Stochastic-Volatility, Jump-Diffusion Optimal Portfolio Problem with Jumps in Returns and Volatility: A Computational Control Application

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Overview

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3. Portfolio Stochastic Dynamic Programming.
4. CRRA Canonical Solution to Optimal Portfolio Problem.
5. Jump-Bankruptcy Less Restrictive for Finite-Support Jump-Amplitudes.
6. Computational Considerations and Results.
7. Conclusions.

A. Histogram of S&P500 %Log-Returns 1980-2000^{-}:

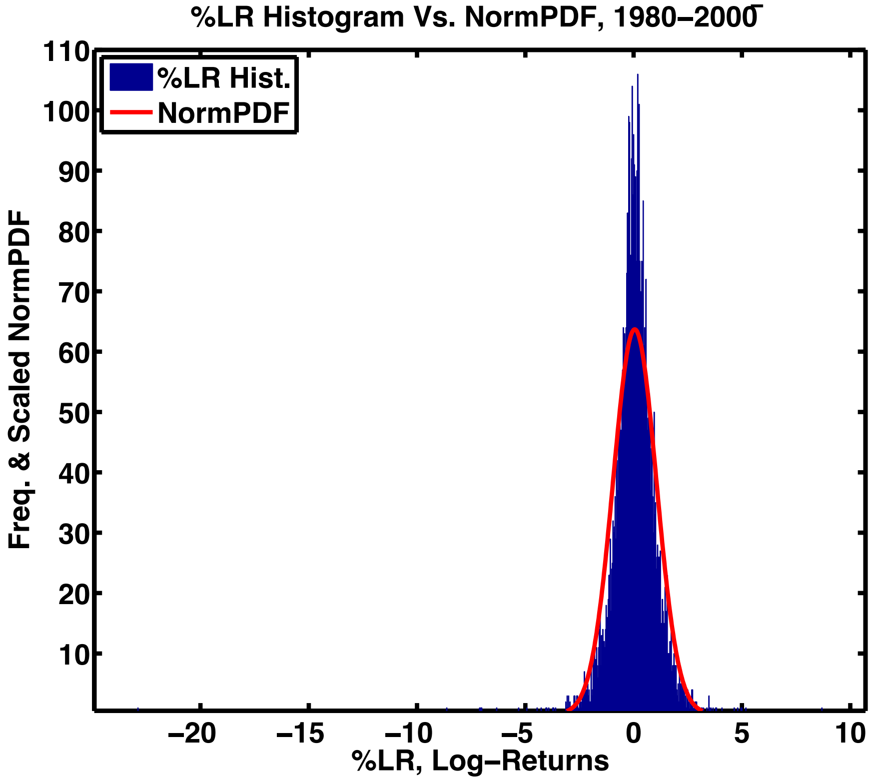
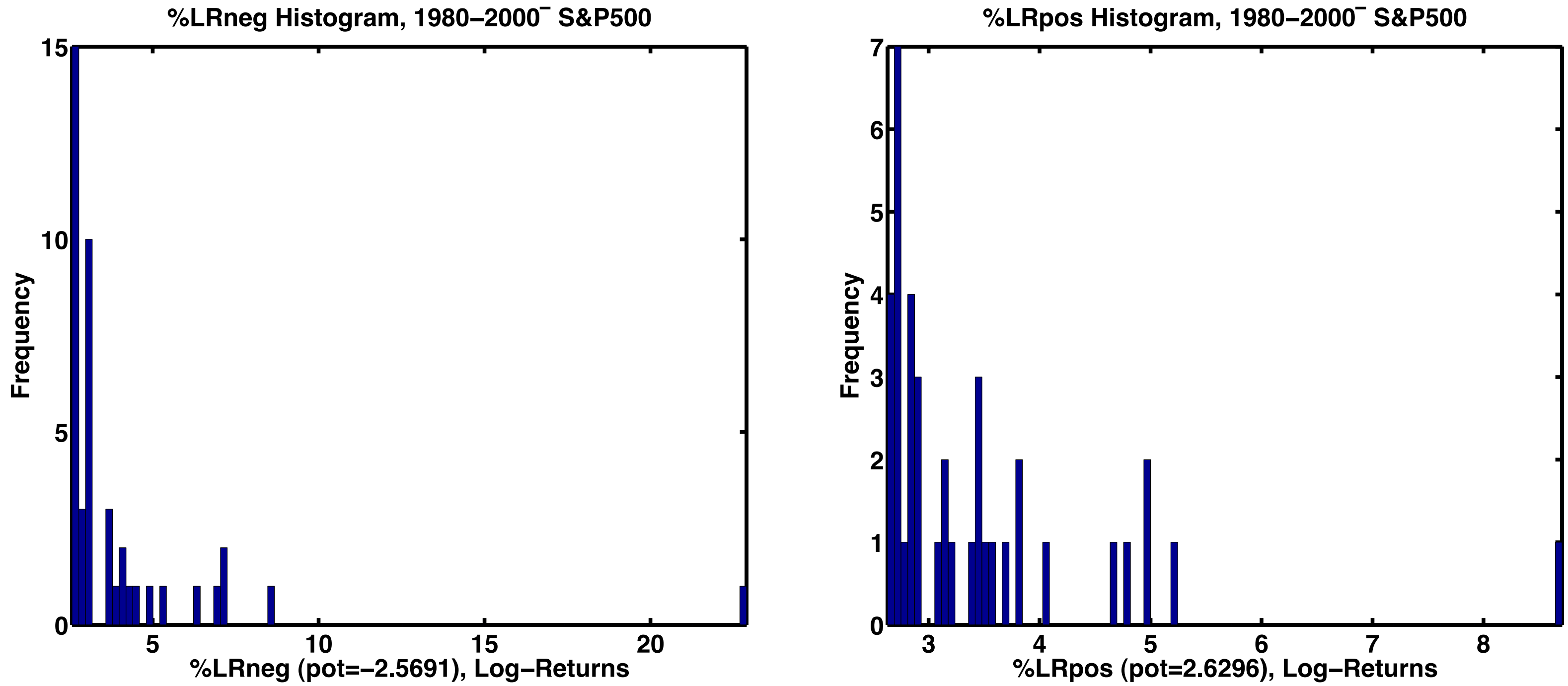


Figure 1: S&P500 Daily Log-Return Adjusted Closings from 1980 (pre-1987) to 2000 with barely visible long-tails. The **red** is a pure normal density with the same mean and variance of the data and scaled to match the data count, but ***missing extreme tails and many small returns***.

B. Extreme Tail Events for %Log-Returns (1980-2000$^-$):



(a) Extreme Negative Tails.

(b) Extreme Positive Tails.

Figure 2: Extreme Negative and Positive Log-Return Tail Events, with Peaks Over Thresholds POT $\simeq -2.6$ and $+2.6$, respectively. In normal distribution terms this means tail probability less than 5.0e-3%. These represent the ***significant crashes or bonanzas*** during the time period.

