Applied Stochastic Processes and Control for Jump-Diffusions: Modeling, Analysis and Computation

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Chapter 2 Diffusions

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Chapter 2

Stochastic Integration for Diffusions

> *My major aim in this was to find facts which would guarantee as much as possible the existence of atoms of definite finite size.*
> —Albert Einstein (1879-1955) in the first of four "Annus Mirabilis" papers in the Annals der Physik during 1905, concerning Brownian motion.

> *Brownian motion, as described by Bachelier in 1900 and Einstein in 1905, was provided a rigorous mathematical definition by Wiener (1984-1964) in Wiener (1923, 1930) by proving the existence of an appropriate measure on a space of functions-of-time.*
> —Harry M. Markowitz in the forward to [241].

Jump-diffusion stochastic differential equations (SDEs) with initial conditions are of the form,

$$dX(t) = f(X(t),t)dt + g(X(t),t)dW(t) + h(X(t),t)dP(t)\ ,\ \ X(0) = x_0\ , \tag{2.1}$$

where the Poisson process $dP(t)$ supplies the jumps and the Wiener process $dW(t)$ supplies the diffusion. Initial value problem (2.1), unlike the ordinary differential equations (ODEs) with initial conditions, are symbolic equations. They are not fully defined until the method of integration for solving a stochastic differential equation (SDE) is specified, given the coefficient functions $\{f(x,t), h(x,t), g(x,t)\}$. More precisely, the SDE (2.1) is not fully specified until the methods of integration for the three types of integrals in the formal integral solution,

$$X(t) = x_0 + \int_0^t f(X(s),s)ds + \int_0^t g(X(s),s)dW(s) + \int_0^t h(X(s),s)dP(s)\ , \tag{2.2}$$

with respect to t, $W(t)$ and $P(t)$, respectively, have been defined. Until then, the stochastic integral equation or SIE (2.2) is as symbolic as the SDE in (2.1), since the evaluation of the second and third integrals in (2.2) is very sensitive to the method of integration used due to the random and singular properties of $dW(t)$ and $dP(t)$. It will be necessary to re-examine the foundations for ordinary or Riemann integration to motivate the inclusion of integrands with randomness, non-smoothness and jump discontinuities contributed by the stochastic processes $W(t)$ and $P(t)$ to the state process $X(t)$. This re-examination of integration will also be useful for subsequent numerical approximations of the new definitions as well as providing a basis for new types of integrals that will arise.

In this chapter, the integrals of the second type in (2.2), i.e.,

$$\int_0^t g(X(s),s)dW(s)\ ,$$

where the integration is with respect to the diffusion process $W(t)$, will be treated primarily. However, the short treatment of ordinary integration will be sufficient for integrals of the first type, i.e.,

$$\int_0^t f(X(s),s)ds$$

where the integration is with respect to the time t and the stochastic process $X(t)$ is only in the integrand. The third type of integral will be treated in the next chapter.

When considering higher approximations or other difficult behavior in the numerical solution of ordinary differential equations, it is often necessary to work with the corresponding integral equation. Similarly, the proper form for solving stochastic differential equations (which can be, in general, considered as a symbolic concept anyway) is the exact and numerical analysis of the corresponding stochastic integral equation.

Once the foundations for stochastic integrations have been made, as they would be for ordinary integration in a good calculus course, and the definition is illustrated for few simple examples, then some simpler formal chain rules will be developed that will make calculations of integral, where possible, much easier. This chapter on stochastic integration of diffusions, and a similar one on jumps that follows, presents the basis for the stochastic differential equation models of this book. Although the level of analysis is much higher than would be expected for an applied text, it is important to have a good reference source when treating new types of problems that do not fit the current models or theories to facilitate the modification of the current theories.

2.1 Ordinary or Riemann Integration

The theory of ordinary or Riemann integration is quickly reviewed as an intermediate step to build up the treatment of stochastic integration. Let the ordinary integral be symbolically defined as

$$I[f](t) = \int_0^t f(s)ds \; , \tag{2.3}$$

where $f(t)$ is a continuous function on $0 \le t \le T$, but continuity is really more than what would be needed in general here. For general functions f, the integral interval $[0, t]$ is partitioned into $n+1$ subintervals, $[t_i, t_{i+1}]$ of width $\Delta t_i \equiv t_{i+1} - t_i > 0$ for $i = 0 : n$, i.e., a grid of $n + 2$ points such that

$$0 = t_0 < t_1 < t_2 < \cdots < t_n < t_{n+1} = t \; . \tag{2.4}$$

On each subinterval an approximation point $t_i^* \equiv t_{i+\theta_i} \equiv t_i + \theta_i \Delta t_i$ is selected with $0 \le \theta_i \le 1$ provided that the θ_is are chosen so that the t_is are distinct as in (2.4), and the area on the subinterval is approximated by the simplest geometry, a rectangle of width Δt_i and height $f_i^* \equiv f_{i+\theta_i} \equiv f(t_{i+\theta_i})$, with area $f(t_{i+\theta_i})\Delta t_i$. Next let the grid size be specified as $\delta t_n \equiv \max_{i=0:n}[\Delta t_i]$ such that $\delta t_n \to 0^+$ as $n \to \infty$ to insure that all subintervals shrink to zero in the limit as $n \to \infty$. Finally, let

$$I_n^{(\theta_i)}[f](t) \equiv \sum_{i=0}^{n} f_{i+\theta_i} \Delta t_i \tag{2.5}$$

be the discrete approximation of the integral and define constructively the **Riemann integral** as

$$I[f](t) = \lim_{\substack{n\to\infty \\ \delta t_n \to 0}} \left[I_n^{(\theta_i)}[f](t) \right] \; , \tag{2.6}$$

provided the limit exists. It is important to note that the limit is independent of θ_i, $0 \le \theta_i \le 1$.

Usually, only a constant value of θ_i is used in practice, so let $\theta_i = \theta$. Also, for simplicity, the grid partition will be assumed to be evenly spaced, so that $\Delta t_i = \Delta t$, with nodes starting at t_0 and successive nodes at $t_{i+1} = t_i + \Delta t$, but integrand approximation points at $t_{i+\theta} = t_i + \theta \Delta t$, for $i = 0 : n$. Also, $t_i = i * \Delta t$ for $i = 0 : (n + 1)$. Since the step size is constant, then

$$\delta t_n = \Delta t = (t_{n+1} - t_0)/(n + 1) = t/(n + 1) \to 0^+ \; ,$$

as $n \to +\infty$, so the extra condition that $\delta t_n \to 0^+$ is not needed.

Fortunately, the limiting definition (2.6) does not have to be used much in ordinary calculus, but the **Riemann sum** (2.5) can be used for simply numerically approximating integrals. When $\theta = 0$ and $t_{i+\theta} = t_i$, the left hand endpoint of the ith subinterval, the numerical forward integration rule is called the **left rectangular rule** or **Euler's explicit method** or tangent-line method for ordinary differential equations. When $\theta = 1$ and $t_{i+\theta} = t_{i+1}$, the right hand endpoint of the ith subinterval, the numerical backward integration rule is called the **right rectangular rule** or implicit **backward Euler's method** for ordinary differential equations. When $\theta = 1/2$ and $t_{i+\theta} = (t_i + t_{i+1})/2$, the midpoint of the ith subinterval, the numerical integration rule is called the **midpoint rectangular rule**, more accurate by an order of magnitude in δt_n provided $f(t)$ is sufficiently differentiable.

Since the process $W(t)$ is continuous with probability one, then integrals of composite functions $f(W(t),t)$ with respect to t can be defined by Riemann integration, i.e.,

$$\int_0^t f(W(s),s)ds = \lim_{n\to\infty}\left[\sum_{i=0}^n f(w(t_i),t_i)\Delta t_i\right] , \tag{2.7}$$

choosing $\theta = 0$ here, though other values would be suitable. Similarly, when the integrand is for the composite process $X(t)$ with implied dependence on the diffusion $W(t)$ and also the jump process $P(t)$ through (2.2), the integral will be defined by Riemann integration, i.e.,

$$\int_0^t f(X(s),s)ds = \lim_{n\to\infty}\left[\sum_{i=0}^n f(X(t_i),t_i)\Delta t_i\right] . \tag{2.8}$$

The Poisson jump process, while discontinuous, is right continuous with left limits, i.e., it is also a piece-wise continuous step function, so fits nicely in the framework of the use of forward integration, which is effectively a sequence of step-function approximations. However, the jumps are stochastic and not predictable, though once a jump is generated through simulation or observation, it will be known.

Sometimes, a deterministic integration is needed with respect to the position on the path $x(t)$. In this case, let the $f(s)ds$ in (2.3) be replaced by $f(x(s),s)dx(s)$, which could also come from the form $f(x(s),s)x'(s)ds$ provided the velocity $v(s) = x'(s)$ or $dx(s) = x'(s)ds$ exists, then this leads to the **Stieltjes integral**, or Riemann-Stieltjes integral, constructive definition:

$$\int_0^t f(x(s),s)dx(s) = \lim_{n\to\infty}\left[\sum_{i=0}^n f(x(t_{i+\theta}),t_{i+\theta})(x(t_{i+1})-x(t_i))\right] , \tag{2.9}$$

provided $x(t)$ is continuous and has bounded variation [164], i.e.,

$$\sum_{i=0}^n |x(t_{i+1})-x(t_i)| < B ,$$

for some constant $B > 0$ for all partitions (2.4) of $[0,t]$ and $f(x(t),t)$ is continuous. (These conditions are stronger than needed and Mikosch [205] gives weaker but more complicated conditions.) Another example is the Stieltjes form for the expectation in terms of the probability distribution $\Phi_X(x)$ in the random variable X,

$$\mathrm{E}_X[f(X)] = \int_{-\infty}^{\infty} f(x)d\Phi_X(x) ,$$

sometimes used to permit the use of more general distributions than would be possible under the usual Riemann integration conditions. The Stieltjes integration form will be modified for the stochastic integration relative to $W(t)$ in the next section.

