Applied Stochastic Processes and Control for Jump-Diffusions: Modeling, Analysis and Computation

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Chapter 1 Jump-Diffusion Introduction

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Chapter 1

Stochastic Jump and Diffusion Processes: Introduction

Life is good for only two things, discovering mathematics and teaching mathematics.
—Siméon Denis Poisson (1781-1840).

I do not regret my attempts, for it is only by trying problems that exceed his powers that the mathematician can ever learn to use these powers to their full extent.
—Norbert Wiener (1894-1964) in *Ex-Prodigy*.

The generation of random numbers is too important to be left to chance.
—Robert Coveyou at `http://www.xs4all.nl/~jcdverha/scijokes/1_5.html#subindex`.

1.1 Poisson and Wiener Processes Basics

This chapter introduces Wiener processes $W(t)$ and simple Poisson jump processes $P(t)$ in differential and integral forms. The Wiener and Poisson processes form the tools of a toolbox to create jump-diffusion process models. Wiener processes also called diffusion or loosely Brownian motion.

The processes $W(t)$ and $P(t)$ are **continuous-time stochastic processes** which basically means they are continuous time-dependent random variables[1]. They are also a special form of stochastic processes called a Markov process that is without memory of all but the prior state and can be simply defined [55], repeating the essential definition given in the previous chapter, as

[1] In this book, the words **stochastic** and **random** have the same meaning, involving probability or chance.

Definition 1.1. *The stochastic process $X(t)$ is a* ***Markov process*** *provided the conditional probability satisfies*

$$\mathrm{Prob}[X(t+\Delta t)=x|X(s),s\leq t]=\mathrm{Prob}[X(t+\Delta t)=x|X(t)]\,,$$

for any $t\geq 0$, any $\Delta t\geq 0$ and x is in the state space.

The stochastic processes serve as useful concepts for modeling random changes in time with stochastic differential equations, similar to the use of ordinary differential equations to model deterministic (non-stochastic) problems. These standard processes have basic infinitesimal moments

$$\mathrm{E}[dW(t)]=0 \text{ and } \mathrm{Var}[dW(t)]=dt \tag{1.1}$$

for the **differential Wiener process** with initial condition $W(0^+)=0$ **with probability one (w.p.o.)**, while

$$\mathrm{E}[dP(t)]=\lambda dt=\mathrm{Var}[dP(t)] \tag{1.2}$$

for the **differential of the simple Poisson counting process** with rate $\lambda>0$ and initial condition $P(0^+)=0$ **with probability one**. The Wiener process is a mathematical idealization of **Brownian motion**, but often the term Brownian motion is used instead of the term Wiener process.

Remark 1.2. *If the W and P processes started at a different initial time other than zero, say at $t=t_0$, then the initial conditions would be changed to $W(t_0^+)=0^+$ and $P(t_0^+)=0^+$, respectively. There is not much special about the zero initial conditions, just convenience and standardization.*

The simplest and very useful view of these differential stochastic processes is to consider them defined as increments, i.e.,

$$dW(t)\equiv W(t+dt)-W(t) \tag{1.3}$$

and

$$dP(t)\equiv P(t+dt)-P(t)\,, \tag{1.4}$$

for infinitesimal increments in time dt. The property that

$$\mathrm{Var}[dW(t)]=\mathrm{E}[(dW(t))^2]=dt \tag{1.5}$$

is motivation for the non-differentiability of the $W(t)$ process since the limit of

$$\sqrt{\mathrm{Var}[dW(t)]}/dt=\sqrt{\mathrm{E}[(dW(t))^2]}/dt=\frac{1}{\sqrt{dt}}\to+\infty \tag{1.6}$$

as $dt\to 0^+$, i.e., the variance of the ratio of differentials $\mathrm{Var}[dW(t)/dt]\to+\infty$ as $dt\to 0^+$. Hence, the differentiability of $W(t)$ is inconsistent with the failure of the variance of the quotient $dW(t)/dt$ in the limit $dt\to 0^+$. Equation (1.6) says that the root mean square (RMS) derivative becomes unbounded as $dt\to 0^+$. This is not a rigorous proof that $W(t)$ is a non-smooth process, although $W(t)$ is a continuous process from (1.1). (For a proof that $W(t)$ is non-differentiable see the theorem below.)

1.2 Wiener Process Basic Properties

The assumptions for the Wiener process, including that of being normally distributed, are the properties:

Properties 1.3. *The* ***standard Wiener process*** $\boldsymbol{W(t)}$

- $W(t)$ *is a* ***continuous process****, since*

$$W(t^+) = W(t) = W(t^-), \quad t > 0\,.$$

- $W(t)$ *has* ***independent increments****, since the Wiener increments*

$$\Delta W(t_i) = W(t_i + \Delta t_i) - W(t_i)$$

are mutually independent for all t_i *on non-overlapping time intervals. The* ***non-overlapping time intervals*** *are defined such that* $t_i \geq 0$, $t_{i+1} = t_i + \Delta t_i$ *and any* $\Delta t_i > 0$ *for* $0 = 1 : n$*, so that*

$$t_i < t_{i+1} \text{ for } i = 0\!:\!n.$$

Noting that $W(t_i) = W(0) + \sum_{j=0}^{i-1} \Delta W(t_j)$*, so depends on all preceding increments, recalling that* $W(0) = 0$ *with probability one at* $t_0 = 0$ *i.e.,*

$$\text{Prob}[\Delta W(t_i) \leq w_i, \Delta W(t_j) \leq w_j] = \text{Prob}[\Delta W(t_i) \leq w_i] \cdot \text{Prob}[\Delta W(t_j) \leq w_j]\,,$$

if $j \neq i$*, such that there is no overlap in the time intervals* $[t_i, t_{i+1})$ *and* $[t_j, t_{j+1})$*. Note that* $\Delta W(t_i)$*, as a forward increment is independent of* $W(t_i)$ *and that* $\Delta W(t_i) \equiv W(t_i + \Delta t_i) - W(t_i)$ *is associated with the time interval* $[t_j, t_j + \Delta t_j)$*, open on the right to be compatible with right continuity of the Poisson process.*

- $W(t)$ *is a* ***stationary process****, since the distribution of the increment* $\Delta W(t) = W(t + \Delta t) - W(t)$*, with* $\Delta t > 0$*, is independent of* t*.*

- $W(t)$ *is a* ***Markov process****, since*

$$\text{Prob}[W(t + \Delta t) = w | W(s), s \leq t] = \text{Prob}[W(t + \Delta t) = w | W(t)]\,,$$

for any $t \geq 0$*, any* $\Delta t \geq 0$*.* (It is helpful to note that $W(t)$ is synonymous with the increment $(W(t) - W(0))$.)

- $W(t)$ *is normally distributed with mean* $\mu = 0$ *and variance* $\sigma^2 = t$, $t > 0$*, i.e.,*

$$\phi_{W(t)}(w) = \phi_n(w; 0, t) = \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{w^2}{2t}\right)\,, \tag{1.7}$$

when $-\infty < w < +\infty$ *and* $t > 0$*. (The actual distribution function for* $W(t)$*,* $\Phi_{W(t)}(w)$*, has been given already in (0.22).)*

- $W(0) = 0$ ***with probability one***, *since* $\phi_{W(0^+)}(w) = \delta(w)$ *from (1.7), i.e., in the limit as* $t \to 0^+$ *(see the Exercise 22 on p. 75 in Section 0.16).*

Thus, the increments $\Delta[W(t+i\Delta t)] \equiv W(t+(i+1)\Delta t) - W(t+i\Delta t)$ for $i = 0, 1, \ldots$ are stationary, independent and identically distributed (IID) as a normal distribution given time step Δt and $t \geq 0$, i.e.,

$$\phi_{\Delta W(t)}(w) = \phi_n(w; 0, \Delta t) = \frac{1}{\sqrt{2\pi\Delta t}} \exp\left(-\frac{w^2}{2\Delta t}\right), \tag{1.8}$$

when $-\infty < w < +\infty$ and $\Delta t > 0$. So the basic moments of the Wiener increments are

$$\mathrm{E}[\Delta W(t)] = 0, \quad \mathrm{Var}[\Delta W(t)] = \Delta t. \tag{1.9}$$

Similarly, by the stationarity property of the $dW(t) = W(t+dt) - W(t)$ differential process when $dt > 0$ has the same probability distribution as the process $W(dt)$ when $t > 0$ and that the distribution from (1.7) is normal with mean $\mu = 0$ and variance $\sigma^2 = dt$,

$$\phi_{dW(t)}(w) = \phi_n(w; 0, dt) = \frac{1}{\sqrt{2\pi dt}} \exp\left(-\frac{w^2}{2dt}\right), \tag{1.10}$$

when $-\infty < w < +\infty$ and $dt > 0$.