1. Introduction.

1.1 Early Background:

- ***Merton pioneered the optimal portfolio and consumption problem*** for geometric diffusions used HARA (hyperbolic absolute risk-aversion) utility in his lifetime portfolio (RES 1969) and general portfolio (JET 1971) papers. However, there were some errors, in particular with bankruptcy boundary conditions and vanishing consumption.

- The optimal portfolio errors are throughly discussed in the collection of papers of Sethi's bankruptcy book (1997). See Sethi's introduction, KLSS paper (MOR 1986) and paper with Taksar (JET 1988).

1.2 Market Jump Properties:

- ***Statistical evidence*** that jumps are significant in financial markets:
  - Stock and Option Prices in Ball and Torous (JFQA 1985);
  - Capital Asset Pricing Model in Jarrow and Rosenfeld (JB 1984);
  - Foreign Exchange and Stocks in Jorion (RFS 1989).
- Log-return market distributions usually ***skewed negative***, $\boldsymbol{\eta_3 \equiv M_3/(M_2)^{1.5} < 0}$ compared to the skew-less normal distribution, if over sufficiently long times.
- Log-return market distributions usually ***leptokurtic***, $\boldsymbol{\eta_4 \equiv M_4/(M_2)^2 > 3}$, if over sufficiently long times, i.e., more peaked than normal.
- Log-return market distribution have ***fatter or heavier tails*** than the normal distribution's exponentially small tails and also ***higher and sharper peaks***.
- ***Stochastic dependence of volatility*** is important.
- ***Time-dependence*** of rate coefficients is important, i.e., non-constant coefficients are important, such as stochastic volatility dynamics.
- ***Infinite jump domain*** is questionable for the optimal portfolio problem.

1.3 Jump-Diffusion Models:

- ***Merton (JFE 1976) in his pioneering jump-diffusion option pricing model*** used IID ***log-normally distributed jump-amplitudes*** with a compound Poisson process. Other authors have also used the normal jump-amplitude model.

- ***Kou*** (Mgt.Sci. 2002, and 2004 with Wang) used the ***IID log-double-exponential*** for option pricing.

- ***Hanson, Westman and Zhu*** (2001-2006) have a number of optimal portfolio papers using various log-return jump-amplitude distributions such as ***IID log-discrete, normal, uniform and double-uniform distributions***.

- ***And many more.***

- ***Return Jump-diffusions give skewness and excess-leptokurtosis*** to market distributions, and ***so do latent stochastic volatility jump-diffusions***.

1.5 Stochastic Jump and Volatility Considerations:

- ***Extreme jumps*** in the market are ***relatively rare (statistical outliers)*** among the large number of daily fluctuations.
- ***NYSE have had circuit breakers installed*** since 1988 to suppress extreme market changes, particularly the ***1987 crash***, with many recent updates due to the ***2007-2009 great recession***, ***May 6, 2010 flash crash*** and ***Dodd-Frank regulation changes***.
- ***Finite range jump distributions*** are consistent with circuit breakers.
- ***Bankruptcy conditions*** also need to be considered for the ***global dependence of jump-integrals of the SVJD PIDE*** as we shall see for the optimal portfolio problem; unlike the ***local dependence option pricing problem***.
- Andersen, Benzoni and Lund (JF 2002) showed that ***both stochastic jump and volatility models are needed to explain equity returns***.
- Eraker, Johannes and Polson (JF 2003) show that ***jumps in volatility are also needed***.

2. Optimal Portfolio Problem and Underlying Stochastic-Volatility, Jump-Diffusion (SVJD) Return Model.

2.1 Stock Price Linear Stochastic Differential Equation (SDE):

$$dS(t) = S(t)\Big(\mu_s(V(t))dt + \sqrt{V}(t)dG_s(t) + Q_s dP_s(t)\Big), \qquad (1)$$

where

- $S(t)$ = ***stock price***, $S(0) = S_0 > 0$;
- $\mu_s(V(t))$ = ***expected rate of return*** in absence of asset jumps;
- $G_s(t)$ = ***stock price diffusion process***, normally distributed such that $\mathrm{E}[dG_s(t)] = 0$ and $\mathrm{Var}[dG_s(t)] = dt$;
- $V(t)$ = ***stochastic variance*** = (stochastic volatility)$^2 = \sigma_s^2(t)$;
- $P_s(t)$ = ***Poisson jump counting process***, Poisson distributed such that $\mathrm{E}[dP_s(t)] = \lambda_s dt = \mathrm{Var}[dP_s(t)]$;

2.1 Continued: Stock Price Dynamics:

- $\boldsymbol{Q_s}$ = ***Poisson jump-amplitude*** underlying ***IID random mark variable***, selected for log-return so that $\boldsymbol{Q_y \equiv Q_{\ln(s)} = \ln(1+Q_s)}$, such that $\boldsymbol{Q_s > -1}$, preserving positive returns;

- Definition of abbreviated compound Poisson jump term:

$$S(t)Q_s dP_s(t) \equiv \sum_{k=P_s(t)+1}^{(P_s+dP_s)(t)} S(T_k^-)Q_{s,k};$$

(Note: $\sum_{k=P_s+1}^{P_s} A_k \equiv 0$, i.e., when there is no jump, $dP_s = 0$.)

- $\boldsymbol{T_k^-}$ is the ***pre-jump time*** and $\boldsymbol{Q_{s,k}}$ is an independent and identically distributed ***(IID) mark*** realization at the kth jump;

- The ***processes*** $\boldsymbol{G_s(t)}$***,*** $\boldsymbol{P_s(t) = P_s(t; Q_s)}$***,*** $\boldsymbol{Q_s}$ ***are independent***, except that $Q_s = Q_{s,k}$ is conditioned on a jump-event at T_k^-.

2.2 Log-Truncated Double-Exponential (TDE) Probability Jump-Amplitude Q_y Mark Distribution:

$$\Phi^{(\mathrm{tde})}_{Q_y}(q) = p_1 \frac{e^{-(q-\mu_{j,y})/\mu_1} - e_1}{|\mu_1|(1-e_1)} I_{\{a \le q \le \mu_{j,y}\}} + \left(p_1 + p_2 \frac{1-e^{-(q-\mu_{j,y})/\mu_2}}{\mu_2(1-e_2)}\right) I_{\{\mu_{j,y} \le q \le b\}} + I_{\{b, \le q < \infty\}},$$

where $a < \mu_{j,y} < b$, $0 \le p_2 = 1 - p_1 \le 1$, $e_1 \equiv e^{-(a-\mu_{j,y})/\mu_1}$ and $e_2 \equiv e^{-(b-\mu_{j,y})/\mu_2}$.