2.2 Stochastic Integration in $W(t)$: The Foundations

As in elementary calculus, the presentation starts with a fairly simple example. The integral that forms the basis for the formulation that follows is the stochastic Stieltjes integral

$$I[W](t) = \int_0^t W(s)dW(s)\ , \tag{2.10}$$

which have a stochastic correction for the simple deterministic calculus Stieltjes integral,

$$I^{((det))}[x](t) = \int_0^t x(s)dx(s) = \frac{1}{2}\int_0^t d(x^2)(s) = \frac{1}{2}\left(x^2(t) - x^2(0)\right). \tag{2.11}$$

This follows from the ordinary calculus chain rule, $d(x^2)(s) = 2x(s)dx(s)$, for differentials, to form an exact differential.

However, in the case of the stochastic integral (2.10), $W(t)$ is a random process, is nowhere differentiable and it can be shown to have unbounded variation. Note that for even spacing $\delta t_n = \Delta t = (t-0)/(n+1)$ for $i = 0 : n$, so that the expected variation, from Table 1.1, is

$$\mathrm{E}\left[\sum_{i=0}^{n}|\Delta W_i|\right] = \sum_{i=0}^{n}\sqrt{2\Delta t/\pi} = (n+1)\sqrt{2t/(\pi(n+1))} = \sqrt{2t(n+1)/\pi} \to +\infty\ ,$$

as $n \to +\infty$, so the variation must be unbounded since the expected variation must not exceed the supremum of the variation and the supremum must be unbounded as well. (See Mikosch [205] for another justification.)

In the first step in finding a constructive definition for the stochastic integral (2.10), with K. Itô [146], a left endpoint rectangular or forward integration rule ($\theta = 0$) is initially used to approximate the integral so that the independent increment property of $W(t)$ is preserved,

$$I_n^{(0)}[W](t) = \sum_{i=0}^{n} W(t_i)\Delta W(t_i) = \sum_{i=0}^{n} W_i \Delta W_i\ , \tag{2.12}$$

with W_i independent of ΔW_i as intended, where the simplifying numerical notations $W_i \equiv W(t_i)$ and $\Delta W_i \equiv \Delta W(t_i) \equiv W(t_{i+1}) - W(t_i)$ have been used. The form (2.12) is not too useful for summing or approximation, but the following two general identities are very useful:

Lemma 2.1. *Let $\{x_i | i = 0 : n+1\}$ be* ***any*** *sequence of numbers, and let $\Delta x_i = x_{i+1} - x_i$ for $i = 0 : n$, then*

$$\sum_{i=0}^{n}\Delta x_i = x_{n+1} - x_0\ , \tag{2.13}$$

$$\sum_{i=0}^{n} x_i \Delta x_i = \frac{1}{2}\left(x_{n+1}^2 - x_0^2 - \sum_{i=0}^{n}(\Delta x_i)^2\right)\ . \tag{2.14}$$

Proof. The first identity (2.13) is trivial, since adding two successive increments cancels the common value of those increments, i.e.,

$$\Delta x_i + \Delta x_{i+1} = (x_{i+1} - x_i) + (x_{i+2} - x_{i+1}) = x_{i+2} - x_i \ .$$

Verifying the second and important identity is much easier by expanding the summand on the right hand side of (2.14) to obtain the left hand side, than vice versa:

$$\begin{aligned}
\tfrac{1}{2}\left(x_{n+1}^2 - x_0^2 - \textstyle\sum_{i=0}^{n}(\Delta x_i)^2\right) &= \tfrac{1}{2}\left(x_{n+1}^2 - x_0^2 - \textstyle\sum_{i=0}^{n}(x_{i+1} - x_i)^2\right) \\
&= \tfrac{1}{2}\left(x_{n+1}^2 - x_0^2 - \textstyle\sum_{i=0}^{n}(x_{i+1}^2 - 2x_i x_{i+1} + x_i^2)\right) \\
&= \tfrac{1}{2}\left(x_{n+1}^2 - x_0^2 - \textstyle\sum_{i=0}^{n} x_{i+1}^2\right. \\
&\qquad \left. +2\textstyle\sum_{i=0}^{n} x_i x_{i+1} - \textstyle\sum_{i=0}^{n} x_i^2\right) \qquad (2.15) \\
&= \tfrac{1}{2}\left(x_{n+1}^2 - x_0^2 - \left(x_{n+1}^2 + \textstyle\sum_{j=0}^{n} x_j^2 - x_0^2\right)\right. \\
&\qquad \left. +2\textstyle\sum_{i=0}^{n} x_i x_{i+1} - \textstyle\sum_{i=0}^{n} x_i^2\right) \\
&= \textstyle\sum_{i=0}^{n} x_i \Delta x_i \ ,
\end{aligned}$$

where

$$\sum_{i=0}^{n} x_{i+1}^2 = \sum_{j=1}^{n+1} x_j^2$$

has been transformed by **change of index** to combine with a similar sum. □

The benefit of the form (2.14) when used as $x_i = W_i$, then the end points are explicitly given by $W_{n+1} = W(t)$ and $W_0 = 0$ with probability one, so the discrete approximation to stochastic integral of $W(t)$ becomes

$$I_n^{(0)}[W](t) = \frac{1}{2}(W^2(t) - \sum_{i=0}^{n}(\Delta W_i)^2) \ . \tag{2.16}$$

Using Table 1.1 again, the expectation of $I_n^{(0)}[W](t)$ is

$$\mathrm{E}\left[I_n^{(0)}[W](t)\right] = \frac{1}{2}(t - \sum_{i=0}^{n}\Delta t_i) = \frac{1}{2}(t - t) = 0 \ ,$$

returning to more general spacing Δt_i, where the (2.13) identity $\sum_{i=0}^{n}\Delta t_i = t_{n+1} - t_0 = t$ has also been used. This result suggests that a reasonable form for the stochastic integral (2.10) answer is

$$I[W](t) \overset{ims}{=} I^{(ims)}[W](t) = \frac{1}{2}(W^2(t) - t) \ , \tag{2.17}$$

where the term $(-\frac{1}{2}t)$ is the correction to the ordinary calculus or Riemann integration answer. However, since the proposed answer is not a true equality, another condition is appropriate for the stochastic nature of the problem and that condition is the mean square limit or mean square convergence:

Definition 2.2. ***Mean Square Limit or Convergence:***

The random variable $I_n^{(0)}(t)$ ***converges in the mean square*** *to the random variable* $I(t)$ *if*

$$\mathrm{E}\left[\left(I_n^{(0)}(t) - I(t)\right)^2\right] \to 0 \tag{2.18}$$

as $n \to \infty$, *assuming that both random variables have bounded mean squares, i.e.*

$$\mathrm{E}\left[(I_n^{(0)})^2(t)\right] < \infty \quad \text{and} \quad \mathrm{E}\left[I^2(t)\right] < \infty\,.$$

If the limit (2.18) exists, then denote the ***mean square limit*** *as*

$$I(t) = \overset{\text{ms}}{\lim_{n\to\infty}}\left[I_n^{(0)}(t)\right]\,.$$

Some related **general stochastic convergence principles**:

Definition 2.3. ***Convergence in Probability:***
The random variable $I_n^{(0)}(t)$ ***converges in probability*** *to the random variable* $I(t)$ *if for every* $\epsilon > 0$,

$$\mathrm{Prob}\left[\left|I_n^{(0)}(t) - I(t)\right| \ge \epsilon\right] \to 0 \tag{2.19}$$

as $n \to \infty$. *If the limit (2.19) exists, then denote the* ***limit in probability*** *as*

$$I(t) = \overset{\textit{prob}}{\lim_{n\to\infty}}[I_n^{(0)}(t)]\,.$$

Definition 2.4. ***Convergence in Mean:***
The random variable $I_n^{(0)}(t)$ ***converges in the mean*** *to the random variable* $I(t)$ *if for every* $\epsilon > 0$,

$$\mathrm{E}\left[\left|I_n^{(0)}(t) - I(t)\right|\right] \to 0 \tag{2.20}$$

as $n \to \infty$. *If the limit (2.20) exists, then denote the* ***limit in the mean*** *as*

$$I(t) = \overset{\text{mean}}{\lim_{n\to\infty}}[I_n^{(0)}(t)]\,.$$

Theorem 2.5. ***Convergence in Mean Square*** $\Longrightarrow$ ***Convergence in Probability:***

$$I(t) = \overset{\text{ms}}{\lim_{n\to\infty}}[I_n^{(0)}(t)] \quad \Longrightarrow \quad I(t) = \overset{\textit{prob}}{\lim_{n\to\infty}}[I_n^{(0)}(t)]\,. \tag{2.21}$$

Similarly:
Convergence in Mean Square $\Longrightarrow$ ***Convergence in Mean:***

$$I(t) = \overset{\text{ms}}{\lim_{n\to\infty}}[I_n^{(0)}(t)] \quad \Longrightarrow \quad I(t) = \overset{\text{mean}}{\lim_{n\to\infty}}[I_n^{(0)}(t)]\,. \tag{2.22}$$

Proof. Let $\epsilon > 0$. Tacitly the mean square expectation of the limit $I(t)$ and the approximation is assumed as conditions for mean square convergence, which implies that $E[|I(t)-I_n^{(0)}(t)|^2] \to 0^+$ as $n \to \infty$. The theorem follows from the **Chebyshev inequality** (0.187) of Exercise 4 on Page 71 which is written in a simplified but convenient form,

$$\text{Prob}[|X| \geq \epsilon] \leq \text{E}[|X|^2]/\epsilon^2\,, \tag{2.23}$$

where $\epsilon > 0$. Let $X = I(t) - I_n^{(0)}(t)$ and thus

$$\text{E}[|I(t) - I_n^{(0)}(t)|^2] \geq \epsilon^2 \text{Prob}[|I(t) - I_n^{(0)}(t)| \geq \epsilon]\,.$$

Hence, as $n \to \infty$, $\text{Prob}[|I(t) - I_n^{(0)}(t)| \geq \epsilon] \to 0^+$ by being squeezed from above by the mean square deviation as it goes to zero, i.e., $I_n^{(0)}(t) \to I(t)$ in probability if $I_n^{(0)}(t) \to I(t)$ in the mean square.