Theorem 1.4. ***Covariance of*** $\mathbf{W(t)}$***:*** *If* $W(t)$ *is a Wiener process, then*

$$\mathrm{Cov}[W(t), W(s)] = \min[t, s]\,. \tag{1.11}$$

Proof. This theorem is a very elementary application of the independent increment and mean zero properties of Wiener or diffusion processes, also demonstrating how application of independent increments rely on the zero mean property. The zero mean property implies that $\mathrm{E}[W(t)] = 0 = \mathrm{E}[W(s)]$. First consider the case $s < t$ and write $W(t) = W(s) + (W(t) - W(s))$, i.e., as independent increments, then

$$\begin{aligned}
\mathrm{Cov}[W(t), W(s)] &= \mathrm{E}[W(t)W(s)] = \mathrm{E}[W^2(s) + W(s)(W(t) - W(s))] \\
&= \mathrm{E}[W^2(s)] + \mathrm{E}[W(s)(W(t) - W(s))] \\
&= \mathrm{Var}[W(s)] + \mathrm{E}[W(s)]\mathrm{E}[(W(t) - W(s))] \\
&= s + 0 \cdot 0 = s\,,
\end{aligned}$$

using the linearity of the expectation operator (0.9), the definition of the variance (0.10) together with the separability of expectations (0.78) for independent increments $W(s)$ and $(W(t) - W(s))$, and finally that $W(s)$ has variance s (0.22, 1.7). In the case $t < s$, then $\mathrm{Cov}[W(t), W(s)] = t$ by symmetry, and combining both cases produces the conclusion $\mathrm{Cov}[W(t), W(s)] = \min[s, t]$, where the function $\min[s, t]$ denotes the minimum of s and t. ∎

When computing diffusion sample paths, i.e., the trajectory of $W(t)$ in time t, it is necessary to break up the time domain, say $[0, T]$ into small increments $\Delta T = T/N$ where N is the number of random samples that will be used, so that each corresponding Wiener increment $\Delta W(t_i)$ will be independent. Since $W(0) = 0$ **with probability one**, let $t_i = i \cdot \Delta T$ for $i = 0 : N$, then

$$W(t_{i+1}) = \sum_{j=0}^{i} \Delta W(t_j)\,.$$

Using MATLAB™ , for instance, an integer state , say 0, is selected with the MATLAB command

```
randn('state',0);
```

where `'state'` is a literal script argument specified that this call is to set the random state of the function `randn`. A column N-vector set of diffusion increments can be computed wholesale by the formula,

```
DW = sqrt(DT)*randn(N,1);
```

where `randn(N,1)` is the $N \times 1$ standard zero-mean, unit-variance normal random generator of MATLAB. The factor `sqrt(DT)` is the Wiener scaling for the square root of the variance (1.9). Then the simulated trajectory can be computed by

```
ts = 0:DT:T;  % time vector ts(1:N+1).
for  i = 1:N
     WS(i+1) = sum((DW(1:i)));
end
```

assuming $WS(1) = 0.0$ in the MATLAB shifted subscript base at one, rather than at zero. Finally, the diffusion sample path can be plotted with

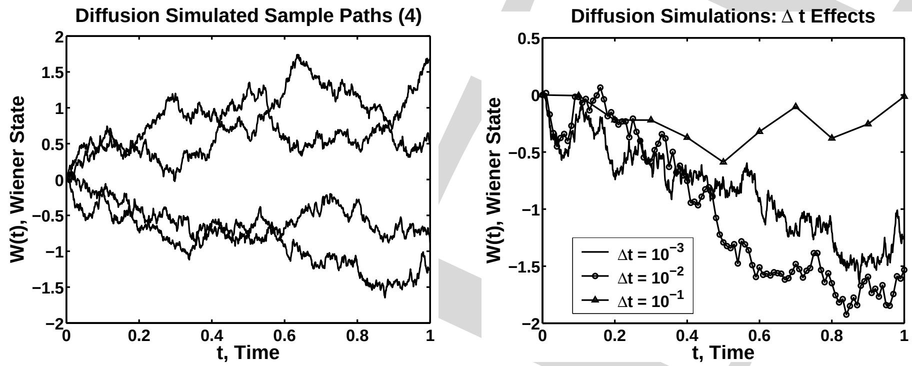
```
plot(ts,WS,'k-');
```

and results for four sample paths are displayed in Fig. 1.1(a) using $N = 1000$, $T = 1.0$ and $k = 1\!:\!4$ `randn` states. The MATLAB program used to generate this part of the figure is given in Program A.7 given in Appendix A.

In Fig. 1.1(b), the variation of the fine structure of the sample path is displayed, with time step size using subsets of the same random sample state. The sample paths in this case differ markedly since the sample subsets are quite different in quantity, being $N = 1000, 100$ and 10 random sample points for $\Delta t = 10^{-3}, 10^{-2}$ and 10^{-1}, respectively, so the different cumulative set of random points leads to quite different random trajectories.

1.3 More Wiener Process Moments

The expectations for the integer powers of the Wiener increment follow from the mean using the Wiener increment normal density (1.8). Only the even integer powers, $m = 2k$, need some calculation since the means will be zero for the odd integer



(a) Four diffusion sample paths.

(b) Diffusion sample paths using different time steps.

Figure 1.1. *In Figure 1.1(a), paths were simulated using MATLAB [206] with* $N = 1000$ *sample points, four* `randn` *states and maximum time* $T = 1.0$. *In Figure 1.1(b), paths were simulated using subsets of the same random state of* `randn` *used for the finer grid* 0.001.

powers due to the even-ness of the density on $(-\infty, +\infty)$, i.e., $\mathrm{E}[(\Delta W(t))^{2k+1}] = 0$ when $k = 0, 1, 2, \ldots,$

$$
\begin{aligned}
\mathrm{E}\left[(\Delta W(t))^m\right] = \mathrm{E}\left[(\Delta W(t))^{2k}\right] &= \int_{-\infty}^{+\infty} \phi_n(w; 0, \Delta t) w^{2k} dw \\
&= \frac{2}{\sqrt{2\pi\Delta t}} \int_0^{+\infty} \exp\left(-\frac{w^2}{2\Delta t}\right) w^{2k} dw \\
&= \frac{(2\Delta t)^k}{\sqrt{\pi}} \int_0^{+\infty} \exp(-u)\, u^{k-1/2} du \\
&= \frac{(2\Delta t)^k \Gamma(k + 1/2)}{\Gamma(1/2)}\,,
\end{aligned}
\tag{1.12}
$$

for $k = 0, 1, 2, \ldots,$ where Γ is the gamma function [2] defined by

$$\Gamma(x) \equiv \int_0^{\infty} e^{-u} u^{x-1} du\,, \quad x > 0\,, \tag{1.13}$$

with initial condition $\Gamma(1) \equiv 1$ and special value $\Gamma(1/2) = \sqrt{\pi}$. The gamma function is like a generalized factorial function, due to the recursive form $\Gamma(x + 1) = x\Gamma(x)$ so that $\Gamma(x + 1) = x!$. The final formula (1.12) satisfies the recursion

$$g_{2k+2}(\Delta t) \equiv \mathrm{E}[(\Delta W)^{2k+2}(t)] = (k + 1/2)(2\Delta t) g_{2k}(\Delta t).$$

Further, note that the final formula (1.12) holds for any integer m when the $\Delta W(t)$ is replaced by the absolute value, i.e.,

$$\mathrm{E}[|\Delta W(t)|^m] = (2\Delta t)^{m/2}\Gamma((m + 1)/2)/\Gamma(1/2). \tag{1.14}$$

The final formula (1.12) satisfies the recursion

$$g_{m+2}(\Delta t) \equiv \mathrm{E}[(\Delta W)^{m+2}(t)] = (m+1)\Delta t g_m(\Delta t)\,,$$

for $m = 0 : \infty$, starting from $g_0(\Delta t) = 1$ or $g_1(\Delta t) = \sqrt{2\Delta t/\pi}$. The results for the first few powers are summarized in Table 1.1:

Table 1.1. *Some expected moments (powers) of absolute value of the Wiener increments.*

m	$\mathrm{E}[\lvert\Delta W(t)\rvert^m]$
0	1
1	$\sqrt{2\Delta t/\pi}$
2	Δt
3	$2\Delta t\sqrt{2\Delta t/\pi}$
4	$3(\Delta t)^2$
5	$8(\Delta t)^2\sqrt{2\Delta t/\pi}$
6	$15(\Delta t)^3$
$\vdots$	$\vdots$
2k	$(2k-1)!!(\Delta t)^k$
2k+1	$k!(2\Delta t)^k\sqrt{2\Delta t/\pi}$

In Table 1.1, the function $(2k-1)!!$ is defined below.

Definition 1.5. ***Double Factorial Function:***

$$(2k-1)!! = (2k-1)\cdot(2k-3)\cdots 1\,, \tag{1.15}$$

denotes the ***double factorial function****, given here for odd arguments. For example,* $1!! = 1$, $3!! = 3$ *and* $5!! = 15$.

For even arguments the double factorial function is proportional to the standard factorial function,

$$(2k)!! = 2^k k!\,.$$

Example 1.6. *These results can be applied to other expected moments, for example,*

$$\begin{aligned}\mathrm{Var}[(\Delta W)^2(t)] &= \mathrm{E}[((\Delta W)^2(t) - \Delta t)^2] \\ &= \mathrm{E}[(\Delta W)^4(t)] - 2\Delta t\mathrm{E}[(\Delta W)^2(t)] + (\Delta t)^2\mathrm{E}[1] \\ &= 2(\Delta t)^2\,, \end{aligned} \tag{1.16}$$

upon expanding the square and using the linear property of the expectation.

The moment calculation in (1.12) can be implemented directly by symbolic computation, for example by Maple.

Example 1.7. ***Wiener Moments by Maple:***

\# ***Maple Functions for Wiener Moments and Wiener Deviation Moments:***

```
> restart : interface(showassumed = 0) : assume(s2 > 0) :
```

```
> fnormal := (x, m, s2)− > exp(−(x − m) ∗ (x − m)/(2 ∗ s2))/sqrt(2 ∗ Pi ∗ s2);
```

$$fnormal := (x, m, s2) \to \frac{e^{\left(1/2\,\frac{(x-m)^2}{s2}\right)}}{\sqrt{2\pi s2}}$$

```
> momentdw := (n, m, s2)− > simplify(int(x^n ∗ fnormal(x, m, s2),
> x = −infinity..infinity));
```

$$momentdw := (n, m, s2)) \to simplify\left(\int_{-\infty}^{\infty} x^n fnormal(x, m, s2)dx\right)$$

```
> momentdevdw := (n, m, s2)− > simplify(int((x − m)^n fnormal(x, m, s2),
> x = −infinity..infinity));
```

$$momentdevdw := (n, m, s2) \to simplify\left(\int_{-\infty}^{\infty} (x-m)^n fnormal(x, m, s2)dx\right)$$

\# *Sample illustrations for moment functions:*

```
> assume(dt > 0) : assume(sigma > 0) :
```

```
> mom6dw := collect(momentdw(6, mu ∗ dt, sigma^2 ∗ dt), dt);
```

$$mom6dw := \mu^6 dt^6 + 15\mu^4\sigma^2 dt^5 + 45\mu^2\sigma^4 + 15\sigma^6 dt^3$$

```
> mom6devdw := momentdevdw(6, mu ∗ dt, sigma^2 ∗ dt);
```

$$mom6dw := 15dt^3$$

```
> mom5absdevdw := momentabsdevdw(5, mu ∗ dt, sigma^2 ∗ dt);
```

$$mom6dw := \frac{8dt^{(5/2)}\sqrt{2}}{\sqrt{\pi}}$$

Remarks 1.8.