- ***Mark Mean:*** $\mu^{(\mathrm{tde})} \equiv \mathrm{E}^{(\mathrm{tde})}_{Q_y}[Q_y] = p_1 \mu^{(\mathrm{tde})}_1 + p_2 \mu^{(\mathrm{tde})}_2 \equiv p_1(\mu_{j,y} + \mu_1 - (a - \mu_{j,y})e_1/(1-e_1)) + p_2(\mu_{j,y} + \mu_2 - (b - \mu_{j,y})e_2/(1-e_2))$;
- ***Mark Variance:*** $\left(\sigma^{(\mathrm{tde})}\right)^2 \equiv \mathrm{Var}^{(\mathrm{tde})}_{Q_y}[Q] = p_1 \left(\sigma^{(\mathrm{tde})}_1\right)^2 + p_2 \left(\sigma^{(\mathrm{tde})}_2\right)^2$.
- ***Mark motivation:*** The truncated double-exponential distribution unlinks the different behaviors in crashes and rallies.

2.3 Stochastic-Volatility with Jumps Model

(Eraker, Johannes and Polson, JF 2003 (jump-Heston model):

$$dV(t) = \kappa_v(\theta_v - V(t))\,dt + \sigma_v\sqrt{V(t)}dG_v(t) + Q_v dP_v(t), \quad (2)$$

with

- $V(t) \geq \min(V(t)) > 0^+$, $V(0) = V_0 \geq \min(V(t)) > 0^+$;
- ***Log-rate*** $\kappa_v > 0$; ***mean-reversion level*** $\theta_v > 0$; ***volatility of volatility (variance)*** $\sigma_v > 0$;
- $G_v(t) =$ ***variance diffusion process***, normally distributed such that $\mathrm{E}[dG_v(t)] = 0$ and $\mathrm{Var}[dG_v(t)] = dt$, with ***correlation*** $\mathrm{Corr}[dG_s(t), dG_v(t)] = \rho dt$;
- Note: ***SDE (2) is singular for transformations*** as $V(t) \to 0^+$ due to the square root, unlike SDE (1) for $S(t)$ where the singularity is removable through the log transformation, but Itô-Taylor chain rule or simulation applications might not be valid unless

$$\Delta t \ll \sqrt{\epsilon_v} \ll 1, \quad \epsilon_v = \min(V(t)) > 0.$$

2.4 Stochastic Volatility Compound Poisson Process:

- $dP_v(t) =$ ***stochastic volatility Poisson counting process***, with joint mean and variance, $\mathrm{E}[dP_v(t)] = \lambda_v dt = \mathrm{Var}[dP_v(t)]$ with $\lambda_v > 0$. This completes the ***SVIJ model*** of Eraker, Johannes and Polson (2003), the stochastic volatility model with independent jumps, assuming that generally that $\lambda_v \neq \lambda_s$.
- ***Stochastic Volatility Jump-Amplitude Exponential Distribution:***

$$\Phi_{Q_v}(q) = \left(1 - e^{-q/\mu_{j,v}}\right) I_{\{0 \le q < +\infty\}},$$

where $\mathrm{E}_{Q_v}[Q] = \mu_{j,v} > 0$ and $\mathrm{Var}_{Q_v}[Q] = \mu_{j,v}^2$.

- ***Mark motivation:*** While the range of jumps in the asset returns can have a significant effect on the range of the asset stock fraction in the Merton-type optimal portfolio problem, the ***jumps in the stochastic volatility do NOT restrict the range stock fraction***, so the semi-infinite range of the jumps of the stochastic volatility is a useful approximation.

2.5 *Log-Return Stock Model:*

- ***Log-Transform of Singular Diffusion to Regular Diffusion:*** Let $Y(t) = \ln(S(t))$, then the removable singularity of the stock geometric Brownian diffusion with linear jumps is transformed to a regular jump-diffusion, as usual.
- ***Log-Return Jump-Diffusion (The Usual Econometric Model):***

$$dY(t) = \mu_y dt + \sqrt{V(t)}dG_s(t) + Q_y dP_s(t). \qquad (3)$$

- ***Transformed Drift:*** $\mu_y = \mu_s(V(t)) - V(t)/2$, following Eraker, Johannes and Polson (2003) whose estimations implied that the log-return drift $\widehat{\mu}_y \simeq$ constant.
- ***Transformed Jump-Amplitude Mark:*** $Q_y = \ln(1+Q_s)$, with $a \leq Q_y \leq b$ for bounded log-return jumps.

2.6 Wealth Portfolio with Bond, Stock and Consumption:

- ***Portfolio:*** Riskless asset or *bond* at price $B(t)$ and Risky asset or *stock* at price $S(t)$ (1), with ***instantaneous*** portfolio change fractions $U_b(t)$ and $U_s(t)$, respectively, such that $U_b(t) = 1 - U_s(t)$.
- ***Exponential Bond Price Process:***

$$dB(t) = r(t)B(t)dt\ , \quad B(0) = B_0\ . \tag{4}$$

- ***SVJD Portfolio Wealth Process $W(t)$, Less Consumption $C(t)$ with Self-Financing:***

$$\begin{aligned} dW(t) = W(t)\Big(r(t)dt + U_s(t)\Big(&(\mu_y + V(t)/2 - r(t))dt \\ &+ \sqrt{V(t)}dG_s(t) + \left(e^{Q_y} - 1\right)dP_s(t)\Big)\Big) - C(t)dt\ , \end{aligned} \tag{5}$$

subject to constraints $W(0) = W_0 > 0$, $W(t) > 0$, $v = V(t) > 0$, $0 < C(t) \le C_0^{(\max)}W(t)$ and $U_0^{(\min)} \le U_s(t) \le U_0^{(\max)}$, while allowing extra shortselling $(U_s(t) < 0)$ and extra borrowing $(U_b(t) < 0)$.

2.7 Portfolio Optimal Objective (The Control Application):

$$J^*(w,v,t) = \max_{\{u,c\}}\Big[\mathrm{E}\Big[e^{-\overline{\beta}(t,t_f)}\mathcal{U}_w(W(t_f)) + \int_t^{t_f} e^{-\overline{\beta}(t,\tau)}\mathcal{U}_c(C(s))\,d\tau \Big| W(t)=w, V(t)=v, U_s(t)=u, C(t)=c\Big]\Big]. \quad (6)$$

where

- ***Cumulative Discount:*** $\overline{\beta}(t,s) = \int_t^s \beta(\tau)d\tau$, where $\beta(t)$ is the instantaneous discount rate.
- ***Consumption and Final Wealth Utility Functions:*** $\mathcal{U}_c(c)$ and $\mathcal{U}_w(w)$ are bounded, strictly increasing and concave.
- ***Variable Classes:*** State variables are w and v, while control variables are u and c.
- ***Final Condition:*** $J^*(w,v,t_f) = \mathcal{U}_w(w)$.

2.8 Absorbing Natural Boundary Condition:

Approaching bankruptcy as $w \to 0^+$, then by the consumption constraint as $c \to 0^+$ and by the objective (6),

$$J^*(0^+, v, t) = \mathcal{U}_w(0^+)\, e^{-\overline{\beta}(t,t_f)} + \mathcal{U}_c(0^+) \int_t^{t_f} e^{-\overline{\beta}(t,s)} ds. \qquad (7)$$

- This is the simple variant of what Merton gave as a correction in his 1990 book for his 1971 optimal portfolio paper.
- However, KLASS 1986 and Sethi with Taksar 1988 pointed out that it was necessary to enforce the non-negativity of wealth and consumption.