Similarly, the Schwartz (Cauchy-Schwartz) inequality (0.188) of Exercise 5 on Page 72, truncated to one variable,

$$\text{E}^2[X] \leq \text{E}[X^2]$$

can be used to show that convergence in the mean square implies convergence in the mean, i.e., $I_n^{(0)}(t) \to I(t)$ in the mean if $I_n^{(0)}(t) \to I(t)$ in the mean square. □

The expectation of the proposed random variable answer is

$$\text{E}\left[I[W](t)\right] = \text{E}\left[\frac{1}{2}(W^2(t) - t)\right] = \frac{1}{2}(t - t) = 0\,,$$

the same as for the approximation.

In order to focus on the crucial term and to simplify the demonstration of the mean square limit, which is conjectured to be t, consider the following lemma:

Lemma 2.6. *Let*

$$J_n^{(0)}(t) \equiv \sum_{i=0}^{n} (\Delta W_i)^2\,, \tag{2.24}$$

then

$$t = \overset{\text{ms}}{\lim_{n\to\infty}}[J_n^{(0)}(t)]\,. \tag{2.25}$$

Proof. The mean t of $J_n^{(0)}(t)$ is absorbed into the summation by (2.13) with $x_i = t_i$, the square of the mean square argument leads to a double sum which is separated into diagonal parts ($j = i$) and off-diagonal parts ($j \neq i$), allowing the splitting of the expectations using the independent increment property, so

$$
\begin{aligned}
\mathrm{E}\left[\left(J_n^{(0)}(t)-t\right)^2\right] &= \mathrm{Var}\left[J_n^{(0)}(t)\right] \\
&= \mathrm{E}\left[\left(\sum_{i=0}^{n}(\Delta W_i)^2 - t\right)^2\right] \\
&= \mathrm{E}\left[\left(\sum_{i=0}^{n}\left((\Delta W_i)^2 - \Delta t_i\right)\right)^2\right] \\
&= \mathrm{E}\left[\sum_{i=0}^{n}\left((\Delta W_i)^2 - \Delta t_i\right)\sum_{j=0}^{n}\left((\Delta W_j)^2 - \Delta t_j\right)\right] \\
&= \sum_{i=0}^{n}\mathrm{E}\left[\left((\Delta W_i)^2 - \Delta t_i\right)^2\right] \\
&\qquad + \sum_{i=0}^{n}\mathrm{E}\left[(\Delta W_i)^2 - \Delta t_i\right]\sum_{\substack{j=0\\ j\neq i}}^{n}\mathrm{E}\left[(\Delta W_j)^2 - \Delta t_j\right] \\
&= \sum_{i=0}^{n}\mathrm{Var}\left[(\Delta W_i)^2\right] + 0\cdot 0 = \sum_{i=0}^{n}\left(\mathrm{E}\left[(\Delta W_i)^4\right] - E^2\left[(\Delta W_i)^2\right]\right) \\
&= \sum_{i=0}^{n}\left(3(\Delta t_i)^2 - (\Delta t_i)^2\right) = 2\sum_{i=0}^{n}(\Delta t_i)^2\,,
\end{aligned}
$$

the last couple of steps relying on the results of Table 1.1. Since $\Delta t_i \leq \delta t_n = \max_j[\Delta t_j]$, then

$$
\mathrm{E}\left[\left(J_n^{(0)}(t)-t\right)^2\right] = 2\sum_{i=0}^{n}(\Delta t_i)^2 \leq 2\delta t_n\sum_{i=0}^{n}\Delta t_i = 2t\delta t_n \to 0
$$

as $n \to \infty$ showing that

$$
t = \underset{n\to\infty}{\overset{\mathrm{ms}}{\lim}}\,[J_n^{(0)}(t)]\,.
$$

Clearly both $J_n^{(0)}(t)$ and t have bounded mean squares for bounded t. Hence, $J_n^{(0)}(t) = I_n^{(0)}[dW](t)$, in our functional notation. ◻

Lemma 2.7.

$$
\frac{1}{2}\left(W^2(t) - t\right) = \underset{n\to\infty}{\overset{\mathrm{ms}}{\lim}}\left[I_n^{(0)}[W](t)\right] \tag{2.26}
$$

where $t < \infty$ *and*

$$I_n^{(0)}[W](t) = \sum_{i=0}^{n} W_i \Delta W_i \, .$$

Proof. Note that

$$\mathrm{E}[((W^2(t) - t)/2)^2] = \mathrm{E}[W^4(t) - 2tW^2(t) + t^2]/4 = (3t^2 - 2t^2 + t^2)/4 = t^2/2 \, ,$$

again using the convenient Table 1.1, so $(W^2(t) - t)/2$ has a bounded mean square so long at t is bounded. Similarly, one can show that $I_n^{(0)}[W](t)$ has a bounded mean square. The mean square convergence of $I_n^{(0)}[W](t)$ is obvious since $J_n^{(0)}(t)$ converged in the mean square to t and $J_n^{(0)}(t)$ is the only term that depends on the grid variable n. In fact,

$$\mathrm{E}\left[\left(I[W](t) - I_n^{(0)}[W](t)\right)^2\right] = \frac{1}{4}\mathrm{E}\left[\left(t - J_n^{(0)}(t)\right)^2\right] \to 0^+ \, ,$$

as $n \to \infty$, so converges for the same reason that $J_n^{(0)}(t)$ did in the mean square. This mean square relation follows due to the affine difference in forms $I_n^{(0)}[W](t) = (W^2(t) - J_n^{(0)}(t))/2$ in (2.16) with (2.24) and $I[W](t) \overset{ims}{=} (W^2(t) - t)/2$ in (2.17), no longer a proposed answer. In more general cases the decomposition of $I_n^{(0)}[W](t)$ will not be so simple as that between $I_n^{(0)}[W](t)$ and the part $J_n^{(0)}(t)$. □

Definition 2.8. *Denote the* ***Itô mean square (ims) limit stochastic integral*** *corresponding to the stochastic integral form*

$$I[g](t) = \int_{t_0}^{t} g(W(s), s) dW(s)$$

with associated forward integration (left rectangular rule or Euler's method) approximation

$$I_n^{(0)}[g](t) \equiv \sum_{i=0}^{n} g(W(t_i), t_i)(W(t_{i+1}) - W(t_i))$$

by

$$I^{(ims)}[g](t) = \underset{n \to \infty}{\overset{\mathrm{ms}}{\lim}} \left[I_n^{(0)}[g](t) \right] \tag{2.27}$$

where $0 \le t_0 \le t$, *assuming the integrand process* $g(W(t), t)$ *has a bounded mean integral of its square, i.e.,*

$$\mathrm{E}\left[\int_{t_0}^{t} g^2(W(s), s) ds\right] < \infty \, ,$$

and the grid partitioning satisfies

$$0 \leq t_0 < t_1 < \cdots < t_{n+1} = t \tag{2.28}$$

with

$$\delta t_n = \max_{i=0:n}[\Delta t_i \equiv t_{i+1} - t_i] \ll 1$$

as $n \to \infty$.

Provided the Itô mean square limit (2.27) exists,

$$I[g](t) \overset{ims}{=} I^{(ims)}[g](t)\ . \tag{2.29}$$

In addition, the definition holds, since the independent increments property remains valid in a more general case, namely, if the function g *depends on the past and present history of the Wiener process,*

$$\mathcal{W}(t) = \{W(r), 0 \leq r \leq t\}\ ,$$

i.e., $g = g(\mathcal{W}(t), t)$, *in which case,* g *is called* ***non-anticipatory*** *or* ***adapted*** *to the process set* $\mathcal{W}(t)$.

Remarks 2.9.

- *For most of the sequel, general functions with dependence on* $W(t)$ *and* t, *i.e.,* $g(W(t), t)$, *will be used in stochastic diffusion integrals, but the reader can easily extend results to functions of the type* $g(\mathcal{W}(t), t)$ *adapted to* $\mathcal{W}(t)$.
- *If the Itô mean square limit (2.27),*

  $$I_n^{(0)}[g](t) \to I[g](t) = I^{(ims)}[g](t)$$

  in the mean square *as* $n \to \infty$, *exists, then by Theorem 2.5*

  $$I_n^{(0)}[g](t) \to I[g](t)$$

  in probability *as* $n \to \infty$.
- *In our notation,* $I[g](t) = I^{(ims)}[g](t)$ *denotes the mean square limit of the Itô forward integration approximation* $I_n^{(0)}[g](t)$ *with* $\theta = 0$ *meaning that the integral* g *is evaluated at* t_i *on the* i*th step. They denote particular evaluations or approximations of the purely symbolic* $I[g](t)$ *integral representation which can also have other evaluations using other integration rules with values of* θ *or using other rules relying on non-rectangular approximations.*

Thus, summarizing the results for the crucial simple example when $g(W(t), t) = W(t)$ is the following theorem:

Theorem 2.10. ***Itô Fundamental Mean Square Stochastic Integrals:***

$$\int_0^t (dW)^2(s) \overset{ims}{=} t\ . \tag{2.30}$$

and

$$\int_0^t W(s)dW(s) \overset{ims}{=} I^{(ims)}[W](t) = \frac{1}{2}\left(W^2(t) - t\right) \ . \tag{2.31}$$

Sketch of Proof. Some more heuristic justification is given here.

- In ordinary deterministic integral calculus, the symbol $\int_0^t (dx)^2(s)$ would be considered nonsense, but in Itô stochastic integration the symbol

  $$\int_0^t (dW)^2(s) \overset{ims}{=} \underset{n\to\infty}{\overset{\text{ms}}{\lim}} \left[\sum_{i=0}^{n} (\Delta W)^2(t_i)\right] = t \ ,$$

  makes perfect sense, since the Itô mean square limit is well defined and leads to the Itô correction to the ordinary calculus rule for the differential of $x^2(t)$, i.e., $x(t)dx(t) = \frac{1}{2}d(x^2)(t)$.