- *The results can also be applied to expected moments of Wiener differential process, $dW(t) = W(t+dt) - W(t)$, by replacing single appearances Δt's by dt, i.e., $\Delta t \to dt$ is assumed, and neglecting terms of $\mathrm{O}^2(\Delta t)$ as $\Delta t \to 0^+$ since they are treated as negligible compared to terms of $\mathrm{ord}(\Delta t)$ as $\Delta t \to 0^+$.*

- *Sometimes to keep the steps in a derivation simple, the infinitesimal dt will be treated as being an infinitesimally small object such that as $dt \to 0^+$, $(dt)^2 \ll 1$ or $(dt)^2 < \mathrm{ord}(dt)$ and similarly for higher powers of dt. However, when there are no order Δt terms in the answer, then, as in (1.16), the proper leading order (by definition nonzero) would be of interest. Expected moments of $W(t)$ also follow by replacing Δt by t, except the higher powers of t would not be negligible compared to the first power, unless t is small.*

1.4 Wiener Process Non-Differentiability

Theorem 1.9. ***Non-differentiability of*** $\mathbf{W(t)}$***:***
For any fixed $x > 0$ and $t > 0$,

$$\mathrm{Prob}\left[\lim_{\Delta t \to 0^+}\left[\left|\frac{\Delta W(t)}{\Delta t}\right| > x\right]\right] = 1. \tag{1.17}$$

Proof. Let $x > 0$ be fixed, $t > 0$, $0 < \Delta t \ll 1$, then interchanging limit with probability operations since time is deterministic and using the normal distribution of the increment $\Delta W(t) = W(t + \Delta t) - W(t)$ in (1.8),

$$\begin{aligned}
\mathrm{Prob}\left[\lim_{\Delta t \to 0^+}\left[\left|\frac{\Delta W(t)}{\Delta t}\right| > x\right]\right] &= \lim_{\Delta t \to 0^+}\left[\mathrm{Prob}\left[\left|\frac{\Delta W(t)}{\Delta t}\right| > x\right]\right] \\
&= \lim_{\Delta t \to 0^+}\left[\mathrm{Prob}\left[|\Delta W(t)| > x\Delta t\right]\right] \\
&= \lim_{\Delta t \to 0^+}\left[\frac{2}{\sqrt{2\pi\Delta t}}\int_{x\Delta t}^{\infty}\exp\left(-\frac{w^2}{2\Delta t}\right)dw\right] \\
&= \lim_{\Delta t \to 0^+}\left[\sqrt{\frac{2}{\pi}}\int_{x\sqrt{\Delta t}}^{\infty}\exp\left(-\frac{v^2}{2}\right)dv\right] \\
&= \sqrt{\frac{2}{\pi}}\int_0^{\infty}\exp\left(-\frac{v^2}{2}\right)dv = 1\,,
\end{aligned}$$

for any $x > 0$ and $t > 0$ fixed. Note that the error is

$$\sqrt{\frac{2}{\pi}}\int_0^{x\sqrt{\Delta t}}\exp\left(-\frac{v^2}{2}\right)dv \le \sqrt{\frac{2}{\pi}}\int_0^{x\sqrt{\Delta t}}1\,dv = \sqrt{\frac{2}{\pi}}x\sqrt{\Delta t} \ll 1\,,$$

since $\exp(-v^2/2) \le 1$. Further note that we can take x as large as we please, as long as it is fixed, so that $\Delta W(t)/\Delta t$ must be unbounded as $\Delta t \to 0^+$ **with probability one** for each t. Hence, the Wiener process $W(t)$ is non-differentiable or non-smooth with probability on for each t. (See also Mikosch [205], for a similar proof using less direct methods.) □

1.5 Wiener Process Expectations Conditioned on Past

Example 1.10. ***Illustration of Independent Increments and Markov Properities for Wiener Process:***

- $\mathrm{E}[W(t)|W(r), 0 \le r \le s] = W(\min[s,t])$.
Note that the conditioning set $\{W(r), 0 \le r \le s\}$ *denotes the past when* $t > s \ge 0$*, viewing* $W(t)$ *as the sum of two independent increments* $(W(s) - W(0)) + (W(t) - W(s))$*, noting that* $W(0) = 0$*. However, when* $0 \le t \le s$*, then the increment* $W(t) \equiv (W(t) - W(0))$ *is a constant relative to the conditioning set, so the result depends on the relation between* t *and* s *using the rule* $\mathrm{E}[f(X)|X] = f(X)$ *given in Chapter 0 on Page 28. Hence,*

$$\begin{aligned}
\mathrm{E}[W(t)|W(r), 0 \le r \le s] &= \left\{ \begin{array}{l} W(t),\ 0 \le t \le s \\ \mathrm{E}[W(s) + (W(t) - W(s))|W(r), 0 \le r \le s], \\ \qquad 0 \le s < t \end{array} \right\} \\
&= \left\{ \begin{array}{l} W(t),\ 0 \le t \le s \\ \mathrm{E}[W(s)|W(r), 0 \le r \le s] + \mathrm{E}[(W(t) - W(s))], \\ \hfill 0 \le s < t \end{array} \right\} \\
&= \left\{ \begin{array}{l} W(t),\ 0 \le t \le s \\ W(s) + 0,\ 0 \le s < t \end{array} \right\} = \left\{ \begin{array}{l} W(t),\ 0 \le t \le s \\ W(s),\ 0 \le s < t \end{array} \right\} \\
&= W(\min[s,t])\,,
\end{aligned}$$

where the independent increment property was used along with the zero mean property of the increment, $\mathrm{E}[\Delta W(t)] = 0$ *and the completely conditioned rule that* $\mathrm{E}[f(X)|X] = f(X)$*. The function* $\min[s,t]$ *denotes the minimum of* s *and* t*. The linear property of the conditional expectation was also used.*

When $0 \le s < t$ *then the formula,*

$$\mathrm{E}[W(t)|W(r), 0 \le r \le s] = W(s)\,, \tag{1.18}$$

signifies that the average information conditioned on the past data, $\{W(r), r \in [0,s]\}$*, is given by the most recent past data* $W(s)$*, which may imply a significant reduction in uncertainty for the present data,* $W(t)$*.*

The form of the expectation result (1.18) is the principal characteristic form for a ***martingale*** $X(t)$*,*

$$\mathrm{E}[X(t)|X(r), 0 \le r \le s] = X(s)\,, \tag{1.19}$$

where $X(t) = f(W(t))$ *for instance. The martingale is an abstract model of a fair game (see the beginning preliminary chapter of Mikosch [205] for a clear description of martingales, but in an abstract presentation; martingales will be described at the end of this book in Chapter 13 with full qualifications).*

- $\mathrm{E}[W^2(t)|W(r), 0 \le r \le s] = W^2(\min[s,t]) + (t-s)H(t-s)$,
where $H(X)$ *is the Heaviside step function (0.154). This result is derived similarly to the prior result for the conditional mean, but much more algebra*

is required, although many of the small details of the prior derivation are omitted.

$$\begin{aligned}
\mathrm{E}[W^2(t)|W(r), 0 \le r \le s] &= \left\{ \begin{array}{l} W^2(t),\ 0 \le t \le s \\ \mathrm{E}[(W(s) + (W(t) - W(s)))^2 | W(r), 0 \le r \le s], \\ \hfill 0 \le s < t \end{array} \right\} \\
&= \left\{ \begin{array}{l} W^2(t),\ 0 \le t \le s \\ W^2(s) + 2W(s)\mathrm{E}[(W(t) - W(s))] + \mathrm{E}[(W(t) - W(s))^2], \\ \hfill 0 \le s < t \end{array} \right\} \\
&= \left\{ \begin{array}{l} W^2(t),\ 0 \le t \le s \\ W^2(s) + 2W(s) \cdot 0 + (t - s),\ 0 \le s < t \end{array} \right\} \\
&= \left\{ \begin{array}{l} W^2(t),\ 0 \le t \le s \\ W^2(s) + (t - s),\ 0 \le s < t \end{array} \right\} \\
&= W^2(\min[s, t]) + (t - s)H(t - s).
\end{aligned}$$

Here, the increment variance $\mathrm{Var}[\Delta W(t)] = \Delta t$ *has been used.*

The general technique for powers $W^m(t)$ *when* $s < t$ *with conditioning on* $W(s)$ *is to use the decomposition into independent increments* $W(t) = W(s) + (W(t) - W(s))$ *and then expand the power of* m *by the binomial expansion (0.148)*

$$(W(s) + (W(t) - W(s)))^m = \sum_{k=0}^{m} \binom{m}{k} W^k(s)(W(t) - W(s))^{m-k},$$

and then use independence of the increments and conditioning to calculate for each term,

$$\begin{aligned}
&\mathrm{E}\left[\binom{m}{k} W^k(s)(W(t) - W(s))^{m-k} \middle| W(r), 0 \le r \le s \right] \\
&\quad = \binom{m}{k} W^k(s)\mathrm{E}\left[(W(t) - W(s))^{m-k}\right],
\end{aligned}$$

relying on Table 1.1 for the remaining expectation.

The term normal distribution is more often used in mathematics and statistics, while the term Gaussian distribution may be used more often in other sciences and engineering.

1.6 Poisson Process Basic Properties

Since the Poisson process suffers from positive jumps of integer magnitude the Poisson process is also discontinuous, which makes the differentiability problems of the Poisson process of secondary importance. For this reason, the Poisson process is also called a **counting process** or **point process**. Thus, the analytical problems are even more severe than for the Wiener process, since the singularities of the Poisson process arise at the zeroth order with the value of $P(t)$ jumping, while those of $W(t)$ arise at the first order derivative. However, the jumps of the Poisson process have a modeling benefit over the Wiener process in that the Poisson process is useful for applications with disasters or crashes and those with bonanzas or rallies.