3. Portfolio Stochastic Dynamic Programming .

3.1 Portfolio Stochastic Dynamic Programming PIDE:

$$
\begin{aligned}
0 = \; & J_t^*(w,v,t) - \beta(t) J^*(w,v,t) + \mathcal{U}_c(c^*) \\
& + \left(r(t) + (\mu_y + v/2 - r(t)) u^*\right) w J_w^*(w,v,t) \\
& - c^* J_w^*(w,v,t) + \tfrac{1}{2} v (u^*)^2 w^2 J_{ww}^*(w,v,t) + \kappa_v(\theta_v - v) J_v^*(w,v,t) \\
& + \tfrac{1}{2}\sigma_v^2 v J_{vv}^*(w,v,t) + \rho\sigma_v v u^* w J_{wv}^*(w,v,t) \\
& + \lambda_s \int_a^b \phi_y^{(\mathrm{tde})}(q) \Big(J^*(\underline{K(u,q)w}, v, t) - J^*(w,v,t) \Big)\, dq \\
& + \lambda_v \int_0^\infty \phi_v(q) \left(J^*(w, (1+q)v, t) - J^*(w,v,t) \right) dq,
\end{aligned}
\tag{8}
$$

where $u^* = u^*(w,v,t) \in \left[U_0^{(\min)}, U_0^{(\max)}\right]$ and $c^* = c^*(w,v,t) \in \left[0, C_0^{(\max)} w\right]$ are the optimal controls if they exist, while J_w^*, J_v^*, J_{ww}^*, J_{wv}^* and J_{vv}^* are the continuous partial derivatives with respect to wealth w and/or v when $0 \le t < t_f$. Note that $\underline{K(u,q)w} = \underline{(1 + (e^q - 1)u^*(w,v,t))w}$ is a wealth argument.

3.2 Positivity of Wealth with the Stock Jump Distribution:

Since $K(u,q)w = (1+(e^{q}-1)u^{*}(w,v,t))w$ is a wealth argument in (8), it must satisfy the wealth positivity condition, so

$$K(u,q) \equiv 1 + (e^{q} - 1)u > 0$$

on the support $[a,b]$ of the jump-amplitude density $\phi_{Q_y}(q)$.

Lemma 3.2 ***Bounds on Optimal Stock Fraction due to Positivity of Wealth Jump Argument:***

(a) If the support of $\phi_{Q_y}(q)$ is the ***finite*** interval $q \in [a,b]$ with $a < \mu_{j,y} < b$ and $a < 0 < b$, then $u^{*}(w,v,t)$ is restricted by (8) to

$$\frac{-1}{e^{b}-1} < u^{*}(w,v,t) < \frac{1}{1-e^{a}}, \tag{9}$$

where $Q_s = e^{Q_y} - 1$ is the stock jump-amplitude relative to price.

(b) If the support of $\phi_{Q_y}(q)$ is fully ***infinite***, i.e., $(-\infty, +\infty)$, then $u^{*}(w,v,t)$ is restricted by (8) to

$$0 < u^{*}(w,v,t) < 1. \tag{10}$$

3.2 Remarks: Non-Negativity of Wealth and Jump Distribution:

- Recall that u is the stock fraction, so that ***short-selling and borrowing will be overly restricted in the infinite support case (10)*** where $a \to -\infty$ and $b \to +\infty$, ***unlike the finite case (9)*** where $-\infty < a < 0 < b < +\infty$.
- So, unlike option pricing, ***finite support of the mark density makes a big difference*** in the optimal portfolio and consumption problem!
- Thus, it would ***not be practical to use either normally or double-exponentially distributed marks in the optimal portfolio and consumption problem*** with a bankruptcy condition.
- For TDE parameters, $[a,b] = [-0.08396, +0.02226]$, then the overall u^* range for the S&P500 data used is

$$[u_{\min}, u_{\max}] = [-44.43, +12.42] \subseteq \left(\frac{-1}{(e^b - 1)}, \frac{+1}{(1 - e^a)}\right).$$

3.3 Unconstrained Optimal or Regular Control Policies:

In absence of control constraints and in presence of sufficient differentiability, the dual policy, implicit critical conditions are

- ***Regular Consumption*** $c^{(\mathrm{reg})}(w,v,t)$ ***{Implicitly}:***

$$\mathcal{U}'_c(c^{(\mathrm{reg})}(w,v,t)) = J^*_w(w,v,t). \tag{11}$$

- ***Regular Portfolio Fraction*** $u^{(\mathrm{reg})}(w,v,t)$ ***{Implicitly}:***

$$\begin{aligned} vw^2 J^*_{ww}(w,v,t)u^{(\mathrm{reg})}(w,v,t) = &-(\mu_y + v/2 - r(t))wJ^*_w(w,v,t) \\ &-\rho\sigma_v vwJ^*_{wv}(w,v,t) \\ &-\lambda(t)w\int_a^b \phi_{Q_y}(q)(e^q - 1)J^*_w\left(K\left(u^{(\mathrm{reg})}(w,v,t),q\right)w,v,t\right)dq. \end{aligned} \tag{12}$$

4. CRRA Canonical Solution to Optimal Portfolio Problem.

4.1 CRRA Utilities:

- ***Constant Relative Risk-Aversion (CRRA $\subset$ HARA) Power Utilities:***

$$\mathcal{U}_c(x) = \mathcal{U}(x) = \mathcal{U}_w(x) = \left\{ \begin{array}{ll} x^\gamma/\gamma, & \gamma \neq 0 \\ \ln(x), & \gamma = 0 \end{array} \right\}, \quad x \geq 0, \gamma < 1. \tag{13}$$

- ***$\Longleftarrow$ Relative Risk-Aversion (RRA):***

$$RRA(x) \equiv -\mathcal{U}''(x)/(\mathcal{U}'(x)/x) = (1-\gamma) > 0, \quad \gamma < 1,$$

i.e., negative of ratio of marginal to average change in marginal utilility ($\mathcal{U}'(x) > 0$ & $\mathcal{U}''(x) < 0$) is a constant.

- ***CRRA Canonical Separation of Variables:***

$$J^*(w,v,t) = \mathcal{U}(w)J_0(v,t), \quad J_0(v,t_f) = 1, \tag{14}$$

i.e., if valid, then wealth state dependence is known and only the variance-time dependent factor $J_0(v,t)$ need be determined.

4.2 Canonical Simplifications with CRRA Utilities:

- ***Regular Consumption Control is Linear in Wealth:***

$$c^{(\mathrm{reg})}(w,v,t) = w \cdot c_0^{(\mathrm{reg})}(v,t) \equiv w/J_0^{1/(1-\gamma)}(v,t), \quad (15)$$

where $c_0^{(\mathrm{reg})}(v,t)$ is a wealth fraction, with optimal consumption

$$c_0^*(v,t) = \max\left[\min\left[c_0^{(\mathrm{reg})}(v,t), C_0^{(\max)}\right], 0\right]$$

per w.