- In fact, this leads to a corresponding symbolic **Itô mean square** version for differentials,

  $$(dW)^2(t) \underset{sym}{\overset{ims}{=}} dt \ . \tag{2.32}$$

  and

  $$W(t)dW(t) \underset{sym}{\overset{ims}{=}} \frac{1}{2}(d(W^2)(t) - dt). \tag{2.33}$$

  Formally, we might rewrite (2.33), assuming the symbol "$\underset{sym}{\overset{ims}{=}}$" denoted a commutative operation,

  $$d(W^2)(t) \overset{ims}{=} 2W(t)dW(t) + dt \ . \tag{2.34}$$

  Using the formal increment definition of the differential (1.3), $dW(t) \equiv W(t+dt) - W(t)$ or the alternate form $W(t+dt) = W(t) + dW(t)$, then a quick calculation leads to

  $$\begin{aligned} d(W^2)(t) &\equiv W^2(t+dt) - W^2(t) = (W+dW)^2(t) - W^2(t) \\ &= (W^2 + 2WdW + (dW)^2 - W^2)(t) \\ &\underset{sym}{\overset{ims}{=}} 2WdW(t) + dt \ , \end{aligned} \tag{2.35}$$

  using a little bit of algebra and the symbolic fact that $(dW)^2 \underset{sym}{\overset{ims}{=}} dt$, formally justifying (2.34), demonstrating a fast technique that would be useful when fast answers are needed.

□

Remarks 2.11.

- *The Itô mean square result symbolized by* $(dW)^2(t) \underset{sym}{\overset{ims}{=}} dt$ *represents a remarkable paradox, since the differential* $(dW)^2(t)$ *is deterministic because* dt *is deterministic, but* $dW(t)$ *is stochastic or random.*

- *In the deterministic continuously differential case, the corresponding quadratic of a differential,* $(dx)^2(t)$ *would be negligible relative to terms of order* dt*. If the integral of such a term were consider the limit of its finite difference approximation would be zero:*

$$\begin{aligned}
\int_0^t (dx)^2(s) &= \lim_{n\to\infty}\left[\sum_0^n (\Delta x_i)^2\right]\\
&= \lim_{n\to\infty}\left[\sum_0^n (x_{i+1}-x_i)\Delta x_i\right]\\
&= \lim_{n\to\infty}\left[\sum_0^n x_{i+1}\Delta x_i\right] - \lim_{n\to\infty}\left[\sum_0^n x_i\Delta x_i\right]\\
&= \lim_{n\to\infty}\left[I_n^{(1)}[x](t)\right] - \lim_{n\to\infty}\left[I_n^{(0)}[x](t)\right]\\
&= I[x](t) - I[x](t) = 0,
\end{aligned}$$

  since the regular integral of $\int_0^t x(s)dx(s)$ *is independent in the limit of the particular approximation parameter used, whether* $\theta = 1$ *or* $\theta = 0$ *as in the above final lines.*

- *See also Exercise 1 which is to demonstrate that the the density,* $\phi_{dW(t)}(w)$*, for* $dW(t)$ *is the sum of two delta functions in the generalized sense that considerably constrains functions of* $dW(t)$*.*

- *Computational confirmation of the Itô's fundamental mean square stochastic integrals is the subject of Exercise 3 for the* $(dW)^2(t)$ *integrand in (2.30) and Exercise 4 for the* $(WdW)(t)$ *integrand in (2.31). For example, Fig. 2.1 is an illustration of the computational confirmation of the Itô fundamental forward integration approximating sum*

$$\int_0^t (dW)^2(s) \overset{ims}{=} t \simeq \sum_{i=0}^n (\Delta W_i)^2\ ,$$

  with $n = 10^4$*. The confirmation is remarkable considering it is a pointwise comparison of the approximating sum with the exact Itô answer* t*, and not a demonstration of convergence in the Itô mean square limit. The sample size has to be sufficiently large, else the approximating sum tends away from the* t *answer due to the slope of the tangent line bias, that is also a feature of deterministic ODE applications of Euler's method.*

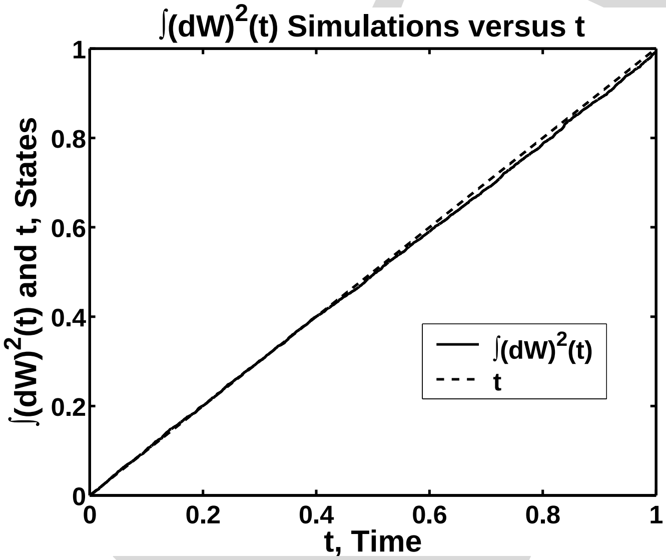

Figure 2.1. *Simulated sample path for the Itô forward integration approximating sum of* $\int(dW)^2(t) \overset{ims}{=} t \simeq \sum_i(\Delta W_i)^2$ *for* $n = 10^4$ *MATLAB* **randn** *sample size.*

- *The general code for simulating the stochastic diffusion integral with integrand* $g(W(t),t)$ *by the Itô forward integration approximation*

$$I[g](t) = \int_{t_0}^{t} g(W(s),s)ds \simeq \sum_{i=0}^{n} g_i \Delta W_i \ ,$$

in an abbreviated MATLAB fragment might be

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
function intdwdw
% Example MATLAB code for integral of (dW)^2.
clc % clear variables;
t0 = 0.0; tf = 1.0;
n = 1.0e+4; nf = n + 1; % set time grid: (n+1) subintervals
dt = (tf-t0)/nf;  %       and (n+2) points;
% replace these particular values according the application;
t(1) = t0; % set initial time at i = 1 for MATLAB;
W(1) = 0.0; % set initial diffusion noise condition;
sqrtdt = sqrt(dt); % dW(i) noise time scale so E[dW] = 0;
sumdw2(1) = 0.0; % set initial sum variable;
kstate = 1; randn('state',kstate); % Set randn state
%    for repeatability;
dW = sqrtdt*randn(nf,1); % simulate (n+1)-dW(i)'s sample;
```