In summary, **Poisson process** $P(t)$ **is a discontinuous process** and satisfies the following properties:

Properties 1.11. ***Simple Poisson Process $P(t)$:***

- *$P(t)$ has **unit jumps**, since if the value of $P(t)$ jumps at time $T_k > 0$, then*
$$P(T_k^+) = P(T_k^-) + 1\,,$$
where $P(T_k^+)$ denotes the limit from the right and $P(T_k^-)$ the limit from the left, so $P(t)$ is discontinuous, increasing and has instantaneous jumps.

- *$P(t)$ is **right-continuous**, since*
$$P(t^+) = P(t) \geq P(t^-), \quad t > 0\,. \tag{1.20}$$

- *$P(t)$ has **independent increments**, since the Poisson increments*
$$\Delta P(t_i) \equiv P(t_i + \Delta t_i) - P(t_i)$$
are mutually independent for all t_i on non-overlapping time intervals defined such that $t_i \geq 0$, $t_{i+1} = t_i + \Delta t_i$ and any $\Delta t_i > 0$ for $0 = 1 : n$ so that
$$t_i < t_{i+1} \text{ for } i = 0 : n\,,$$
noting that $P(t_i) = P(0) + \sum_{j=0}^{i-1} \Delta P(t_j)$, depending on all preceding increments, recalling that $P(0) = 0$ with probability one at $t_0 = 0$, i.e.,
$$\mathrm{Prob}[\Delta P(t_i) \leq p_i, \Delta P(t_j) \leq p_j] = \mathrm{Prob}[\Delta P(t_i) \leq p_i] \cdot \mathrm{Prob}[\Delta P(t_j) \leq p_j]\,,$$
if $j \neq i$, such that there is no overlap in the time intervals $(t_i, t_{i+1}]$ and $(t_j, t_{j+1}]$. Note that $\Delta P(t_i)$, as a forward increment is independent of $P(t_i)$ and recall that $\Delta P(t_i) \equiv P(t_i + \Delta t_i) - P(t_i)$ is associated with the time interval $[t_j, t_j + \Delta t_j)$, open on the right since the process $P(t_i)$ is right continuous.

- *$P(t)$ is a **stationary process**, since the distribution of the increment $\Delta P(t) = P(t + \Delta t) - P(t)$ is independent of t.*

- *$P(t)$ is a **Markov process**, since*
$$\mathrm{Prob}[P(t + \Delta t) = k\,|P(s), s \leq t] = \mathrm{Prob}[P(t + \Delta t) = k\,|P(t)]\,,$$
for any $t \geq 0$, any $\Delta t \geq 0$. (It is helpful to note that $P(t)$ is synonymous with the increment $(P(t) - P(0))$.)

- *$P(t)$ is **Poisson distributed** with mean $\mu = \lambda t$ and variance $\sigma^2 = \lambda t$, $t > 0$, i.e.,*
$$\Phi_{P(t)}(k; \lambda t) = \mathrm{Prob}[P(t) = k] \equiv p_k(\lambda t) = e^{-\lambda t}\frac{(\lambda t)^k}{k!}\,, \tag{1.21}$$
for integer values $k = 0, 1, 2, \ldots$, with constant $\lambda > 0$ and $t \geq 0$.

- $P(0^+) = 0^+$ ***with probability one***, *since from (1.21),* $p_k(0^+) = \delta_{k,0}$, *i.e., in the limit as* $t \to 0^+$.

See also Çinlar [55] or Snyder and Miller [247] for a more essential list of assumptions.

Thus, for $P(t)$, the increments $\Delta[P(t+i\Delta t)] \equiv P(t+(i+1)\Delta t) - P(t+i\Delta t)$ for $i = 0, 1, \ldots$ are independent and identically distributed (IID) given time step $\Delta t > 0$ and $t \geq 0$.

By the stationarity property of the incremental Poisson process $\Delta P(t) = P(t+\Delta t) - P(t)$ has the same discrete distribution as $P(\Delta t)$ in (1.21), so has the parameter $\lambda\Delta t$ instead of the λt in (0.50), i.e.,

$$\Phi_{\Delta P(t)}(k; \lambda\Delta t) = \mathrm{Prob}[\Delta P(t) = k] = p_k(\lambda\Delta t) = e^{-\lambda\Delta t}\frac{(\lambda\Delta t)^k}{k!}\,, \tag{1.22}$$

for $k = 0, 1, 2, \ldots$, $t \geq 0$ and $\Delta t \geq 0$.

Similarly, by the stationarity property of the differential, $dP(t) = P(t+dt) - P(t)$, for Poisson process has the same discrete distribution as $P(dt)$ in (1.21), except that $dP(t)$ has the parameter λdt instead of the λt in (0.50) for $P(t)$. Thus $dP(t)$ has the distribution,

$$\Phi_{dP(t)}(k; \lambda dt) = \mathrm{Prob}[dP(t) = k] = p_k(\lambda dt) = e^{-\lambda dt}\frac{(\lambda dt)^k}{k!}\,, \tag{1.23}$$

for $k = 0, 1, 2, \ldots$, $t \geq 0$ and $dt \geq 0$. The distribution (1.23) might be considered as a limiting version of the more basic and proper incremental version in (1.22).

The simulation of the simple Poisson process $P(t)$ is usually based upon simulating the time between jumps, the inter-arrival time $T_{k+1} - T_k$, since the inter-arrival time can be shown to be exponentially distributed as sketched in Chapter 0.

Lemma 1.12. ***Exponential Distribution of Time Between Jumps:***
Let $P(t)$ be a simple Poisson process with fixed jump frequency λ and let T_j denote the jth jump time, then the distribution of the ***inter-jump time*** $\Delta T_j \equiv T_{j+1} - T_j$ *for $j = 0, 1, 2, \ldots$, defining $T_0 \equiv 0$, conditioned on T_j, is*

$$\Phi_{\Delta T_j}(\Delta t) = \mathrm{Prob}[\Delta T_j \leq \Delta t \mid T_j] = 1 - e^{-\lambda\Delta t}\,. \tag{1.24}$$

Proof. The basic idea of this proof is that the probability of the time between jumps $\Delta T_j = T_{j+1} - T_j$ less than Δt, conditioned on the prior jump time T_j, will be the same as the probability that there be at least one jump in the time interval, which is the same as one minus the probability that there are no jumps in the time interval, i.e.,

$$\begin{aligned}\mathrm{Prob}[\Delta T_j \leq \Delta t \mid T_j] &= 1 - \mathrm{Prob}[\Delta T_j > \Delta t \mid T_j]\\ &= 1 - \mathrm{Prob}[\Delta P(T_j) = 0 \mid T_j]\,.\end{aligned}$$

However, by the stationary property of the simple Poisson process $P(t)$ the probability of the difference does not depend on the common time T_j, but on the difference in time ΔT_j,

$$\begin{aligned}\mathrm{Prob}[\Delta T_j \le \Delta t \mid T_j] &= 1 - \mathrm{Prob}[P(\Delta t) - P(0) = 0] \\ &= 1 - \mathrm{Prob}[P(\Delta t) = 0] = 1 - p_0(\lambda\Delta t) \\ &= 1 - e^{-\lambda\Delta t} = \Phi_e(\Delta t; 1/\lambda)\,,\end{aligned}$$

where the fact that $P(0) = 0$ with probability one has been used, Poisson distribution $p_k(\lambda\Delta t)$ is given in (1.22) and the exponential distribution $\Phi_e(t;\mu)$ is given in (0.40). □

Using MATLAB with the efficient and fundamental distribution transformation from uniform to exponential distribution (0.42) , a uniformly distributed pseudo-random number generator can be used. These numbers can be generated wholesale, in vector form, for plotting or other applications, using a given K samples and the Poisson parameter value `lambda`, by the following code fragment,

```
xu = rand(K,1); T(1) = 0; kv(1) = 0;
for k = 1:K, kv(k+1) = k;
    T(k+1) = T(k) - log(xu(k))/lambda;
end
plot(kv,T,'k-');
```

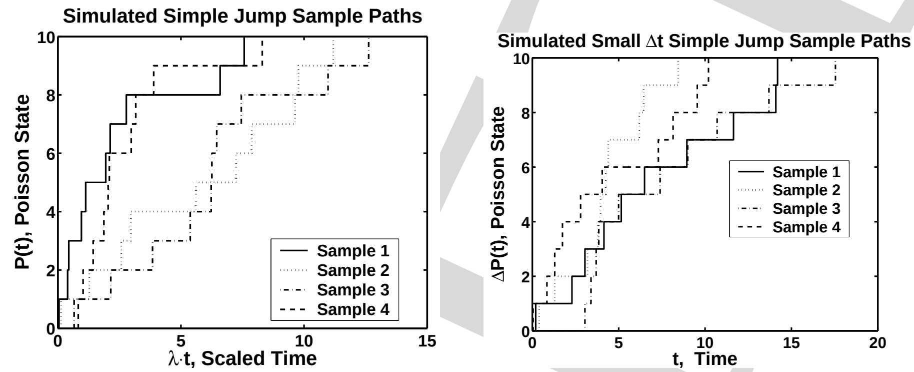
where `log` is the MATLAB natural logarithm notation. See the comments about (0.44) explaining why the proper term $\log(xu(k))$ is used here rather than the less efficient term $\log(1 - xu(k))$.

Since the natural time variable for Poisson is scaled as $\lambda * t$, four sample paths for $P(t)$ are illustrated in Fig. 1.2(a) versus the dimensionless time $\lambda * t$. The variation with the jump rate λ can be deduced since higher frequencies ($\lambda > 1$) compress the time axis and lower frequencies ($\lambda < 1$) expand the time axis. Note that the exponentially distributed inter-jump or inter-arrival times must used for simulating $P(t)$ since the Poisson distribution is not useful in simulating the jump times directly. The MATLAB source code for the left figure is given in Program A.9 in Appendix A.