- ***Regular Portfolio Fraction Control is Independent of Wealth:***

$$\begin{aligned} u^{(\mathrm{reg})}(w,v,t) \equiv\ & u_0^{(\mathrm{reg})}(v,t) \\ =\ & \frac{1}{(1-\gamma)v}\Big(\mu_y + v/2 - r(t) + \rho\sigma_v v(J_{0,v}/J_0)(v,t) \quad (16) \\ & + \lambda_s I_1\left(u_0^{(\mathrm{reg})}(v,t)\right)\Big), \end{aligned}$$

in fixed point form, where

$$u^* = u_0^*(v,t) = \max\left[\min\left[u_0^{(\mathrm{reg})}(v,t), U_0^{(\max)}\right], U_0^{(\max)}\right],$$

and $\quad I_1(u) \equiv \int_a^b \phi_{Q_y}(q)(e^q - 1)K^{\gamma-1}(u,q)dq.$

4.3 CRRA Time-Variance Dependent Component in Formal "Bernoulli" PIDE ($\gamma \neq 0; \gamma < 1$):

$$\mathbf{0} = J_{0,t}(v,t) + (1-\gamma)\left(g_1(v,t)J_0(v,t) + g_2(v,t)J_0^{\frac{\gamma}{\gamma-1}}(v,t)\right) + g_3(v,t)J_{0,v} + \tfrac{1}{2}\sigma_v^2 v J_{0,vv}, \quad (17)$$

where

- ***Bernoulli Coefficients*** $g_1(v,t)$, $g_2(v,t)$, ***and*** $g_3(v,t)$:
  $g_1(v,t) = g_1(v,t;u_0^*(v,t))$, $g_2(v,t) = g_2\left(v,t;c_0^*(v,t),c_0^{(\mathrm{reg})}(v,t)\right)$, and $g_3(v,t) = g_3(v,t;u_0^*(v,t))$, introduce implicit nonlinear dependence on $u_0^*(v,t)$, $c_0^*(v,t)$ and $c_0^{(\mathrm{reg})}(v,t)$, so iterations are required.
- ***Formal (Implicit) Solution using Bernoulli transformation,*** $x(v,t) = J_0^{1/(1-\gamma)}(v,t)$ ***, improving interations:***

$$\mathbf{0} = x_t(v,t) + g_1(v,t)x(v,t) + g_4(v,t), \quad x(v,t_f) = 1,$$

$$J_0(v,t) = \left[e^{\overline{g}_1(v,t,tf)} + \int_t^{t_f} g_4(v,\tau)e^{\overline{g}_1(v,t,\tau)}d\tau\right]^{1-\gamma}, \quad (18)$$

4.3 Continued, Coefficient Functions for Reference Only:

where the coefficient function are

$$g_1(v,t) \equiv \frac{1}{1-\gamma}\left(-\beta(t)+\gamma\left(r(t)+(\mu_y+v/2-r(t))u_0^*(v,t)\right)\right.$$
$$-0.5(1-\gamma)v(u_0^*)^2(v,t)+\lambda_s(I_2(u_0^*(v,t))-1)$$
$$\left.+\lambda_v((I_3[J_0]/J_0)(v,t)-1)\right),$$

$$\overline{g}_1(v,t,\tau) \equiv \int_t^\tau g_1(v,s)ds.$$

$$I_2(u) \equiv \int_a^b \phi_{Q_y}(q)K^\gamma(u,q)dq, \quad I_3[J_0](v,t)) \equiv \int_0^\infty \phi_{Q_v}(q)J_0((1+q)v,t)dq,$$

$$g_2 \equiv \frac{1}{1-\gamma}\left(\left(\frac{c_0^*(v,t)}{c_0^{(\mathrm{reg})}(v,t)}\right)^\gamma - \gamma\left(\frac{c_0^*(v,t)}{c_0^{(\mathrm{reg})}(v,t)}\right)\right),$$

$$g_3(v,t) = +\kappa_v(\theta_v-v)+\gamma\rho\sigma_v v u_0^*(v,t),$$

$$g_4(v,t) = g_2(v,t)+g_3(v,t)x_v(v,t)+\frac{1}{2}\sigma_v^2(t)v\left(x_{vv}-\gamma((x_v)^2/y)\right)(v,t).$$

4.4 CRRA Time-Variance Dependent Component in Formal "Bernoulli" PDE ($\gamma = 0$; Kelly Criterion) :

In this medium risk-averse case of the logarithmic CRRA utility, the formal, implicit canonical solution has two terms,

$$\boldsymbol{J^*(w,v,t) = \ln(w)J_0(v,t) + J_1(v,t),} \tag{19}$$

with final boundary conditions $J_0(v,t) = 1$ and $J_1(v,t) = 0$, The regular controls satisfy,

$$\begin{aligned} c^{(\mathrm{reg})}(w,v,t) = & \; wc_0^{(\mathrm{reg})}(v,t) \equiv w/J_0(v,t), \\ u^{(\mathrm{reg})}(w,v,t) = & \; u_0^{(\mathrm{reg})}(v,t) \equiv \frac{1}{v}\Big(\mu_y + v/2 - r(t) + \rho\sigma_v (J_{0,v}/J_0)(v,t) \\ & +\lambda_s I_1\Big(u_0^{(\mathrm{reg})}(v,t)\Big)\Big). \end{aligned}$$

4.4 $\gamma = 0$ case continued:

The $\ln(w)$ and w-independent coefficients satisfies the implicit, uni-directionally-coupled PIDEs,

$$0 = J_{0,t}(v,t) - \beta(t)J_0(v,t) + g_0(v,t),$$

$$0 = J_{1,t}(v,t) - \beta(t)J_1(v,t) + \widetilde{g}_2(v,t),$$

with formal solutions

$$J_0(v,t) = e^{-\overline{\beta}(t;t_f)} + \int_t^{t_f} e^{-\overline{\beta}(t;\tau)} g_0(v,\tau)d\tau,$$

$$J_1(v,t) = \int_t^{t_f} e^{-\overline{\beta}(t;\tau)} \widetilde{g}_2(v,\tau)d\tau,$$

where

$$g_0(v,t) \equiv 1 + \kappa_v(\theta_v - v)J_{0,v}(v,t) + \tfrac{1}{2}\sigma_v^2 v J_{0,vv}(v,t),$$

$$\begin{aligned}\widetilde{g}_2(v,t) \equiv & -\ln(J_0(v,t)) - 1 + \Big(r(t) + (\mu_y + v/2 - r(t))u_0^*(v,t) \\ & -0.5v(u_0^*)^2(v,t) + \lambda_s I_2^{(0)}(u_0^*(v,t))\Big) J_0(v,t) \\ & +\kappa_v(\theta_v - v)J_{1,v}(v,t) + 0.5\sigma_v^2 v J_{1,vv}(v,t),\end{aligned}$$

and where $I_2^{(0)}$ has $\ln(K)$ replacing K^γ in I_2.