```
t = t0:dt:tf; % get time vector t;
for i = 1:nf % simulate integral sample path.
    W(i+1) = W(i) + dW(i); % sum diffusion noise;
    sumdw2(i+1) = sumdw2(i) + (dW(i))^2; % sum whole integrand;
end
plot(t,sumdw2,'k-',t,t,'k--','LineWidth',2); % plot sum;
title('\int(dW)^2(t) Simulations versus t');
ylabel('\int(dW)^2(t) and t, States');
xlabel('t, Time');
legend('\int(dW)^2(t)','t',0);
% End Code
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
```

- *The form for the simulation of the Wiener increment process $\Delta W(t)$ by a standard normal distribution Z scaled by $\sqrt{\Delta t}$ in the above code fragment is based upon the following change of variables (or change of measure) result, showing that both $\Delta W(t)$ and $\sqrt{\Delta t}Z$ have the same distribution:*

Theorem 2.12. Wiener Simulations by Standard Normal:
Let Z be a random variable with a standard normal distribution, $\Phi_Z(z) = \Phi_n(z;0,1)$, then

$$\Phi_{\Delta W(t)}(w) = \Phi_{\sqrt{\Delta t}Z}(w)\,, \tag{2.36}$$

where $\Delta t > 0$.

Proof. *From properties of the normal distribution,*

$$\Phi_Z(z) = \mathrm{Prob}[Z \le z] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{z} e^{-y^2/2}dy$$

and

$$\begin{aligned}\Phi_{\Delta W(t)}(w) &= Prob[\Delta W(t) \le w] = \frac{1}{\sqrt{2\pi\Delta t}}\int_{-\infty}^{w} e^{-v^2/(2\Delta t)}dv \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{w/\sqrt{\Delta t}} e^{-y^2/2}dy = Prob\left[z \le w/\sqrt{\Delta t}\right] \\ &= \mathrm{Prob}\left[\sqrt{\Delta t}Z \le w\right] = \Phi_{\sqrt{\Delta t}Z}(w)\,,\end{aligned}$$

since $\mathrm{Prob}[aZ \le w] = \mathrm{Prob}[Z \le w/a]$ *provided* $a > 0$. ☐

- *See also the full version of this MATLAB code in Section A.11 of the Appendix A for actual type-set figure.*

- *See also Fig. 4.1 in Chapter 4 illustrating the application to* $g(W(t),t) = \exp(W(t)-t/2)$ *that yields an exact differential in the Itô mean square sense.*
- *Computational simulation is another way to get fast answers when they are needed.*

However, the Itô stochastic integration of exact differentials is easy as the following theorem shows.

Theorem 2.13. ***Fundamental Theorem of Itô Stochastic Diffusion Calculus:***
Let $g(w)$ *be continuous and* $G(w)$ *be continuously differentiable. Then*

(a)

$$d\left[\int_0^t g(W(s))dW(s)\right] \overset{ims}{=} g(W(t))dW(t) \tag{2.37}$$

and

(b)

$$\int_0^t dG(W(s)) \overset{ims}{=} G(W(t)) - G(0)\ , \tag{2.38}$$

for $0 \le t$.

Proof. The first part of the fundamental theorem (a) benefits from the Itô forward integration approximation and continuity of g, but mostly from the continuity of W. Consider the increment version for sufficiently small increments Δt,

$$\begin{aligned}\Delta\left[\int_0^t g(W(s))dW(s)\right] &= \left(\int_0^{t+\Delta t} - \int_0^t\right) g(W(s))dW(s) \\ &= \int_t^{t+\Delta t} g(W(s))dW(s) \\ &\simeq g(W(t))\Delta W(t) \\ &\to g(W(t))dW(t)\end{aligned}$$

as $\Delta t \to 0^+$, using the continuity of both g and W.

For second part of the fundamental theorem (b), using the Itô stochastic integration Definition 2.8,

$$\begin{aligned}\int_0^t dG(W(s)) &\overset{ims}{=} \overset{\text{ms}}{\lim_{n\to\infty}} \left[\sum_{i=0}^{n}(G(W(t_{i+1})) - G(W(t_i)))\right] \\ &= \overset{\text{ms}}{\lim_{n\to\infty}} \left[\sum_{i=0}^{n}(\Delta G(W(t_i)))\right] = \overset{\text{ms}}{\lim_{n\to\infty}} \left[G(W(t_{n+1})) - G(W(t_0))\right] \\ &= \overset{\text{ms}}{\lim_{n\to\infty}} \left[G(W(t)) - G(0)\right] = G(W(t)) - G(0)\ ,\end{aligned}$$

upon using the facts that $t_0 = 0$, $t_{n+1} = t$, and for any sum over all increments is the total increment from (2.13) of Lemma 2.1. Assuming that $G(W(t))$ is bounded on $[0, t]$ should be all that is needed. Thus, for exact derivatives, Itô stochastic integration and ordinary deterministic or Riemann integration agree. See Kolmogorov and Fomin [164] or Protter [228] about the importance of bounded variation as well, but these details are beyond the scope of this book. □

Remarks 2.14.

- *The first part (a) relates the integral to the differential formulation and the second part (b) is useful since it is one of the main ways of finding stochastic integrals which are not often found in closed form. Usually, part (b) is used to reduce a more complicated stochastic integral to a closed form plus a simpler, perhaps Riemann, integral.*
- *Note that in the proof of part (a), there is a difference in the exact* ***increment of an integral*** *and its approximate increment for small Δt. Using a more general form in some process $X(t)$ for the integral, the exact increment has the form*

$$\Delta[I[G]](t) \equiv I[G](t + \Delta t) - I[G](t) = \int_t^{t+\Delta t} G(X(s), s) dX(s)$$

that holds for arbitrary Δt as long as the integral can be defined, while the approximate integral has the form

$$\Delta[I[G]](t) \simeq G(X(t), t)\Delta X(t) \ ,$$

for sufficiently small Δt. So which form is used in an application depends on the application and the size of the time increment Δt.

When dealing with Itô stochastic integrals more general functions of Markov stochastic processes such as $g(W(t))$, $g(W(t), t)$ or $g(X(t), t)$, where $X(t)$ may itself be a stochastic process that is a functional of $W(t)$ and also $P(t)$, some more information may be needed. In particular, some more assumptions or some theorems beyond the scope of this applied book may be be needed to demonstrate the mean square convergence of the stochastic integrals. Typically, the usual assumptions [13, 161, 205, 228] require that the integrand function, say $Y(t) = g(X(t), t)$, has a piece-wise-constant, right-continuous approximation that is compatible with the Itô forward summation approximation and that permits satisfaction of the mean square limit criterion. Such assumptions are unnecessary when there is a explicit function of $W(t)$ since, as will be seen, the mean square limit property can be verified directly. However, when a general function is considered with little information then this extra piece-wise constant assumption will be necessary.

Assumption 2.15. ***Piece-Wise-Constant Approximations (PWCA) for General Mean Square Limits***

1. *Let $Z(t)$ be a* ***piece-wise-constant, right-continuous*** *stochastic process such that*

$$Z_n(s) = \{\zeta_i; \tau_i \le s < \tau_{i+1};\ \text{for } i = 0:n\}\ , \tag{2.39}$$

   where the times τ_i belong to a partition of $[0,t]$ such that $\tau_0 = 0$ and $\tau_{n+1} = t$, so $Z_n(t) = \zeta_{n+1}$ if needed, but does not contribute to the integral. The ζ_i are a sequence of discrete stochastic processes depending on the past Wiener processes $\mathcal{W}_i = \{W(s) \mid 0 \le s \le \tau_i\}$, i.e., adapted to $\mathcal{W}_i$ for $i = 0 : n+1$. Let $\mathcal{W}$ be the set of all $\mathcal{W}_i$.

2. *Let $Y(t)$ be a stochastic process depending on $\mathcal{W}$ where $Y(s)$ can be approximated by the piece-wise-constant, right-continuous stochastic process $Z_n(s)$ for $0 \le s \le t$ such that*

$$\lim_{n\to\infty} \mathrm{E}\left[\int_0^t \left(Y(s) - Z_n(s)\right)^2 ds\right] \to 0 \tag{2.40}$$

   as $n \to +\infty$.

Theorem 2.16. ***Mean of Itô Stochastic Integral:***

$$\mathrm{E}\left[\int_{t_0}^t g(W(s),s)dW(s)\right] \overset{ims}{=} 0\,, \quad 0 \le t_0 \le t\ , \tag{2.41}$$

assuming the mean square integrability condition

$$\mathrm{E}\left[\int_{t_0}^t g^2(W(s),s)ds\right] < \infty\ , \tag{2.42}$$

and the PWCA Mean Square Limits Assumption 2.15 for $Y(t) = g(W(t),t)$.

Proof. Only heuristic justification will be given here to keep this presentation simple. For more elaborate justification using sequences of approximate step function sums, consult the works of Arnold [13], Schuss [240],Øksendal [218], Mikosch [205] or Steele [251].

- Using the Itô mean square limit (2.27), then we have the formal finite sum approximation using partition (2.28),

$$\int_{t_0}^t g(W(s),s)dW(s) \simeq \sum_{i=0}^n g(W(t_i),t_i)(W(t_{i+1}) - W(t_i)) = \sum_{i=0}^n g_i \Delta W_i\ ,$$

  where $g_i = g(W(t_i),t_i)$ and $\Delta W_i = W(t_{i+1}) - W(t_i)$. Since the right hand side sum is finite, the operations of expectation and summation can be interchanged, so

$$\begin{aligned}\mathrm{E}\left[\int_{t_0}^{t} g(W(s),s)dW(s)\right] &\simeq \sum_{i=0}^{n}\mathrm{E}[g_i\Delta W_i] = \sum_{i=0}^{n}\mathrm{E}[g_i]\mathrm{E}[\Delta W_i] \\ &= \sum_{i=0}^{n}\mathrm{E}[g_i]\cdot 0 = 0\,,\end{aligned}$$

the last line using the independent increments and zero mean properties.

- The final justification requires justifying the interchange of the expectation operator, a Riemann integral, and the mean square limit operator. The underlying integrability assumption can be rewritten using Itô's forward integration choice leads to the approximation,

$$\mathrm{E}\left[\int_{t_0}^{t} g^2(W(s),s)ds\right] = \int_{t_0}^{t}\mathrm{E}\left[g^2(W(s),s)\right]ds \simeq \sum_{i=0}^{n}\mathrm{E}[g_i^2]\Delta t_i\,.$$

