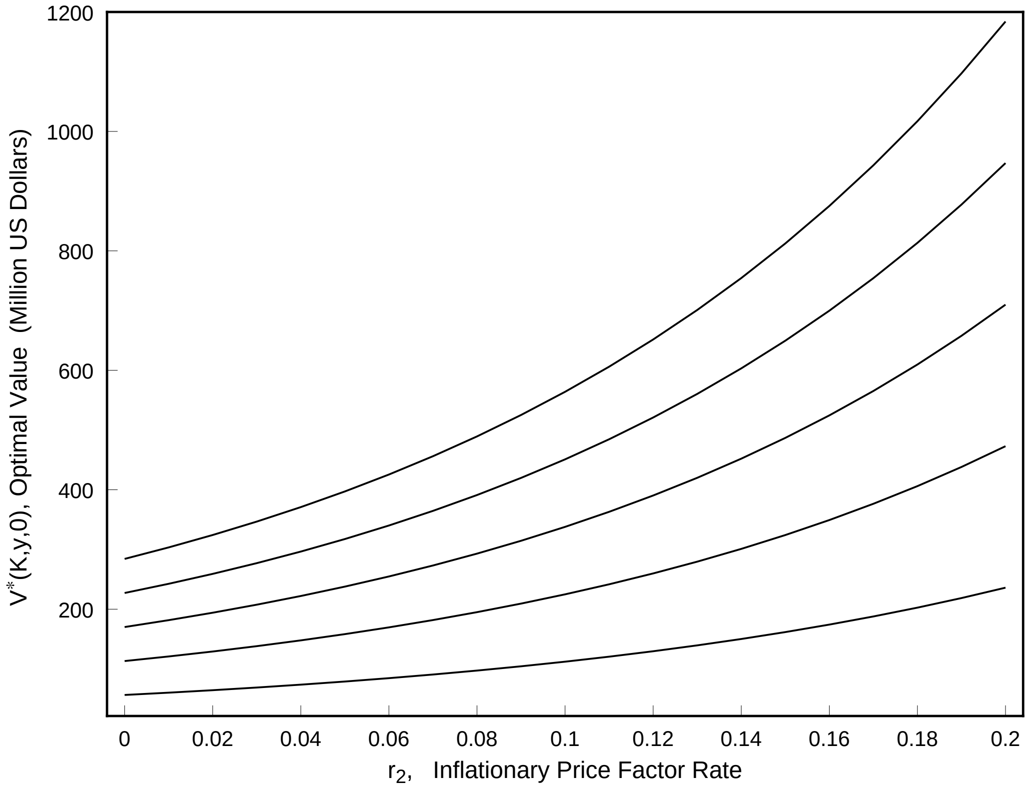

Figure 6: Sensitivity of optimal current value, $V^*(K, y, 0)$, to the inflation price factor rate r_2, with curves parameterized by the scaled inflation price factor, $y \cdot \exp(-r_2 \cdot T)$, ranging from 1.0 at the top to 0.2 at the bottom in steps of 0.2, with time fixed at the initial value $t = 0.0$, and with population size fixed carrying capacity $x = K$.

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