In Fig. 1.2(b) are the corresponding sample paths for the incremental Poisson process $\Delta P(t)$ when the time increments between jumps are sufficiently small so that the zero-one jump law, discussed more extensively in Theorem 1.19 in Section 1.7, applies and the time between jumps is uniformly distributed with asymptotic probability $\lambda\Delta t$ for the next jump and $(1 - \lambda\Delta t)$ for zero jumps, since

$$\mathrm{Prob}[T_{k+1} - T_k \le \Delta t \mid T_k] = 1 - e^{-\lambda\Delta t} \sim \lambda\Delta t\,,$$

provided $\lambda\Delta t \ll 1$, i.e., small, taking $\Delta t = 0.05$ and $\lambda = 1.0$. The small time increment process can be numerically simulated by a standard uniform number generator like MATLAB's `rand` and the **method of acceptance-rejection**



(a) Four Poisson jump $P(t)$ sample paths. (b) Incremental Poisson jump $\Delta P(t)$ sample paths using different time steps.

Figure 1.2. *In Figure 1.2(a), Simulated sample paths for the simple Poisson Process $P(t)$ versus the dimension-less time λt using four different MATLAB [206] random states for four different sample paths and the exponential distribution of the time between jumps. In Figure 1.2(b) is a similar illustration for the incremental simple Poisson process simulations versus t with $\lambda = 1.0$ and $\Delta t = 0.05$, based upon the zero-one jump law implemented with a uniform distribution paths were simulated using subsets of the same random state of* `randn` *used for the finer grid* 0.001.

[226, 96] such that the open interval $(0, 1)$ is partitioned into a centered interval of length $\lambda\Delta t$ and the complement of $(0, 1)$. When a uniformly generated point lands in the centered interval then a jump is counted, while there is no jump if it lands in the complement. The centered interval, $[(1-\lambda\Delta t)/2, (1+\lambda\Delta t)/2]$ is used to avoid the bias of open interval property of pseudo-random number generators where the neighborhood of the end points is excluded by a very small amount that is the order of the **machine epsilon** (the smallest positive floating number that is significant when added to one, theoretically, 2^{-53} in IEEE standard double precision). A sufficiently large sample should approximately satisfy the zero-one jump law probabilities, since the rejection method is equivalent to the unit step function applied $U(X_u; (1-\lambda\Delta t)/2, (1+\lambda\Delta t)/2)$ to the uniform variate X_u approximately generated by `rand` and the expectation is

$$\begin{aligned}\mathrm{E}\left[U(X_u; (1-\lambda\Delta t)/2, (1+\lambda\Delta t)/2)\right] &= \int_0^1 U(u; (1-\lambda\Delta t)/2, (1+\lambda\Delta t)/2)du \\ &= \int_{(1-\lambda\Delta t)/2}^{(1+\lambda\Delta t)/2} du = \lambda\Delta t\,.\end{aligned}$$

The MATLAB source code for the right figure is given in Program A.10 of Appendix A.

Theorem 1.13. ***Covariance of*** $\mathbf{P(t)}$***:*** *If* $P(t)$ *is a Poisson process, then*

$$\mathrm{Cov}[P(t), P(s)] = \lambda \min[t, s] \,. \tag{1.25}$$

Proof. This theorem is a very elementary application of the independent increment property of Poisson or Markov jump processes, also demonstrating how application of independent increments rely on the zero mean properties. For the Poisson process, unlike the standardized diffusion process, the zero mean property comes from using the Poisson deviation or centered Poisson processes $\widehat{P}(t) = P(t) - \lambda t$ where $\mathrm{E}[P(t)] = \lambda t$, such that $\mathrm{E}[\widehat{P}(t)] = 0 = \mathrm{E}[\widehat{P}(s)]$. First consider the case $s < t$ and write

$$\begin{aligned}\widehat{P}(t) &= \widehat{P}(s) + (\widehat{P}(t) - \widehat{P}(s)) = \widehat{P}(s) + ((P(t) - P(s)) - \lambda(t-s)) \\ &= \widehat{P}(s) + ((P(t) - P(s)) - \mathrm{E}[(P(t) - P(s))]) \,,\end{aligned}$$

i.e., as independent increments and using the time increment $\Delta t = t - s$, noting that subtracting the deterministic term λt preserves the independent increment property. Then

$$\begin{aligned}\mathrm{Cov}[P(t), P(s)] &= \mathrm{E}[\widehat{P}(t)\widehat{P}(s)] = \mathrm{E}[\widehat{P}^2(s) + \widehat{P}(s)((P(t) - P(s)) - \mathrm{E}[(P(t) - P(s))])] \\ &= \mathrm{E}[\widehat{P}^2(s)] + \mathrm{E}[\widehat{P}(s)((P(t) - P(s)) - \mathrm{E}[(P(t) - P(s))])] \\ &= \mathrm{Var}[P(s)] + \mathrm{E}[\widehat{P}(s)]\mathrm{E}[((P(t) - P(s)) - \mathrm{E}[(P(t) - P(s))])] \\ &= \lambda s + 0 \cdot 0 = \lambda s \,,\end{aligned}$$

using the linearity of the expectation operator (0.9), the definition of the variance (0.10) together with the independence of the expectations (0.78) for independent increments $\widehat{P}(s)$ and $((P(t) - P(s)) - \mathrm{E}[(P(t) - P(s))])$, and finally that $\widehat{P}(s)$, with $P(s)$, has variance λs (1.21). In the case $t < s$, then $\mathrm{Cov}[P(t), P(s)] = \lambda t$ by symmetry, and both cases together produce the conclusion $\mathrm{Cov}[P(t), P(s)] = \lambda \min[s, t]$. ▯

1.7 More Poisson Process Moments

The expectations for the integer powers of the Poisson increment follow from the mean over the Poisson distribution (1.22) and summed by differentiation of the exponential series (0.53):

Lemma 1.14. ***Poisson Sums by Differentiation:***

$$\begin{aligned}\mathrm{E}[(\Delta P)^m(t)] &= e^{-\lambda\Delta t} \sum_{k=0}^{\infty} \frac{(\lambda\Delta t)^k k^m}{k!} &(1.26\text{a}) \\ &= \left[e^{-u} \left(u \frac{d}{du} \right)^m e^u \right] \bigg|_{u=\lambda\Delta t} \,, &(1.26\text{b})\end{aligned}$$

for $m = 0, 1, 2, \ldots$.

The result (1.26b) can be shown by induction from the definition (1.26). Either the direct summation form (1.26) or the differentiation form (1.26b) can be implemented by symbolic computation, for example the summation definition form can be coded in Maple as

Example 1.15. ***Poisson Moment Summations by Maple:***

\# ***Maple Functions for Poisson Moments and Poisson Deviation Moments:***

$> \mathrm{fpoisson} := (\mathrm{k},\mathrm{u}) -> \exp(-\mathrm{u}) * \mathrm{u}^{\mathrm{k}}/\mathrm{k}!;$

$$fpoisson := (k, u) \to \frac{e^{(-u)} u^k}{k!}$$

$> \mathrm{momentdp} := (\mathrm{n},\mathrm{u}) -> \mathrm{simplify}(\mathrm{sum}('\mathrm{k}^{\mathrm{n}}\mathrm{fpoisson}(\mathrm{k},\mathrm{u})','\mathrm{k}' = 0..\mathrm{infinity}));$

$$momentdp := (n, u) \to simplify\left(\sum_{'k'=0}^{\infty} 'k^n fpoisson(k, u)'\right)$$

$> \mathrm{momentdevdp} := (\mathrm{n},\mathrm{u}) -> \mathrm{simplify}(\mathrm{sum}('(\mathrm{k}-\mathrm{u})^{\mathrm{n}}\mathrm{fpoisson}(\mathrm{k},\mathrm{u})',$
$> '\mathrm{k}' = 0..\mathrm{infinity}));$

$$momentdevdp := (n, u) \to simplify\left(\sum_{'k'=0}^{\infty} '(k-u)^n fpoisson(k, u)'\right)$$

\# *Sample illustrations for 5th moment of both moment functions:*

$> \mathrm{mom5dp} := \mathrm{momentdp}(5, \mathrm{lambda} * \mathrm{dt});$

$$mom5dp := \lambda dt(1 + 15\lambda dt + 25\lambda^2 dt^2 + 10\lambda^3 dt^3 + \lambda^4 dt^4)$$

$> \mathrm{mom5devdp} := \mathrm{momentdevdp}(5, \mathrm{lambda} * \mathrm{dt});$

$$mom5dp := 10\lambda^2 dt^2 + \lambda dt$$

The results for the first few powers are summarized in Table 1.2: The second column of this table can be quickly calculated by recursion, since if $u = \lambda\Delta t$ and $g_m(u) = \mathrm{E}[(\Delta P)^m(t)]$, then it can be shown that $g_{m+1}(u) = u\cdot(g_m(u)+g'_m(u))$. See Exercise 6 on Page 105 for the asymptotic form of $\mathrm{E}[(\Delta P)^m(t)]$. The expectation of a general function, $\mathrm{E}[f(\Delta P(t))]$, in terms of an infinite series of the finite differences of $f(0)$, which terminates if $f(\Delta P(t))$ is an integer power of $\Delta P(t)$, is the topic of Exercise 7 on Page 105.

Table 1.2. *Some expected moments (powers) of Poisson increments and their deviations.*

m	$\mathrm{E}[(\Delta P)^m(t)]$	$\mathrm{E}[(\Delta P(t) - \lambda\Delta t)^m]$
0	1	—
1	$\lambda\Delta t$	0
2	$\lambda\Delta t(1 + \lambda\Delta t)$	$\lambda\Delta t$
3	$\lambda\Delta t(1 + 3\lambda\Delta t + (\lambda\Delta t)^2)$	$\lambda\Delta t$
4	$\lambda\Delta t(1 + 7\lambda\Delta t + 6(\lambda\Delta t)^2 + (\lambda\Delta t)^3)$	$\lambda\Delta t(1 + 3\lambda\Delta t)$
5	$\lambda\Delta t(1 + 15\lambda\Delta t + 25(\lambda\Delta t)^2 + 10(\lambda\Delta t)^3 + (\lambda\Delta t)^4)$	$\lambda\Delta t(1 + 10\lambda\Delta t)$

These tabulated results can be applied to other expected moments, for example,

$$\begin{aligned}\mathrm{Var}[\Delta P(t)] &= \mathrm{E}[(\Delta P(t) - \lambda\Delta t)^2] \\ &= \mathrm{E}[(\Delta P)^2(t)] - 2\lambda\Delta t\mathrm{E}[\Delta P(t)] + (\lambda\Delta t)^2\mathrm{E}[1] = \lambda\Delta t\,,\end{aligned}$$

upon expanding the square and using the linear property of the expectation. See the third column of Table 1.2. The results can also be applied to expected moments of Poisson differential process as an increment process, $dP(t) = P(t+dt) - P(t)$, by replacing Δt by dt and neglecting terms of $\mathrm{O}^2(dt)$ since they are treated as negligible compared to term of $\mathrm{ord}(dt)$, dt being infinitesimally small. Expected moments of $P(t)$ also follow by replacing Δt by t, except the higher powers of t would not be negligible compared to the first power, unless t is small.