5. Computational Considerations and Results.
5.1 Parameter Data

- Due to the complexity of the data, the elaborate parameter estimates of Eraker, Johannes and Polson (2003) for their SVIJ model from the S&P 500 index returns from the beginning of 1980 to the end of 1999, including the extreme market stresses in 1987, 1997 and 1998. Their methods of estimation include Bayesian oriented Markov chain Monte Carlo simulations.
- In our notation, the original parameter estimates of Eraker, Johannes and Polson (2003) are given in Table 1 along with converted estimate in units appropriate for the PIDE formulation (i.e., annualized and non-percentage units):

Table 1: ***SVIJ Parameter Estimates***

Estimated Parameter	SVIJ Original	SVIJ Converted	Scale Factors*
μ_y	0.0506	0.1275	$\times 252/100$
κ_v	0.0250	6.3000	$\times 252$
θ_v	0.5585	0.01407	$\times 252/100^2$
σ_v	0.0896	0.2258	$\times 252/100$
ρ	-0.5040	-0.5040	$\times 1$
λ_y	0.0046	1.1592	$\times 252$
μ_{jy}	-3.0851	-0.030851	$\times 1/100$
σ_{jy}	2.9890	0.02989	$\times 1/100$
λ_v	0.0055	1.3860	$\times 252$
μ_{jv}	1.7980	0.04531	$\times 252/100^2$
σ_{jv}	1.7980	0.04531	(exponential dist.)
ρ_{jvy}	0	0	(independent jumps)

- The standard 252 trading days was used to convert daily units to annual units, while division by 100 cancels percentage scaling. The conversion factors follow from the comments on a few key parameters in Eraker, Johannes and Polson (2003) and preserving dimensional consistency with the driving SDEs, (1) and (2).
- Since the log-return jump-amplitude distributions are different from those of Eraker, Johannes and Polson (2003), it is necessary to convert the log-return normal jump-amplitude basic moments to the basic moments of the truncated double-exponential distribution here to take advantage of Eraker, Johannes and Polson's very large scale estimation, assuming that a consistent matching of half-range and full range moments will be suitable for our purpose. The computational results for the TDE parameters to three significant figures were

$$p_1 = 0.5,\ \mu_1 \simeq -0.0863,\ \mu_2 \simeq +0.0863,\ a \simeq -0.0840,\ b \simeq +0.0223.$$

Interest Primes Rates $r(t)$ and Discount Rates $\beta(t)$, 1980-2000^{-} (Federal Reserve Statistical Release H.15):

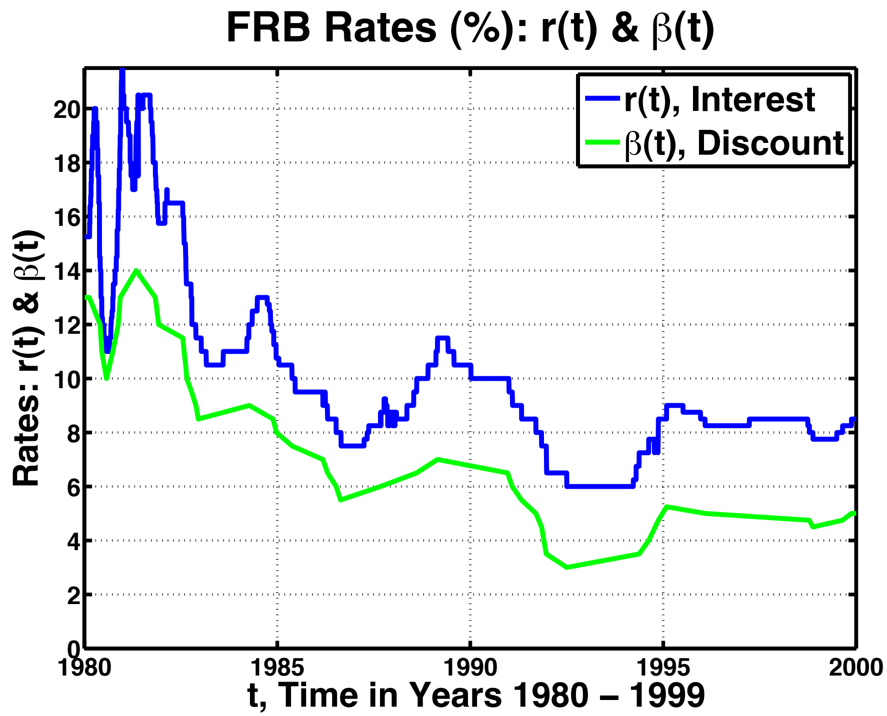
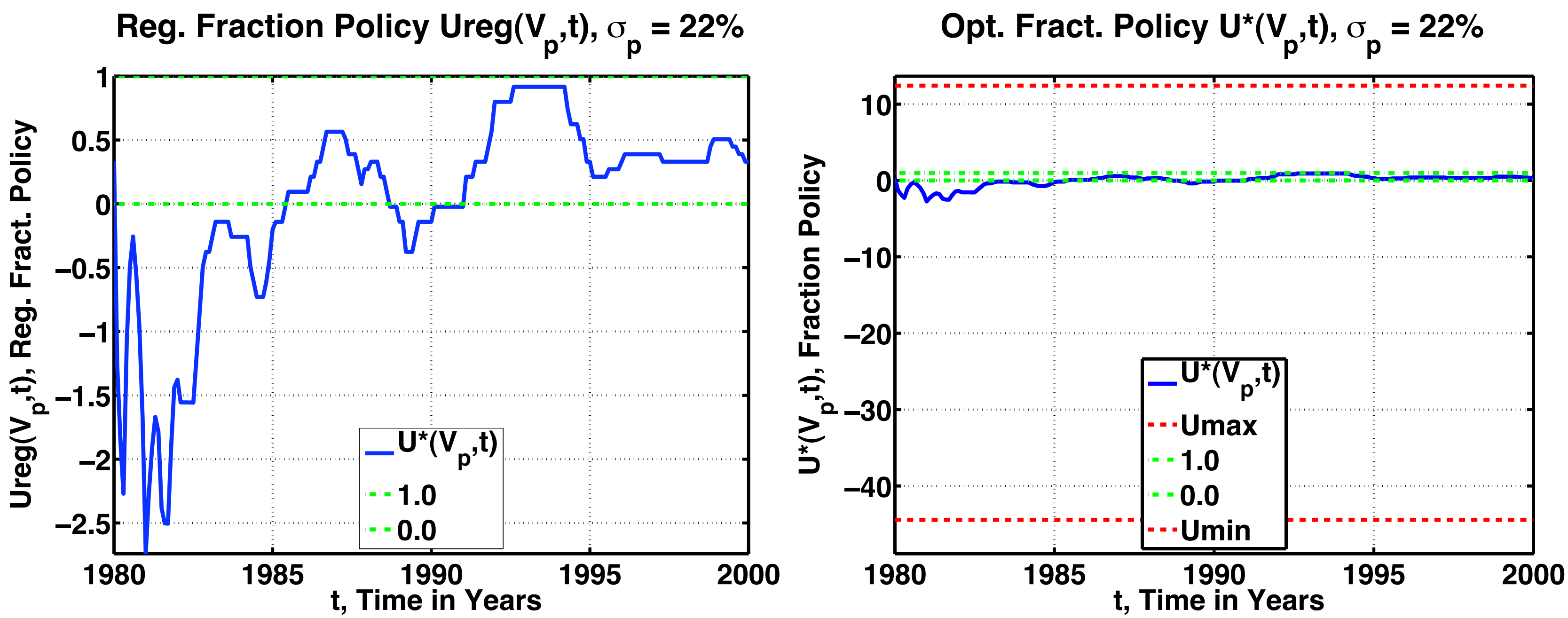