- This approximation can be compared with the expected absolute value of original Itô approximated sum of interest followed by a one-component Schwarz's inequality $\left(\overset{\text{csi}}{\le}\right)$, to put it into a usable quadratic form and rearrangement into independent increments $\left(\underset{inc}{\overset{ind}{=}}\right)$,

$$\begin{aligned}
\mathrm{E}\left[\left|\int_{t_0}^{t} g(W(s),s)dW(s)\right|\right] &\simeq \mathrm{E}\left[\left|\textstyle\sum_{i=0}^{n} g_i\Delta W_i\right|\right] \\
&\overset{\text{csi}}{\le} \sqrt{\mathrm{E}\left[\textstyle\sum_{i=0}^{n} g_i\Delta W_i\cdot\sum_{j=0}^{n} g_j\Delta W_j\right]} \\
= \sqrt{\mathrm{E}\left[\textstyle\sum_{i=0}^{n} g_i^2(\Delta W_i)^2 + \sum_{i=0}^{n} g_i\Delta W_i\left(\sum_{j=0}^{i-1}+\sum_{j=i+1}^{n}\right)g_j\Delta W_j\right]} \\
&\underset{inc}{\overset{ind}{=}} \left[\textstyle\sum_{i=0}^{n}\mathrm{E}\left[g_i^2\right]\mathrm{E}\left[(\Delta W_i)^2\right]\right. \\
+\textstyle\sum_{i=0}^{n}\left(\sum_{j=0}^{i-1}\mathrm{E}\left[g_ig_j\Delta W_j\right]\mathrm{E}\left[\Delta W_i\right]+\sum_{j=i+1}^{n}\mathrm{E}\left[g_ig_j\Delta W_i\right]\mathrm{E}\left[\Delta W_j\right]\right)&\Big]^{0.5} \\
&= \sqrt{\textstyle\sum_{i=0}^{n}\mathrm{E}\left[g_i^2\right]\Delta t_i + 0}\,,
\end{aligned}$$

where the zero mean and Δt_i variance properties of ΔW_i were used in the last step. The **expectation Schwarz (Cauchy-Schwarz) inequality**

$$\mathrm{E}[|XY|] \le \sqrt{\mathrm{E}[X^2]\cdot\mathrm{E}[Y^2]} \tag{2.43}$$

has been used with $X = \sum_{i=0}^{n} g_i\Delta W_i$ and $Y = 1$ to relate the magnitude of the sum to the square root of the sum of squares. Hence, in the mean square sense as $n \to +\infty$, we formally have the expected absolute value of the stochastic diffusion integral is majorized by the square root of the integral of the expected square of the integrand,

$$\mathrm{E}\left[\left|\int_{t_0}^{t} g(W(s),s)dW(s)\right|\right] \le \sqrt{\int_{t_0}^{t} E\left[g^2(W(s),s)\right]ds}\,. \tag{2.44}$$

It has been assumed that the sums are bounded on the bounded interval $[t_0, t]$, so that, in absence of stochasticity, we can expect uniform convergence of the sums and that the operations of expectation and mean square limit can be interchanged.

- Note, that this mean zero (2.41) for the Itô stochastic integral result depends heavily on the Itô forward or left endpoint integration choice, and as will be seen later, the mean zero result will not hold for other rectangular integration rule choice.

- Under similar conditions, a quadratic or "ims-covariance" version of this theorem holds for interchanging expectation and mean square limit.

$\square$

Theorem 2.17. ***Itô-Covariance of Stochastic Integral***

$$\begin{aligned} &\mathrm{E}\left[\int_{t_0}^{t} f(W(s),s)dW(s) \int_{t_0}^{t} g(W(r),r)dW(r)\right] \\ &\quad \overset{ims}{=} \int_{t_0}^{t} \mathrm{E}\left[f(W(s),s)g(W(s),s)\right] ds \,, \end{aligned} \tag{2.45}$$

for $0 \le t_0 \le t$, assuming that $f(W(t),t)$ and $g(W(t),t)$ satisfy the mean square integrability condition (2.42) and the PWCA Mean Square Limits Assumption 2.15 for $Y(t) = g(W(t),t)$.

Proof. Again, heuristic justifications are presented here. Replacing the expectation of the Itô integral product with that of the corresponding product of finite sum approximations leads to

$$J_2(t) = \mathrm{E}\left[\int_{t_0}^{t} f(W(s),s)dW(s)\int_{t_0}^{t} g(W(r),r)dW(r)\right] \simeq \sum_{i=0}^{n}\sum_{j=0}^{n} \mathrm{E}[f_i \Delta W_i g_j \Delta W_j] \,,$$

but the independent increments are inter-mingled in the sums and the argument of the expectation of $f_i \Delta W_i g_j \Delta W_j$. However, if $j < i$ then the increment ΔW_i will be independent of f_i, g_j and ΔW_j, while if $j > i$ then ΔW_j will be independent of f_i, g_j and ΔW_i, and for $i = j$ the usual independent increment form is obtained. Thus, taking these independence properties to split the double sum three ways and using independent increment properties leads to

$$\begin{aligned} J_2(t) \;\simeq\; & \sum_{i=0}^{n} \mathrm{E}[f_i g_i]\mathrm{E}[(\Delta W_i)^2] + \sum_{i=0}^{n}\sum_{j=0}^{i-1} \mathrm{E}[f_i g_j \Delta W_j]\mathrm{E}[\Delta W_i] \\ & s + \sum_{i=0}^{n}\sum_{j=i+1}^{n} \mathrm{E}[f_i g_j \Delta W_i]\mathrm{E}[\Delta W_j] \\ = \; & \sum_{i=0}^{n} \mathrm{E}[f_i g_i]\Delta t_i \\ \xrightarrow{ims} \; & \int_{t_0}^{t} \mathrm{E}\left[f(W(s),s)g(W(s),s)\right] ds \,, \end{aligned}$$

giving the desired conclusion except for replacing the approximately equals ($\simeq$) by the mean square limit as $n \to \infty$

Upon replacing the function f by g, leads to the immediate corollary for the "ims-variance" of the Itô stochastic integral in the following. ☐

Corollary 2.18. ***Itô-Variance of Stochastic Integral:***

$$\mathrm{E}\left[\left(\int_{t_0}^{t} g(W(s),s)dW(s)\right)^2\right] \overset{ims}{=} \int_{t_0}^{t} \mathrm{E}\left[g^2(W(s),s)\right] ds \,, \tag{2.46}$$

for $0 \le t_0 \le t$*, assuming that* $g(W(t),t)$ *satisfies the mean square integrability condition (2.42).*

Result (2.46) is also called *Itô isometry* or *martingale isometry*.

Theorem 2.19. ***Itô Stochastic Integral Simple Rules:***
Let g*,* g_1 *and* g_2 *satisfy the mean square integrability condition (2.42) on* $0 \le t_0 \le t$ *and the PWCA Mean Square Limits Assumption 2.15, while letting* c_1 *and* c_2 *be constants.*

- ***Operator Linearity:***

$$\begin{aligned} & \int_{t_0}^{t} [c_1 g_1(W(s),s) \;+\; c_2 g_2(W(s),s)] dW(s) \\ & \qquad \overset{ims}{=} c_1 \int_{t_0}^{t} g_1(W(s),s)dW(s) + c_2 \int_{t_0}^{t} g_2(W(s),s)dW(s) \,. \end{aligned}$$

- ***Additivity over Subintervals:***

$$\int_{t_0}^{t} g(W(s),s)dW(s) \overset{ims}{=} \int_{t_0}^{r} g(W(s),s)dW(s) + \int_{r}^{t} g(W(s),s)dW(s)$$

for $0 \le t_0 \le r \le t$*.*

- ***Continuity of Sample Paths for***

$$I[g](t) = \int_{t_0}^{t} g(W(s), s)dW(s) ,$$

with probability one.

Proof. The first two are clearly true by examining the forward integration approximation. For the last item note that

$$\Delta I[g](t) = I[g](t+\Delta t) - I[g](t) = \int_{t}^{t+\Delta t} g(W(s), s)dW(s) \overset{ims}{=} g(W(t), t)\Delta W(t) \to 0$$

with probability one as $\Delta t \to 0^+$. □

For later use in formal stochastic calculations, it will be helpful to know how to handle powers of $dW(t)$ greater than square powers. The critical problem is to know when to truncate a differential expansion, such as that for $exp(dW(t))$, at a power of $dW(t)$ beyond which the higher powers are zero in the sense of the Itô mean square limit. For example, $\exp(dW(t))$ can be formally expanded by Taylor series as

$$\exp(dW(t)) = 1 + dW(t) + (dW)^2(t)/2! + (dW)^3(t)/3! + (dW)^4(t)/4! + \cdots$$

and it turns out we can justify stopping at the quadratic term for the mean square limit. The consequence will be the famous Itô stochastic chain rule discussed for jump-diffusions in Chapter 4 and will lead to more rapid calculations. The main purpose of the current chapter is setting up the foundational justification for this chain rule.

Lemma 2.20. *Powers of* $dW(t)$***:***
Let the integer $m \geq 3$.

$$\int_0^t (dW)^m(s) \overset{ims}{=} 0 \tag{2.47}$$

or in symbolic differential notation

$$(dW)^m(t) \underset{sym}{\overset{ims}{=}} 0 . \tag{2.48}$$

Proof. Let $m \geq 3$ and

$$I[(dW)^{m-1}](t) = I(t; m) \equiv \int_0^t (dW)^m(s) \simeq I_n^{(0)}(t; m) = \sum_{i=0}^{n} (\Delta W_i)^m. \tag{2.49}$$

The expectation of the Itô approximate sum $I_n^{(0)}(t;m)$ yields different formulae for odd values, $m = 2k-1$ for $k \geq 2$,

$$\mathrm{E}[I_n^{(0)}(t;2k-1)] = \sum_{i=0}^{n} \mathrm{E}\left[(\Delta W_i)^{2k-1}\right] = 0\;,$$

while for even values, $m = 2k$ for $k \geq 2$,

$$\begin{aligned}\mathrm{E}[I_n^{(0)}(t;2k)] &= \sum_{i=0}^{n} \mathrm{E}\left[(\Delta W_i)^{2k}\right] = (2k-1)!!\sum_{i=0}^{n}(\Delta t_i)^k \\ &\leq (2k-1)!!t(\delta t_n)^{k-1} \to 0\;,\end{aligned}$$

as $n \to \infty$, where $(2k-1)!!$ is the double factorial function (1.15). Odd or even m, $m \geq 3$, the results suggest that the Itô mean square value is given by

$$I(t;m) \overset{ims}{=} I^{(ims)}(t;m) \equiv \lim_{n\to\infty}[I_n^{(0)}(t;m)] = 0\;.$$

The justification requires confirmation of mean square convergence,

$$\lim_{n\to\infty} \mathrm{E}\left[(I_n^{(0)}(t;m) - I^{(ims)}(t;m))^2\right] = \lim_{n\to\infty} \mathrm{E}\left[(I_n^{(0)})^2(t;m)\right]\;.$$

For odd values, $m = 2k-1$, separating out the diagonal part of the quadratic to separate the independent increments,

$$\begin{aligned}\mathrm{E}\left[(I_n^{(0)})^2(t;2k-1)\right] &= \sum_{i=0}^{n} \mathrm{E}\left[(\Delta W_i)^{2(2k-1)} + \sum_{j\neq i}(\Delta W_i)^{2k-1}(\Delta W_j)^{2k-1}\right] \\ &= (4k-3)!!\sum_{i=0}^{n}(\Delta t_i)^{2k-1} \\ &\leq (4k-3)!!t(\delta t_n)^{2k-2} \to 0\;,\end{aligned}$$