Theorem 1.16. ***Approximate Zero-One Jump Law:***
As $\Delta t \to 0^+$ and $\lambda > 0$ as well as bounded, then

$$\mathrm{Prob}[\Delta P(t) = 0] = 1 - \lambda\Delta t + \mathrm{O}^2(\lambda\Delta t)\,, \tag{1.27}$$

$$\mathrm{Prob}[\Delta P(t) = 1] = \lambda\Delta t + \mathrm{O}^2(\lambda\Delta t)\,, \tag{1.28}$$

$$\mathrm{Prob}[\Delta P(t) > 1] = \mathrm{O}^2(\lambda\Delta t)\,, \tag{1.29}$$

$$\mathrm{Prob}[(\Delta P)^2(t) = \Delta P(t)] = 1 - \frac{1}{2}(\lambda\Delta t)^2 + \mathrm{O}^3(\lambda\Delta t)\,. \tag{1.30}$$

Proof. Taking the Poisson increment distribution (1.22) and expanding it asymptotically using primarily the exponential series expansion (0.53) for $\Delta t \ll 1$ yields,

$$\begin{aligned}\mathrm{Prob}[\Delta P(t) = 0] &= e^{-\lambda\Delta t} = 1 - \lambda\Delta t + \frac{1}{2}(\lambda\Delta t)^2 + \mathrm{O}^3(\lambda\Delta t)\,, \\ \mathrm{Prob}[\Delta P(t) = 1] &= e^{-\lambda\Delta t}\lambda\Delta t = \lambda\Delta t - (\lambda\Delta t)^2 + \mathrm{O}^3(\lambda\Delta t)\,, \\ \mathrm{Prob}[\Delta P(t) > 1] &= 1 - \mathrm{Prob}[\Delta P(t) = 0] - \mathrm{Prob}[\Delta P(t) = 1] \\ &= \frac{1}{2}(\lambda\Delta t)^2 + \mathrm{O}^3(\lambda\Delta t)\,.\end{aligned}$$

Since $\mathrm{O}^2(\lambda\Delta t) + \mathrm{O}^3(\lambda\Delta t) = \mathrm{O}^2(\lambda\Delta t)$, the first three equations are proved. The last equation (1.30) follows from the fact that $x^2 = x$ is only true if $x = 0$ or $x = 1$, so

$$\begin{aligned}\mathrm{Prob}[(\Delta P)^2(t) = \Delta P(t)] &= \mathrm{Prob}[\Delta P(t) = 0] + \mathrm{Prob}[\Delta P(t) = 1] \\ &= 1 - \mathrm{Prob}[\Delta P(t) > 1] = 1 - \frac{1}{2}(\lambda\Delta t)^2 + \mathrm{O}^3(\lambda\Delta t)\,.\end{aligned}$$

The significance of this result is that if $\lambda\Delta t$ is sufficiently small and terms of order $(\lambda\Delta t)^2$ can be neglected, then only jumps of zero or one are very likely, i.e., very probable. □

Remarks 1.17.

- *In some other texts, the three small Poisson increment properties, Eqs. (1.27, 1.28, 1.29), are used as an elementary definition of the simple Poisson process. Here, we have started at a higher level of definition to facilitate the use of the Poisson process in applications.*

- *Combining the asymptotic probability relations (1.28) for $\Delta P(t) = 1$ and (1.29) for $\Delta P(t) > 1$ leads to*

$$\mathrm{Prob}[\Delta P(t) > 1] \ll \mathrm{Prob}[\Delta P(t) = 1]$$

*when $\lambda\Delta t \ll 1$. This asymptotic relationship characterizes the **orderliness** property of Poisson process (see Snyder and Miller [247]).*

With this result, the corresponding results for differential Poisson processes follow. First, a definition to specify that the square of a differential as been neglected.

Definition 1.18. ***Equality to Precision-dt:***
Let $f(dt;x)$ and $g(x)$ be bounded functions for $dt \geq 0$ and parameter x. Write

$$f(dt;x) \stackrel{\mathrm{dt}}{=} g(x)dt \tag{1.31}$$

if

$$f(x,dt) = g(x)dt + \mathrm{o}(dt)$$

as $dt \to 0^+$ and fixed x.

Theorem 1.19. ***Zero-One Jump Law:***

Let $dt > 0$ and let λ be positive and bounded, then

$$\mathrm{Prob}[dP(t) = 0] \overset{\mathrm{dt}}{=} 1 - \lambda dt\,, \tag{1.32}$$

$$\mathrm{Prob}[dP(t) = 1] \overset{\mathrm{dt}}{=} \lambda dt\,, \tag{1.33}$$

$$\mathrm{Prob}[dP(t) > 1] \overset{\mathrm{dt}}{=} 0\,, \tag{1.34}$$

$$\mathrm{Prob}[(dP)^2(t) = dP(t)] \overset{\mathrm{dt}}{=} 1\,, \tag{1.35}$$

$$\mathrm{Prob}[(dP)^m(t) = dP(t)] \overset{\mathrm{dt}}{=} 1\,, m > 0\,. \tag{1.36}$$

Proof. The proof follows easily from the increment approximate Theorem 1.16 upon neglecting all terms $\mathrm{O}^2(\lambda\Delta t)$. The last equation in precision-dt (1.36) holds for the same reason that the prior equation (1.35) holds as long as $m > 0$. Note that $(dP)^m(t) = dP(t)$ is obviously valid for $dP(t) = 0$, but if $dP(t) \neq 0$ then division by $dP(t)$ is permissible so $(dP)^{m-1}(t) = 1$ and we must have $P(t) = 1$, one being the only real root in this real problem. The rules (1.32-1.36) will come in very handy for simplifying powers of $dP(t)$ in the Poisson jump calculus later in this text. ◻

This **zero-one jump law** immediately leads to the following corollary for Poisson differential distribution and expectations:

Corollary 1.20. ***Zero-One Distribution and Expectation:***

$$\Phi_{dP(t)}(k) = p_k(\lambda dt) \overset{\mathrm{dt}}{=} (1 - \lambda dt)\delta_{k,0} + \lambda dt \delta_{k,1}\,, \tag{1.37}$$

is a generalized representation of the differential Poisson distribution and

$$\mathrm{E}[f(dP(t))] \overset{\mathrm{dt}}{=} (1 - \lambda dt) f(0) + \lambda dt f(1)\,, \tag{1.38}$$

is the expectation, provided $f(p)$ is a bounded and continuous function.

The Poisson zero-one jump law is a special case of a **Bernoulli distribution**, concerning Bernoulli trials which have only two outcomes, here with failure probability $p = 1 - \lambda dt$ for zero jump or success probability $1 - p = \lambda dt$ for one jump, provided λdt is small compared to unity.

Properties 1.21. ***Temporal Poisson process:***

- *For the* ***temporal or non-stationary Poisson process*** *$P(t)$ the jump rate is time dependent, $\lambda = \lambda(t)$, so that $P(t)$ is no longer simple or stationary, but non-stationary. First consider the differential process $dP(t)$ replacing the simple Poisson jump-rate λdt by the time-dependent one,*

$$d\Lambda(t) \equiv \lambda(t) dt\,. \tag{1.39}$$

Letting $\Lambda(0) = 0$ *initially, then*

$$\Lambda(t) = \int_0^t \lambda(s)ds\,, \tag{1.40}$$

with increment

$$\Delta\Lambda(t) \equiv \Lambda(t+\Delta t) - \Lambda(t) = \int_t^{t+\Delta t} \lambda(s)ds\,. \tag{1.41}$$

Thus, $\Delta\Lambda(t) \sim \lambda(t)\Delta t$ *only when* $\Delta t \ll 1$*, i.e., is small.*

- *The temporal Poisson distribution for the differential Poisson process remains unchanged from the fixed jump rate Poisson, except for* $\lambda = \lambda(t)$ *and*

$$\begin{aligned}\Phi_{dP(t)}(k;\lambda(t)dt) &= \mathrm{Prob}[dP(t) = k]\\ &= p_k(\lambda(t)dt) = e^{-\lambda(t)dt}\frac{(\lambda(t)dt)^k}{k!}\,,\end{aligned} \tag{1.42}$$

for $k = 0, 1, 2, \ldots,$ *with* $t \geq 0$ *and temporal parameter* $\lambda(t) > 0$*.*

However, the ***Poisson distribution property*** *(1.21) of the Poisson process needs to be changed for the temporal increment process* $\Delta P(t)$ *(1.22) using the modified parameter* $\Delta\Lambda(t)$*,*

$$\begin{aligned}\Phi_{\Delta P(t)}(k;\Delta\Lambda(t)) &= \mathrm{Prob}[\Delta P(t) = k]\\ &= p_k(\Delta\Lambda(t)) = e^{-\Delta\Lambda(t)}\frac{(\Delta\Lambda(t))^k}{k!}\,,\end{aligned} \tag{1.43}$$

for $k = 0, 1, 2, \ldots,$ *with* $t \geq 0$*,* $\Delta t \geq 0$ *and temporal parameter* $\Delta\Lambda(t)$*. Thus, the temporal Poisson process is also a* ***time-inhomogeneous*** *process. The Poisson increment distribution is fundamental for the temporal Poisson process. Note that* $\Lambda(t)$ *will be nondecreasing if* $\lambda(t) > 0$ *and continuous.*

Finally, since the full temporal Poisson process $P(t)$ *is the increment* $P(t) - P(0) = P(t)$*, then it has the distribution*

$$\begin{aligned}\Phi_{P(t)}(k;\Lambda(t)) &= \mathrm{Prob}[P(t) = k]\\ &= p_k(\Lambda(t)) = e^{-\Lambda(t)}\frac{(\Lambda(t))^k}{k!}\,,\end{aligned} \tag{1.44}$$

inherited from (1.43).