Figure 3: FRB prime rates $r(t)$ and discount rates $\beta(t)$ for $t \in [1980, 2000^{-}]$.

5.2 Computational Considerations:

- The primary problem is having stable computations and much smaller time-steps Δt are needed compared to variance-steps ΔV, since the computations are ***drift-dominated*** over the diffusion coefficient, in that the mesh coefficient associated with $J_{0,v}$ can be hundreds times larger than that associated with $J_{0,vv}$ for the variance-diffusion.
- ***Drift-upwinding*** is needed so the finite differences for the drift-partial derivatives follow the sign of the drift-coefficient, while central differences are sufficient for the diffusion partials.
- ***Iteration calculations in time, controls and volatility*** are sensitive to small and negative deviations, as well as the form of the iteration in terms of the formal implicitly-defined solutions.

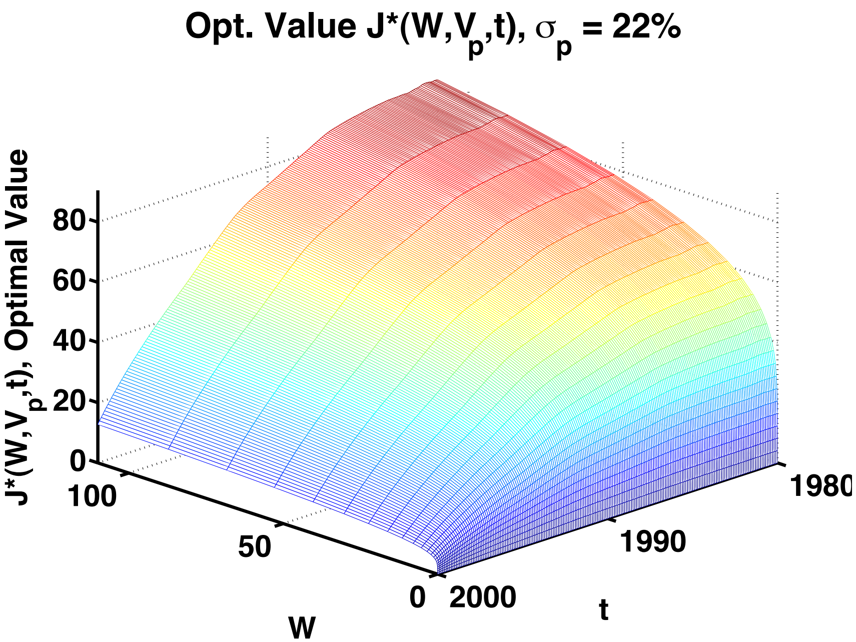
5.3 Results for Regular $u^{(\text{reg})}(v_p, t)$ and Optimal $u^*(v_p, t)$ Portfolio Fraction Policies, $\sigma_p = \sqrt{v_p} = 22\%$:



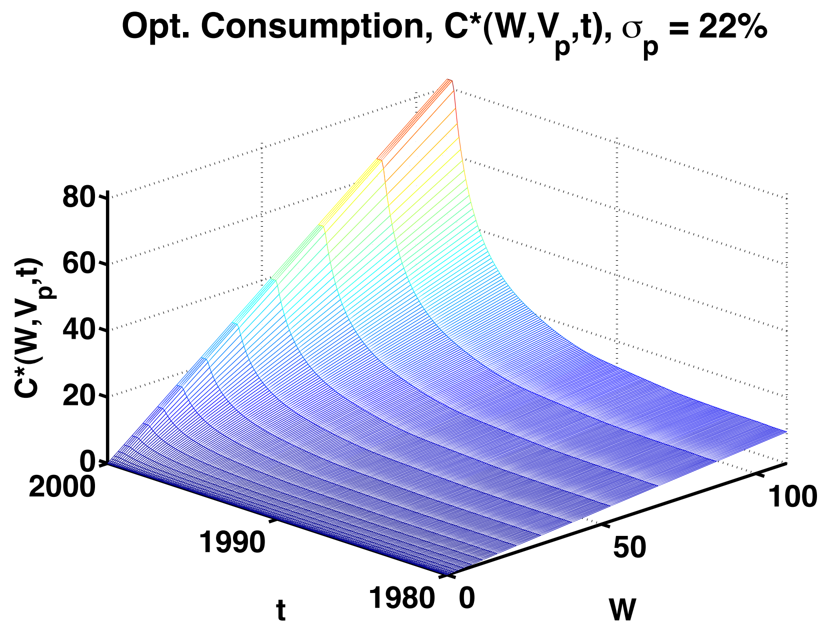
(a) Regular fraction policy $u^{(\text{reg})}(v_p, t)$. (b) Optimal fraction policy, $u^*(v_p, t)$.

Figure 4: Regular and optimal portfolio stock fraction policies, $u^{(\text{reg})}(v_p, t)$ and $u^*(v_p, t)$ at $\sigma_p = \sqrt{v_p} = 0.22 = 22\%$ on $t \in [1980, 2000)$, while $u^*(v_p, t) \in [-44.4, 12.4]$. Jump-bankruptcy bounds are included only in (b).