as $n \to \infty$, off-diagonal odd power terms do not contribute. Here $(4k-3)!!$ is the double factorial function (1.15). For even values, $m = 2k$, the off-diagonal terms contribute since they are products of even powers of increments in i and j, so upon completing the double sum over $j \neq i$ and subtracting the completed amount from the single sum,

$$\begin{aligned}\mathrm{E}\left[(I_n^{(0)})^2(t;2k)\right] &= \sum_{i=0}^{n} \mathrm{E}\left[(\Delta W_i)^{4k} + \sum_{j\neq i}(\Delta W_i)^{2k}(\Delta W_j)^{2k}\right] \\ &= ((4k-1)!! - ((2k-1)!!)^2)\sum_{i=0}^{n}(\Delta t_i)^{2k} \\ &\quad +((2k-1)!!)^2\sum_{i=0}^{n}(\Delta t_i)^k\sum_{j=0}^{n}(\Delta t_j)^k \\ &\leq (4k-1)!!t(\delta t_n)^{2k-1} + ((2k-1)!!)^2 t(\delta t_n)^{2k-2}(t-\delta t_n) \to 0\;,\end{aligned}$$

as $n \to \infty$. Thus, denoting the conclusion symbolically, $(dW)^m(t) \overset{ims}{\underset{sym}{=}} 0$, provided $m \geq 3$ to an accuracy with error $\mathrm{o}(dt)$. $\Box$

Another differential product whose Itô mean square limit will be useful is $dt\, dW(t)$ since it arises in the expansions of functions of stochastic differentials:

Lemma 2.21. ***Differential Product*** $dt\, dW(t)$***:***

$$\int_0^t ds\, dW(s) \overset{ims}{=} 0 \tag{2.50}$$

or in symbolic notation

$$dt\, dW(t) \overset{ims}{\underset{sym}{=}} 0\,. \tag{2.51}$$

Proof. Let

$$I[dt](t) = \int_0^t ds\, dW(s) \simeq I_n^{(0)}[dt](t) \equiv \sum_{i=0}^{n} \Delta t_i \Delta W_i, \tag{2.52}$$

with some abuse of the notation by replacing functional argument g by dt. The expectation of the sum $I_n^{(0)}[dt](t)$ yields

$$\mathrm{E}[I_n^{(0)}[dt](t)] = \sum_{i=0}^{n} \mathrm{E}\left[\Delta t_i \Delta W_i\right] = 0\,.$$

The result suggests that the Itô mean square value is given by

$$I[dt](t;m) \overset{ims}{=} \lim_{n\to\infty}[I_n^{(0)}[dt](t;m)] = 0\,.$$

The justification requires confirmation of mean square convergence, separating out the diagonal part of the quadratic to separate the independent increments,

$$\begin{aligned}\mathrm{E}\left[(I_n^{(0)})^2[dt](t)\right] &= \sum_{i=0}^{n} \mathrm{E}\left[(\Delta t_i)^2(\Delta W_i)^2 + \sum_{j\neq i} \Delta t_i \Delta t_j \Delta W_i \Delta W_j\right] \\ &= \sum_{i=0}^{n} (\Delta t_i)^3 \leq t(\delta t_n)^2 \to 0\,,\end{aligned}$$

as $n \to \infty$, off-diagonal do not contribute. Thus, $dt\, dW(t) \overset{ims}{\underset{sym}{=}} 0$ to an accuracy with error $\mathrm{o}(dt)$. $\Box$

Remarks 2.22.

- *Of the Itô differentiable forms that have zero limit in the mean square,* $dtdW(t)$ *is one of the most marginable to approximate due to the randomness of* $dW(t)$, *even though we know* $\mathrm{E}[dtdW(t)] = 0$ *and* $\mathrm{E}[|dW(t)|] = \sqrt{2\Delta t/\pi}$ *from convenient Table 1.1. Hence, the justification of* $\int_0^t dsdW(s) \overset{ims}{=} 0$ *by showing the mean square limit is especially important. Note that for even spacing of time increments, the root mean square of the bound of the mean square approximation above is* $\sqrt{t(\delta t_n)^2} = t\sqrt{t}/(n+1) \to 0$ *as* $n \to \infty$. *However, see Exercise 2 for a more cutting-edge example.*

- *See Exercise 5 for how to computationally confirm the above Lemma 2.21.*

The mean square limits to an accuracy with error $\mathrm{o}(dt)$ are summarized in the following Table 2.1.

Table 2.1. *Some Itô stochastic diffusion differentials with an accuracy with error* $\mathrm{o}(dt)$ *as* $dt \to 0^+$.

Differential Diffusion Form	Itô Mean Square Limit
$dW(t)$	$dW(t)$
dt	dt
$dt\ dW(t)$	0
$(dW)^2(t)$	dt
$(dW)^m(t)$	$0,\ m \geq 3$
$(dt)^\alpha(dW)^m(t)$	$0,\ \alpha > 0,\ \ m \geq 1$

The more general form,

$$(dt)^p(dW)^q(t) \underset{sym}{\overset{ims}{=}} \delta_{p,0}\delta_{q,0} + dW(t)\delta_{p,0}\delta_{q,1} + dt(\delta_{p,1}\delta_{q,0} + \delta_{p,0}\delta_{q,2})\ , \quad (2.53)$$

when p and q are non-negative integers, is left as Exercise 1 on Page 134.

Remark 2.23. *In using Table 2.1, the differential entries are just symbols of the underlying integral basis and care should be taken when applying them to find the mean square representation of differentials, especially when they appear in multiplicative combinations. For instance, one might be tempted to replace* $(dW)^4(t)$ *by* $(dW)^2(t)(dW)^2(t)$, *then replace those terms with* $(dW)^2(t) \underset{sym}{\overset{ims}{=}} dt$ *and getting to* $(dt)^2 \underset{sym}{\overset{ims}{=}} 0$, *which is the correct but crudely found answer for* $(dW)^4(t)$. *Note that for finite increments,* $\mathrm{E}[(\Delta W_i)^4] = 3(\Delta t_i)^2$ *while* $\mathrm{E}^2[(\Delta W_i)^2] = (\Delta t_i)^2$, *differing by a factor of three.*

2.3 Stratonovich and other Stochastic Integration Rules

In this section, other definitions of stochastic integration rules, other than Itô's choice of the forward left endpoint rule, are explored for the purpose of comparison and understanding Itô's choice. This comparison will be illustrated by the simple stochastic integral of $W(t)$.

Let the integration rule approximation point be

$$t_{i+\theta} \equiv t_i + \theta \Delta t_i \ ,$$

where $0 \le \theta \le 1$, so the Itô's rule is when $\theta = 0$ with $\Delta t_i \equiv t_{i+1} - t_i$. Let the interval of integration be $[0, t]$ with partition (2.4). Let the approximate integrand be $W_{i+\theta} \equiv W(t_{i+\theta})$. The technique of splitting terms into independent increments is similar to that for Itô's rule, except that there are extra independent increments,

$$\Delta^\theta W_i \equiv W_{i+\theta} - W_i$$

and its complement

$$\Delta^\theta_c W_i \equiv \Delta W_i - \Delta^\theta W_i = W_{i+1} - W_{i+\theta}$$

for intermediate approximation points when $\theta > 0$, such that $\Delta^\theta W_i + \Delta^\theta_c W_i = \Delta W_i$. We also reuse (2.14) of the reduction Lemma 2.1 for the Itô case in the more general case here:

$$\begin{aligned}
I[W](t) = \int_0^t W(s)dW(s) \simeq I_n^{(\theta)}[W](t) &\equiv \sum_{i=0}^{n} W_{i+\theta}\Delta W_i \\
&= \sum_{i=0}^{n} (W_i + \Delta^\theta W_i)(\Delta^\theta W_i + \Delta^\theta_c W_i) \\
&= \sum_{i=0}^{n} \left(W_i \Delta W_i + (\Delta^\theta W_i)^2 + \Delta^\theta W_i \Delta^\theta_c W_i \right) \\
&= \frac{1}{2}\left(W_{n+1}^2 - \sum_{i=0}^{n} (\Delta W_i)^2 \right) + \sum_{i=0}^{n} (\Delta^\theta W_i)^2 + \sum_{i=0}^{n} \Delta^\theta W_i \Delta^\theta_c W_i \ .
\end{aligned}$$

Since $W_{n+1} = W(t)$ with this $[0, t]$ partition and the mean square limit of $\sum_{i=0}^{n} (\Delta W_i)^2$ has been shown to be t, similarly the mean square limit of $\sum_{i=0}^{n} (\Delta^\theta W_i)^2$ will be the expected value θt, and the last sum will not contribute in the mean being the product of independent increments, the mean square limit corresponding to the Itô Lemma 2.7 can be stated:

Lemma 2.24.

$$\begin{aligned}
\int_0^t W(s)dW(s) \overset{\theta-\mathrm{ms}}{=} I^{(\theta)}[W](t) &= \frac{1}{2}W^2(t) - \left(\frac{1}{2} - \theta\right) t \qquad (2.54) \\
&= \underset{n\to\infty}{\overset{\mathrm{ms}}{\lim}} \left[I_n^{(\theta)}[W](t) \right] \ ,
\end{aligned}$$

Proof. The mean square limit justifications are quite lengthy and somewhat tangent to our goals here, so only the general end result is given with the details left to the reader:

$$\begin{aligned}\mathrm{E}\left[\left(I_n^{(\theta)}[W](t) - I^{(\theta)}[W](t)\right)^2\right] &= 2\left|\frac{1}{2} - \theta\right| \sum_{i=0}^{n} (\Delta t_i)^2 \\ &\le 2\left|\frac{1}{2} - \theta\right| t\delta t_n \to 0\ ,\end{aligned}$$

where $\delta t_n = \max_{i=0:n}[\Delta t_i] \to 0^+$ as $n \to \infty$. □

Remark 2.25. ***Stratonovich and Other Stochastic Integration Rules:*** *The mean square limit is exact, no limit* $n \to \infty$, *required, in the case* $\theta = 1/2$ *where* $t_{i+0.5} = (t_i + t_{i+1})/2$ *is the midpoint of* $[t_i, t_{i+1}]$ *and the integration rule is called the midpoint rule or* ***Stratonovich stochastic integration*** *[255]. For Stratonovich integration,*

$$\int_0^t W(s)dW(s) \overset{\theta-\mathrm{ms}}{=} I^{(0,5)}[W](t) = W^2(t)/2\ ,$$

which is the deterministic integral answer, containing no correction as in the case of Itô's rule. This deterministic property might offer some benefit in some applications, but at the expense of more complicated overlapping dependence of increments in time.

Lemma 2.26.

$$\mathrm{E}\left[I^{(\theta)}[W](t)\right] = \mathrm{E}\left[\frac{1}{2}W^2(t) - \left(\frac{1}{2} - \theta\right)t\right] = \theta t. \tag{2.55}$$

Proof. The result is immediate since $\mathrm{E}[W^2(t)] = t$ from Table 1.1 when $n = 2$ with $|\Delta W|^2(t)$ replaced by $W^2(t)$ and Δt by t. □

Remarks 2.27.

- *When* $\theta \neq 0$, *then the useful Itô* ***expectation-integration interchange property****,*

  $$\mathrm{E}\left[\int_0^t f(W(s), s)dW(s)\right] \overset{ims}{=} \int_0^t \mathrm{E}[f(W(s), s)]\mathrm{E}[dW(s)] = 0$$

  is no longer valid as implied by (2.41). This is a quite nice concrete property, but for abstract analysis it is more crucial since it means, with appropriate qualification on $f(W(t), t)$, *that the Itô integral is a martingale.*