- *The* ***non-stationary behavior*** *follows from the fact that the distribution of the increment (1.43) depends on* t *through the parameter* $\Delta\Lambda(t)$ *or more simply from the* ***Poisson increment expectation*** *given in (0.51) or Table 1.2 with* $\Delta\Lambda(t)$ *replacing the parameter* $\lambda\Delta t$*,*

$$\mathrm{E}[\Delta P(t)] = \Delta\Lambda(t)\,,$$

since it will be, in general, a function of time t. Thus,

$$\mathrm{E}[P(t)-P(t_0)] = \Lambda(t)-\Lambda(t_0) = \int_{t_0}^{t} \lambda(s)ds\,.$$

The ***Poisson increment variance*** *must be the same as its expectation (0.510.51–0.52),*

$$\mathrm{Var}[\Delta P(t)] = \Delta\Lambda(t).$$

However, treating the increment as an integral leads to another form

$$\begin{aligned}
\mathrm{Var}[\Delta P(t)] &= \mathrm{Var}\left[\int_t^{t+\Delta t} dP(s)\right] \\
&= \mathrm{E}\left[\left(\int_t^{t+\Delta t} dP(s) - \Delta\Lambda(t)\right)^2\right] \\
&= \mathrm{E}\left[\left(\int_t^{t+\Delta t} (dP(s) - \lambda(s)ds)\right)^2\right] \\
&= \mathrm{E}\left[\int_t^{t+\Delta t} (dP(s_1) - \lambda(s_1)ds_1)\int_t^{t+\Delta t} (dP(s_2) - \lambda(s_2)ds_2)\right] \\
&= \int_t^{t+\Delta t}\int_t^{t+\Delta t} \mathrm{E}\left[(dP(s_1) - \lambda(s_1)ds_1)(dP(s_2) - \lambda(s_2)ds_2)\right] \\
&= \int_t^{t+\Delta t}\int_t^{t+\Delta t} \mathrm{Cov}[dP(s_1), dP(s_2)]\,. \qquad (1.45)
\end{aligned}$$

Since

$$\mathrm{Var}[\Delta P(t)] = \Delta\Lambda(t) = \int_t^{t+\Delta t} \lambda(s)ds,$$

noting that $dP(s_1)$ and $dP(s_2)$ are independent increments as differentials as long as $s_2 \neq s_1$. So $\mathrm{Cov}[dP(s_1), dP(s_2)] \neq 0$ only if $s_2 = s_1$ when it has the value $\mathrm{Cov}[dP(s_1), dP(s_1)] = \mathrm{Var}[dP(s_1)]$. Consequently,

$$\mathrm{Cov}[dP(s_1), dP(s_2)] \stackrel{\mathrm{gen}}{=} \lambda(s_1)\delta(s_1-s_2)ds_1 ds_2 \qquad (1.46)$$

for arbitrary Δt, so the inner integral of (1.45) will be

$$\int_t^{t+\Delta t} \mathrm{Cov}[dP(s_1), dP(s_2)]\lambda(s_1)ds_1\,.$$

- *The* ***temporal Poisson differential process*** *distribution for $dP(t)$ to precision-dt is*

$$\begin{aligned}
\Phi_{dP(t)}(k; d\Lambda(t)) &= \mathrm{Prob}[dP(t) = k] \\
&= p_k(d\Lambda(t)) \stackrel{\mathrm{dt}}{=} (1-\lambda(t)dt)\delta_{k,0} + \lambda(t)dt\delta_{k,1}\,, \qquad (1.47)
\end{aligned}$$

which simply follows from (1.43) and the corresponding simple process result (1.37).

- *The increasing property of $\Lambda(t)$ ($d\Lambda(t) > 0$) means that it can be used as a substitute "clock" in place of t, but for $\Lambda(t)$ to be a full range clock it is necessary that $\Lambda(t)$ be unbounded, i.e., $\Lambda(t) \to +\infty$ as $t \to +\infty$. Let T_j be the jth jump time of the temporal $P(t)$ and $\Delta T_j \equiv T_{j+1} - T_j$ be the inter-jump time (also called the waiting time or inter-arrival time). Slightly modifying the arguments for the exponential distribution of ΔT_j for the stationary $P(t)$ in (1.24), the non-stationary distribution conditioned on the most recent jump time T_j is*

$$\begin{aligned}
\Phi_{\Delta T_j|T_j}(\Delta t) &\equiv \mathrm{Prob}[\Delta T_j \le \Delta t \mid T_j] \\
&= 1 - \mathrm{Prob}[\Delta T_j > \Delta t \mid T_j] \\
&= 1 - \mathrm{Prob}[\Delta P(T_j) \equiv P(T_j + \Delta t) - P(T_j) = 0 \mid T_j] \\
&= 1 - p_0(\Lambda(T_j + \Delta t) - \Lambda(T_j)) = 1 - p_0(\Delta\Lambda(T_j)) \\
&= 1 - e^{-\Delta\Lambda(T_j)} = 1 - \exp\left(-\int_{T_j}^{T_j+\Delta t} \lambda(t)dt\right) \\
&= \Phi_e(\Delta\Lambda(T_j); 1)\,,
\end{aligned}$$

where $\Phi_e(\Delta\Lambda(T_j); 1)$ is the exponential distribution (0.40) in $\Delta\Lambda(T_j)$ with mean $\mu = 1$, i.e., still exponentially distributed but distribution depends on T_j.

*Note that if $\Lambda(t)$ is finite, then $\Phi_{\Delta T_j|T_j}(\Delta t)$ as derived is **not a proper** probability distribution since $1 - \exp\left(-\int_{T_j}^{+\infty} \lambda(t)dt\right) < 1$ with $\Lambda(+\infty) < +\infty$).*

- *For more general properties see Snyder and Miller [247] for extended information or Çinlar [55].*

1.8 Poisson Process Expectations Conditioned on Past

Example 1.22. ***Illustration of Independent Increments and Markov Properties for Poisson Process:***

- $\mathrm{E}[P(t)|P(r), 0 \le r \le s] = P(\min[s,t]) + \lambda(t-s)H(t-s)$,
where $H(X)$ is the Heaviside step function (0.154). The techniques are similar to those for the Wiener process, except that there is no zero mean, but the mean increment is the same as the increment variance, i.e., $\mathrm{E}[\Delta P(t)] = \lambda\Delta t =$

$\mathrm{Var}[\Delta P(t)]$. *Also, $P(0)$ is zero by definition with probability one.*

$$\begin{aligned}
\mathrm{E}[P(t)|P(r), 0 \le r \le s] &= \left\{ \begin{array}{l} P(t),\ 0 \le t \le s \\ \mathrm{E}[(P(t)-P(s)) + (P(s)-P(0))|P(r), 0 \le r \le s], \\ \hfill 0 \le s < t \end{array} \right\} \\
&= \left\{ \begin{array}{l} P(t),\ 0 \le t \le s \\ \lambda(t-s) + P(s),\ 0 \le s < t \end{array} \right\} \\
&= P(\min[s,t]) + \lambda(t-s)H(t-s).
\end{aligned}$$

When $0 \le s < t$ then the above formula symmetrized using the Poisson deviation process, $(P(t) - \lambda t)$, having zero mean, with $H(t-s) = 1$ for $s < t$, has the form

$$\mathrm{E}[P(t) - \lambda t|P(r), 0 \le r \le s] = P(s) - \lambda s, \tag{1.48}$$

signifies that for the deviation the average information conditioned on the past data, $\{P(r), r \in [0, s]\}$, is given by the most recent past deviation $P(s) - \lambda s$, which may imply a significant reduction in uncertainty for the present data, $P(t)$.

*The form of the result (1.48) is again the principal characteristic form for a **martingale** as was (1.18) with $X(t) = f(P(t))$ or (1.18) for $W(t)$, i.e., an abstract model of a fair game (see the beginning preliminary chapter of Mikosch [205] for a clear description of martingales, but in an elementary abstract presentation; martingales will be described at the end of this book).*

- $\mathrm{E}[P^2(t)|P(r), 0 \le r \le s] = P^2(\min[s,t]) + \lambda(t-s)(1+2P(s)+\lambda(t-s))H(t-s)$. *The derivation is similar to that for the conditional mean above.*

$$\begin{aligned}
\mathrm{E}[P^2(t)|P(r), 0 \le r \le s] &= \left\{ \begin{array}{l} P^2(t),\ 0 \le t \le s \\ \mathrm{E}[((P(t)-P(s)) + (P(s)-P(0)))^2|P(r), 0 \le r \le s], \\ \hfill 0 \le s < t \end{array} \right\} \\
&= \left\{ \begin{array}{l} P^2(t),\ 0 \le t \le s \\ \mathrm{E}[(P(t)-P(s))^2] + 2P(s)\mathrm{E}[(P(t)-P(s))] + P^2(s), \\ \hfill 0 \le s < t \end{array} \right\} \\
&= \left\{ \begin{array}{l} P^2(t),\ 0 \le t \le s \\ \lambda(t-s)(1+\lambda(t-s)) + 2P(s)\cdot\lambda(t-s) + P^2(s), \\ \hfill 0 \le s < t \end{array} \right\} \\
&= P^2(\min[s,t]) + \lambda(t-s)(1+2P(s)+\lambda(t-s))H(t-s).
\end{aligned}$$

Table 1.2 has to be used for $\mathrm{E}[(\Delta P)^2(s)]$ with $\Delta t = (t-s)$

Similar to the techniques used previously for the Wiener process with conditioning on the past, the general technique for powers $P^m(t)$, when $s < t$ with conditioning on $P(s)$, is to use the decomposition into independent increments $P(t) = P(s) + (P(t) - P(s))$ and then expand the power of m by the binomial expansion (0.148)

$$(P(s) + (P(t) - P(s)))^m = \sum_{k=0}^{m} \left(\begin{array}{c} m \\ k \end{array} \right) P^k(s)(P(t) - P(s))^{m-k},$$

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and then use independence of the increments and conditioning to calculate for each term,

$$\mathrm{E}\left[\left(\begin{array}{c} m \\ k \end{array}\right) P^k(s)(P(t)-P(s))^{m-k}\middle| P(r), 0 \le r \le s\right]$$
$$= \left(\begin{array}{c} m \\ k \end{array}\right) P^k(s)\mathrm{E}\left[(P(t)-P(s))^{m-k}\right],$$

relying on Table 1.2 for the remaining expectations.