5.3 Results for Optimal Value $J^*(w, v_p, t)$ and Optimal Consumption $c^*(w, v_p, t)$, $\sigma_p = \sqrt{v_p} = 22\%$:



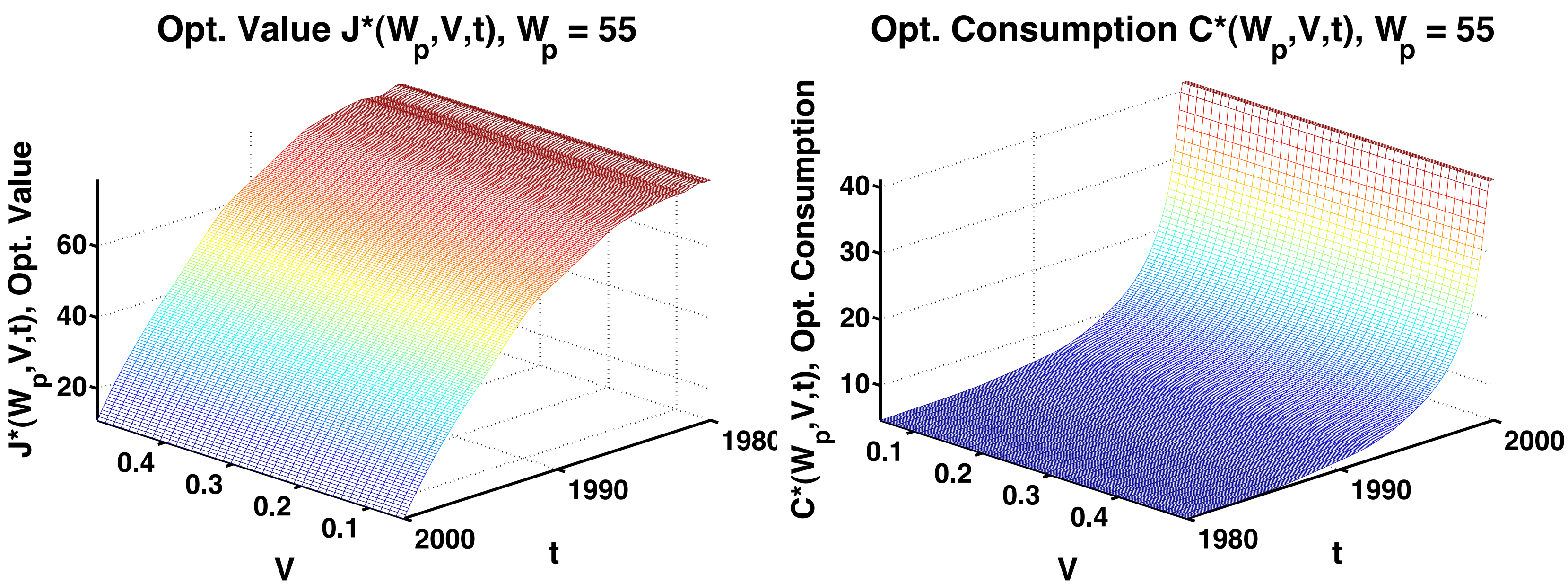
(a) Optimal portfolio value $J^*(w, v_p, t)$.



(b) Optimal consumption policy $c^*(w, v_p, t)$.

Figure 5: Optimal portfolio value $J^*(w, v_p, t)$ and optimal consumption policy $c^*(w, v_p, t)$ at $\sigma_p = \sqrt{v_p} = 0.22 = 22\%$ on $(w, t) \in [0, 110] \times [1980, 2000)$, while $c^*(w, v_p, t) \in [0, 0.75 \cdot w]$.

5.4 Results for Optimal Value $J^*(w_p, v, t)$ and Optimal Consumption $c^*(w_p, v, t)$, $w_p = 55$:



(a) Optimal portfolio value $J^*(w_p, v, t)$. (b) Optimal consumption $c^*(w_p, v, t)$.

Figure 6: Optimal portfolio value $J^*(w_p, v, t)$ and optimal consumption $c^*(w_p, v, t)$ at $w_p = 55$ for $(v, t) \in \times[v_{\min}, 1.0] \times [1980, 2000)$, while $c^*(w_p, v, t) \in [0, 0.75 \cdot w_p]$.

5.5 Results for Optimal Portfolio Fraction $u^*(v,t)$:

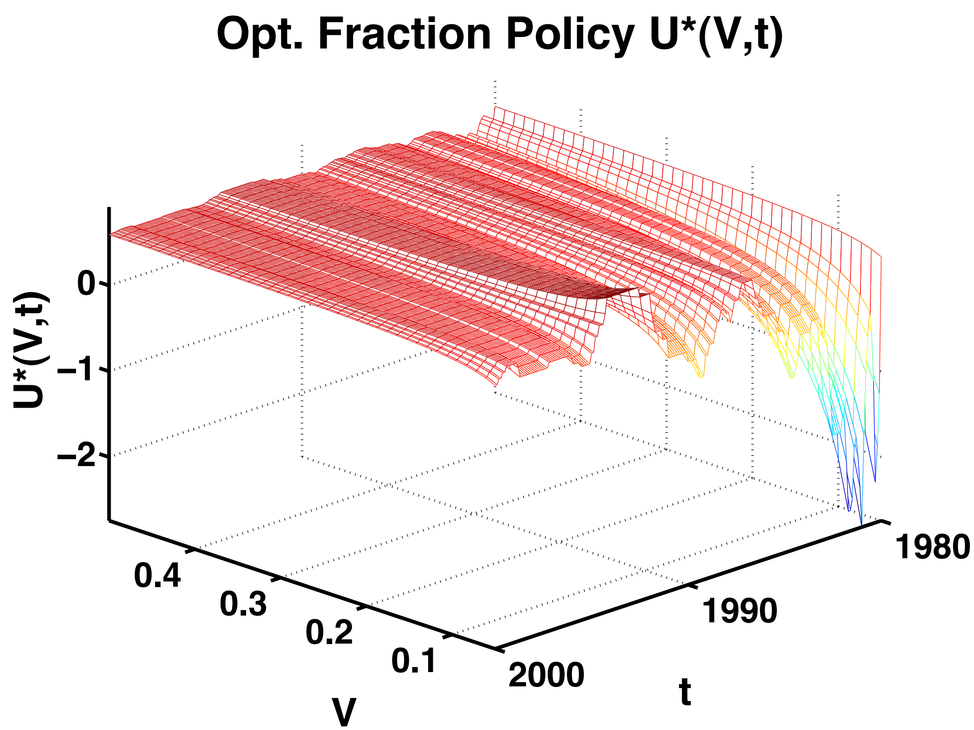


Figure 7: Optimal portfolio fraction policy $u^*(v,t)$ for $(v,t) \in \times[v_{\min}, 1.0] \times [1980, 2000)$, while $u^*(v,t) \in [-44.4, +12.4]$. ***Note the large changes of fraction for small variance, v, tracking Fed Chair Paul Volcker's frequent interest rate swings in 1980s.***

6. Conclusions

- ***Generalized the optimal portfolio and consumption problem for stochastic-volatility jump-diffusions to include jumps in the stochastic volatility/variance*** .
- Confirmed significant effects on ***variation of instantaneous stock fraction policies*** due to time-dependence of interest and discount rates for SJVJD optimal portfolio and consumption models.
- ***Showed jump-amplitude distributions with compact support are much less restricted on short-selling and borrowing*** compared to the infinite support case in the SjVJD optimal portfolio and consumption problem.
- Noted that the CRRA reduced canonical optimal portfolio problem is ***strongly drift-dominated*** for sample market parameter values over the diffusion terms, so at least first order drift-upwinding is essential for stable Bernoulli PIDE computations.