- *Decades ago, there was a larger controversy as to*

whether Itô or Stratonovich stochastic integration

> *should be used. The question sometimes centered about what was more appropriate for the application at hand (see for instance, Turelli [266] for a discussion involving biological applications), but the benefits of Itô's choice of forward integration facilitating the use of independent increments of the processes and the fact that many Stratonovich properties were derived by Itô stochastic calculus have made the Itô calculus dominant.*

2.4 Conclusion

In this chapter, the foundations have been laid for the integrals of the second type in the integrated SDE (2.2), i.e., using the stochastic diffusion integral of Itô of Definition 2.8 extended to the more general case:

Definition 2.28. ***Stochastic Diffusion Integration:***

$$\int_0^t g(X(s),s)dW(s) \overset{ims}{=} \underset{n\to\infty}{\overset{\mathrm{ms}}{\lim}} \left[\sum_{i=0}^{n} g(X(t_i),t_i)dW(t_i)\right] , \tag{2.56}$$

where $X(t)$ in the integrand function g has an implied dependence on the diffusion process $W(t)$, but also depends on the jump process $P(t)$. The integrand process $g(X(t),t)$ is also assumed to have a bounded mean square,

$$\mathrm{E}\left[\int_0^t g^2(X(s),s)ds\right] < \infty ,$$

and satisfy the PWCA Mean Square Limits Assumption 2.15 for $Y(t) = g(X(t),t)$.

However, as previously explained, the Poisson jump process fits within the framework of Itô stochastic integration since it is piece-wise continuous. The stochastic diffusion integration rule (2.56) has been motivated and illustrated by a number of examples using functions and powers of the diffusion process $W(t)$.

2.5 Exercises

In all computational exercises, *Mathematica*, MATLAB, Maple or other programming may be used where appropriate.

1. Formally justify the general form (2.53),

$$(dt)^p(dW)^q(t) \underset{sym}{\overset{ims}{=}} \delta_{p,0}\delta_{q,0} + dW(t)\delta_{p,0}\delta_{q,1} + dt(\delta_{p,1}\delta_{q,0} + \delta_{p,0}\delta_{q,2}) ,$$

when p and q are non-negative integers.

2. Show the limit in the mean square for

$$I\left[(dt)^{\alpha}\right](t) \equiv \int_0^t (ds)^{\alpha} dW(s) \overset{ims}{=} 0\,,$$

provided $\alpha > 0$ and is real.
{*Hint: See Lemma 2.21 for the case* $\alpha = 1$.}

3. Computationally confirm the mean square limit for Itô's most fundamental stochastic integral given as

$$\int_0^t (dW)^2(s) \overset{ims}{=} t\,,$$

by demonstrating that the Itô forward integration approximating sum

$$I_n^{(0)}[dW](t) = \sum_{i=0}^{n} (\Delta W_i)^2$$

gives a close approximation to t for sufficiently large n. Apply a modification of the algorithm of Program A.7 in Appendix A, used in generating Figure 1.1, to the approximation $I_n^{(0)}[dW](t)$. Use $n = 100$ and $n = 10000$ sample sizes, plotting the $I_n^{(0)}[dW](t)$ with the limit t versus t for $t \in [0, 2]$. Plot separately the errors for each n between the approximation sum and the exact IMS answer. Also report the standard deviation (`std` in MATLAB) of the errors for each n. Does the larger value of n make Itô's stochastic integration model more convincing than the smaller value? Does this mean that for this example that the convergence is much stronger than the mean square limit?
{*Caution: In this problem and the next two, you are not asked to verify the mean square limit, but to verify that the forward approximation comes close in this example.*}

4. Computationally confirm the mean square limit for Itô's other very fundamental stochastic integral given as

$$\int_0^t W(s)dW(s) \overset{ims}{=} I^{(\mathrm{ims})}[W](t) = \frac{1}{2}\left(W^2(t) - t\right)$$

by demonstrating that the Itô forward integration approximating sum

$$I_n^{(0)}[W](t) = \sum_{i=0}^{n} W_i \Delta W_i$$

gives a close approximation to $W^2(t) - t)/2$ for sufficiently large n. Apply a modification of the algorithm of Program A.7 in Appendix A, used in generating Figure 1.1, to the approximation $I_n^{(0)}[W](t)$. Use $n = 100$ and $n = 10000$ sample sizes, plotting the approximation $I_n^{(0)}[W](t)$ and the error

$E_n[W](t) = I_n^{(0)}[W](t) - (W^2(t) - t)/2$ versus t for $t \in [0,2]$. Plot separately the errors for each n between the approximation sum and the exact IMS answer. Also report the standard deviation (`std` in MATLAB) of the errors for each n. Does the larger value of n make Itô's stochastic integration model more convincing than the smaller value? Does this mean that for this example that the convergence is much stronger than the mean square limit?

5. Computationally confirm the mean square limit for another of Itô's more obvious fundamental stochastic integrals:

$$\int_0^t ds dW(s) \overset{ims}{=} I^{(\text{ims})}[dt](t) = 0$$

by demonstrating that the Itô forward integration approximating sum

$$I_n^{(0)}[dt](t) = \sum_{i=0}^{n} \Delta t_i \Delta W_i$$

gives a close approximation to 0 for sufficiently large n. Apply a modification of the algorithm of Program A.7 in Appendix A , used in generating Figure 1.1, to the approximation $I_n^{(0)}[dt](t)$. Use $n = 100$ and $n = 10000$ sample sizes, plotting the common value of the approximation and error $I_n^{(0)}[dt](t) = E_n[dt](t)$ and the noise $W(t)$ for $t \in [0,2]$. Plot separately the errors for each n between the approximation sum and the exact IMS answer. Also report the standard deviation (`std` in MATLAB) of the errors for each n. Does the larger value of n make Itô's stochastic integration model more convincing than the smaller value? Does this mean that for this example that the convergence is much stronger than the mean square limit?

6. Computationally check the Itô mean square limit for convergence of the Itô approximating sum of the stochastic integral of $(dW)^2(t)$ to the limit t by directly computing the K-sample mean square

$$S_{i,n}^{(K)} = \frac{1}{K}\sum_{k=1}^{K}\left(\sum_{j=1}^{i}\left(\left(\Delta W_j^{(k)}\right)^2 - \Delta t_j\right)\right)^2 ,$$

where the identity $t = t_{n+1} = \sum_{i=0}^{n} \Delta t_i$ has been used to merge t into the approximating sum. Select $K = 5$ random states or seeds, $n = 10^m$ for $m = 2:5$ sample sizes, constant $\Delta t_i = \Delta t$, $i = n$ and $t = 1$, as an example. Plot $\log_{10}(S_{n,n}^{(K)})$ versus $m = \log_{10}(n)$. What rate of convergence is suggested by this graph?
{*Hint: If* $\Delta t = 10^m$ *and* $S \sim C \cdot (\Delta t)^a$ *then* $\log_{10}(S) \sim a \cdot m + \log_{10}(C)$. *In MATLAB for instance, recall that* `randn('state',k);` *sets the* k *normal random number* `state`.}

7. Show that the non-Itô, approximate backward integration rule ($\theta = 1$) for the stochastic integral

$$\int_{t_0}^{t} W(s)dW(s) \simeq I_n^{(1)}(t) = \sum_{i=0}^{n} W_{i+1}\Delta W_i$$

differs from the Itô rule ($\theta = 0$) by a deterministic factor of t in the mean square limit, i.e.,

$$I_n^{(1)}(t) - I_n^{(0)}(t) \overset{ims}{\longrightarrow} t\ .$$

{*Hint: The mean square limit is not needed if the approximate integral is related to the Itô integral for* $(dW)^2(t)$.}

8. Show that the non-Itô, approximate trapezoidal integration rule, a variant of the Stratonovich integral, for the stochastic integral

$$\int_{t_0}^{t} W(s)dW(s) \simeq I_n^{(trap)}(t) = \frac{1}{2}\sum_{i=0}^{n}(W_i + W_{i+1})\Delta W_i$$

differs from the Itô rule ($\theta = 0$) by a deterministic factor of $t/2$ in the mean square limit, i.e.,

$$I_n^{(trap)}(t) - I_n^{(0)}(t) \overset{ims}{\longrightarrow} \frac{1}{2}t\ .$$

{*Hint: The mean square limit is not needed if the approximate integral is related to the one for* $(dW)^2(t)$.}

9. Demonstrate that the **trapezoidal rule** leads to Stratonovich or regular calculus by approximating the stochastic integral example

$$\int_{t_0}^{t} W^2(s)dW(s)$$

with

$$I_n^{(0)}(t) = \frac{1}{2}\sum_{i=0}^{n}(W_i^2 + W_{i+1}^2)\Delta W_i\ .$$

In particular, show that

$$I_n^{(0)}(t) = \frac{1}{3}(W^3(t_{n+1}) - W^3(t_0)) + \frac{1}{6}\sum_{i=0}^{n}(\Delta W_i)^3\ ,$$

by forming convenient powers of independent increments. Formally, justify that mean square limit is just the first term using elementary mean square properties for the powers of increments $(\Delta W)^p(t_i)$. You are not required to rigorously show mean square convergence, unless you want to show it.

Remark 2.29. *In numerical integration of deterministic integrands, both the midpoint rectangular rule and the trapezoidal rule yield the same order of error estimate when the integrand is sufficiently continuous.*

Suggested References for Further Reading

- Arnold, 1974 [13].
- Gard, 1988 [91].
- Karlin and Taylor, 1981 [158].
- Kloeden and Platen, 1999 [161].
- Kolmogorov and Fomin, 1970 [164].
- Mikosch, 1998 [205].
- Øksendal, 1998 [218].
- Protter, 1990 [228].
- Schuss, 1980 [240].
- Taylor and Karlin, 1998 [260].