1.9 Exercises

1. Show formally that

$$\phi_{dW(t)}(w) \stackrel{\text{gen}}{=} \delta(w) + \frac{1}{2}dt\delta''(w)\,, \tag{1.49}$$

i.e., has a **delta-density** in the generalized sense, by showing that

$$\mathrm{E}[f(dW(t))] = \int_{\infty}^{+\infty} \phi_{dW(t)}(w)f(w)dw \stackrel{\text{dt}}{=} f(0) + \frac{1}{2}dtf''(0)\,,$$

i.e., to precision-dt, neglecting terms $\mathrm{o}(dt)$. Also, show that the integral of the delta-density on the right hand side of (1.49) has the same effect as the integral of the left hand side. Assume that $f(w)$ is three times continuously differentiable and with $f(w)$ and its derivatives vanishing sufficiently at infinity.
{*Hint: Only a formal expansion of $f(w)$ should be needed here. The exponential properties of $\phi_{dW(t)}(w)$ ensure uniformity to allow expansion inside the integral, so that Laplace's or higher order asymptotic method should not be needed.*}

2. Let $\{t_i : t_{i+1} = t_i + \Delta t_i, i = 0 : n, t_0 = 0; t_{n+1} = T\}$ be a variably-spaced partition of the time interval $[0, T]$ with $\Delta t_i > 0$. Show the following properties and justify by giving a reason for every step, such as a property of the process or a property of expectations.

   (a) Let $\Delta W(t_i) \equiv W(t_i + \Delta t_i) - W(t_i)$, then show

   $$\mathrm{Cov}[\Delta W(t_i), \Delta W(t_j)] = \Delta t_i\,\delta_{i,j}\,,$$

   for $i, j = 0 : n$, where $\delta_{i,j}$ is the Kronecker delta (0.54).

   (b) Let $\Delta P(t_i) \equiv P(t_i + \Delta t_i) - P(t_i)$, then show

   $$\mathrm{Cov}[\Delta P(t_i), \Delta P(t_j)] = \lambda\Delta t_i\,\delta_{i,j}\,,$$

   for $i, j = 0 : n$, where λ is a fixed jump rate.

(c) Again let $\Delta W(t_i) \equiv W(t_i+\Delta t_i)-W(t_i)$, but $\Delta^\theta W(t_i) \equiv W(t_i+\theta\Delta t_i)-W(t_i)$ with $0<\theta<1$, then show

$$\mathrm{Cov}[\Delta W(t_i), \Delta^\theta W(t_j)] = \theta\Delta t_i\,\delta_{i,j}\ ,$$

for $i,j=0:n$.

3. (a) Verify the $m=3:4$ entries in Table 1.1 of the text for $\mathrm{E}[|\Delta W(t)|^m]$.
   (b) Verify the $m=3:4$ entries in Table 1.2 of the text for $\mathrm{E}[(\Delta P(t))^m]$ and $\mathrm{E}[(\Delta P(t)-\lambda\Delta t)^m]$.

4. (a) Show that when $0\le s\le t$,

$$\mathrm{E}[W^3(t)|W(r), 0\le r\le s] = W^3(s)+3(t-s)W(s)\ ,$$

justifying every step with a reason, such as a property of the process or a property of conditional expectations.

   (b) Use this result to verify the martingale form (1.18)

$$\mathrm{E}[W^3(t)-3tW(t)|W(r), 0\le r\le s] = W^3(s)-3sW(s)\ .$$

{*Hint: The general technique is to seek the expectation of mth power in the separable form,*

$$\mathrm{E}[M_m(W(t),t)|W(r), 0\le r\le s] = M_m(W(s),s)\ ,$$

where

$$M_m(W(t),t) = W^m(t)+\sum_{k=0}^{m-1}\alpha_k(t)W^k(t)\ ,$$

satisfied for the sequence of coefficient functions $\{\alpha_0(t),\ldots,\alpha_{m-1}(t)\}$ *for the separable form, so that the conditional expectations of the lower order powers*

$$\mathrm{E}[W^k(t)|W(r), 0\le r\le s]$$

can be recursively obtained in the order $k=0:m-1$.}

5. (a) Show that when $0\le s\le t$,

$$\mathrm{E}[W^4(t)|W(r), 0\le r\le s] = W^4(s)+6(t-s)W^2(s)+3(t-s)^2\ ,$$

justifying every step with a reason, such as a property of the process or a property of conditional expectations.

   (b) Use this result to verify the martingale form (1.18)

$$\mathrm{E}[W^4(t)-6tW^2(t)+3t^2|W(r), 0\le r\le s] = W^4(s)-6sW^2(s)+3s^2\ ,$$

together with the form for similar conditional expectation of $W^2(t)$ or that for $W^2(t)-t$.

{*See the Hint in Exercise 4 above.*}

6. Show that
$$E[(\Delta P)^m(t)] = \lambda\Delta t(1 + \mathrm{O}(\lambda\Delta t)) \tag{1.50}$$
for $\lambda\Delta t \ll 1$, by induction for $m \geq 1$.

7. Show that for the Poisson increment process, $\Delta P(t)$, the expectation can be expanded as
$$\mathrm{E}[f(\Delta P(t))] = \sum_{k=0}^{\infty} \frac{(\lambda\Delta t)^k}{k!}\Delta^k[f(0)]\,,$$
assuming that $f(p)$ is a bounded function so that the sum converges. The kth order finite difference is defined inductively such that
$$\Delta^{k+1}[f(i)] \equiv \Delta[\Delta^k[f(i)]]$$
starting from $\Delta^0[f(i)] = f(i)$ and $\Delta^1[f(i)] = \Delta[f(i)] \equiv f(i+1) - f(i)$.
{*Hint: Use the zero-step* $I_0[f(i)] \equiv f(i)$ *and one-step* $I_1[f(i)] \equiv f(i+1)$ *operators, so that* $\Delta = I_1 - I_0$ *and* $\Delta^k = (I_1 - I_0)^k$*, for which the binomial expansion can be used.*}

8. Show that the incremental, temporal Poisson distribution, $p_k(\Delta\Lambda(t))$, satisfies the following differential-difference equation (DDE),
$$\frac{d}{dt}\left[p_k(\Delta\Lambda(t))\right] = \lambda(t)\left(p_k(\Delta\Lambda(t)) - p_{k-1}(\Delta\Lambda(t))\right)\,, \tag{1.51}$$
i.e., differential in t, but difference equation in k.

9. Show the following characteristic or moment generating function formulas, (you may assume that $\theta = \sqrt{-1}\,\zeta$ in the case of characteristic functions can be treated like a real variable):
   (a) for the *Gaussian process*, $G(t) = \mu_0 + \sigma_0 W(t)$, with constant coefficients,
   $$C[G](\theta;t) \equiv \mathrm{E}[\exp(\theta G(t))] = \exp\left(\theta\left(\mu_0 + \theta\sigma_0^2 t/2\right)\right)\;;$$
   (b) for the Poisson process, $\nu_0 P(t)$, with constant jump rate λ_0 and jump amplitude ν_0,
   $$C[\nu_0 P](\theta;t) \equiv \mathrm{E}[\exp(\theta\nu_0 P(t))] = \exp\left(\lambda_0 t\left(\exp(\theta\nu_0) - 1\right)\right)\;;$$
   (c) and finally for the jump-diffusion process with constant coefficients $X(t) = \mu_0 + \sigma_0 W(t) + \nu_0 P(t)$,
   $$C[X](\theta;t) \equiv \mathrm{E}[\exp(\theta X(t))] = \exp\left(\theta\left(\mu_0 + \theta\sigma_0^2 t/2\right) + \lambda_0 t\left(\exp(\theta\nu_0) - 1\right)\right)\,.$$

10. (a) Show that when $0 \leq s < t$ (*see the general result in section 1.8, but verify independently this special result*) that
   $$\mathrm{E}[P^2(t)|P(r), 0 \leq r \leq s] = P^2(s) + 2\lambda(t-s)P(s) + \lambda(t-s)(1 + \lambda(t-s))\,.$$

(b) Find the time polynomials $\alpha_1(t)$ and $\alpha_2(t)$ so that

$$MP_2(t) = P^2(t) + \alpha_1(t)P(t) + \alpha_2(t)$$

is a martingale.

11. (a) Show that when $0 \le s < t$ that

$$\begin{aligned}
&\mathrm{E}[P^3(t)|P(r), 0 \le r \le s] \\
&= P^3(s) + 3\lambda(t-s)P^2(s) + 3\lambda(t-s)(1+\lambda(t-s))P(s) \\
&\quad +\lambda(t-s)(1+3\lambda(t-s)+\lambda^2(t-s)^2)\,,
\end{aligned}$$

justifying every step with a reason, such as a property of the process or a property of conditional expectations.

(b) Use this result to verify the martingale form (1.18)

$$\begin{aligned}
&\mathrm{E}[P^3(t) - 3\lambda t P^2(t) - 3\lambda t(1-\lambda t)P(t) - \lambda t(1 - 3\lambda t + \lambda^2 t^2)|P(r), 0 \le r \le s] \\
&= P^3(s) - 3\lambda s P^2(s) - 3\lambda s(1-\lambda s)P(s) - \lambda s(1 - 3\lambda s + \lambda^2 s^2).
\end{aligned}$$

{*Hint: See the Hint in Exercise 4 in this section for* $W^3(t)$ *conditional expectation.*}

Suggested References for Further Reading

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