Applied Stochastic Processes and Control for Jump-Diffusions: Modeling, Analysis and Computation

Floyd B. Hanson
University of Illinois
Chicago, Illinois, USA

Chapter 3 Jumps

October 4, 2006

Chapter 3

Stochastic Integration for Jumps

A unique feature of this chapter is the greater emphasis on the importance of the lack of continuity that leads to deviations from the chain rule of regular calculus, namely, the discontinuity of Poisson jumps in time and the non-smooth behavior of Wiener. The Poisson jump processes are given in terms of special right-continuous step and impulse functions. Unless otherwise stated, a fixed jump rate λ is assumed. The Poisson jump calculus is also formulated in terms of finite difference algebraic recursions.

3.1 Stochastic Integration in $P(t)$: The Foundations

In this chapter, foundations will be laid for the integrals of the third type in the integrated SDE (2.2), i.e., using the notion of Itô stochastic integral of Definition 2.8 (p. 118) by extending it to the jump case:

Definition 3.1. ***Poisson Jump Stochastic Integration:***

$$\int_0^t h(X(s),s)dP(s) \stackrel{ims}{=} \underset{n\to\infty}{\overset{\mathrm{ms}}{\lim}} \left[\sum_{i=0}^{n} h(X(t_i),t_i)\Delta P(t_i)\right] , \tag{3.1}$$

where $X(t)$ in the integrand function h has an implied dependence on the diffusion process $W(t)$, but also depends on the jump process $P(t)$. The integrand process $h(X(t),t)$ is also assumed to have a ***bounded mean integral of squares****,*

$$\mathrm{E}\left[\int_0^t h^2(X(s),s)ds\right] < \infty , \tag{3.2}$$

and to satisfy the Piece-Wise Constant Approximations (i-PWCA) Mean Square Limits Assumption 2.15 (p. 125) for $Y(t) = h(X(t),t)$, with the usual grid partition specifications on $[0,t]$.

For most problems encountered in practice, there will not be a need for this elaborate but fundamental mean square definition. The definition may be needed as a reference for unusual applications with stochastic jumps.

For instance, if an exact differential in $P(t)$ can be formed, then as with stochastic diffusion integration, i.e., when the variable of integration is the random diffusion process $W(t)$, there will be no need for mean square justification. Since much of the work of stochastic integration was performed in the previous chapter, with some very general results, it will be possible to move though this chapter faster.

Theorem 3.2. ***Fundamental Theorem of Poisson Jump Calculus:***
Let $h(p)$ be continuous and $\mathcal{H}(p)$ be continuously differentiable. Then

(a)

$$d\left(\int_0^t h(P(s))dP(s)\right) \stackrel{ims}{=} h(P(t))dP(t) \tag{3.3}$$

and

(b)

$$\int_0^t d\mathcal{H}(P(s)) \stackrel{ims}{=} \mathcal{H}(P(t)) - \mathcal{H}(0)\ ,\ 0 \le t\ . \tag{3.4}$$

Proof. The proof is almost the same as for the analogous result (2.37, 2.38), except for change in names from $W(t)$ to $P(t)$ and that the issue of unbounded variation need not be considered.

However, the right continuity property of $P(t)$ is essential to account for a jump at t for part (a). Consider the increment version for sufficiently small increments Δt,

$$\begin{aligned}
\Delta\left(\int_0^t h(P(s))dP(s)\right) &= \left(\int_0^{t+\Delta t} - \int_0^t\right) h(P(s))dP(s) \\
&= \int_t^{t+\Delta t} h(P(s))dP(s) \\
&\simeq h(P(t))\Delta P(t) = h(P(t))(P(t+\Delta t) - P(t)) \\
&\to h(P(t))dP(t)
\end{aligned}$$

as $\Delta t \to 0^+$, using the increment definition, subinterval additivity (see (3.21) later in this chapter), the continuity h and piece-wise continuity of P, such that any last minute jump is captured in $\Delta P(t)$ or $dP(t)$.

See the proof of the diffusion part (b) (2.38) for the jump part (b). □

First, consider the most basic jump integral, the integral of $P(t)$ with respect to $P(t)$, namely,

$$I[P](t) = \int_0^t P(s)dP(s)\ ,$$

which will be evaluated directly through precision-dt calculus and indirectly by showing that the defining mean square limit is satisfied.

Theorem 3.3. ***Jump Integral of*** $\int PdP$**:**

$$I[P](t) = \int_0^t P(s)dP(s) \stackrel{\text{dt}}{=} I^{(\text{dt})}[P](t) \equiv \frac{1}{2}(P(P-1))(t) \; , \tag{3.5}$$

is ***precision-dt integral*** *(see the definition of* $\stackrel{\text{dt}}{=}$ *in (1.31) on p. 97) that is also the* ***mean square limit integral****,*

$$I^{(\text{dt})}[P](t) \stackrel{ims}{=} \operatorname*{lim}_{n\to\infty}^{\text{ms}} \left[I_n^{(0)}[P](t)\right] \; , \tag{3.6}$$

where the forward integration approximation is

$$I_n^{(0)}[P](t) = \sum_{i=0}^{n} P(t_i)\Delta P(t_i) \; . \tag{3.7}$$

Proof. Starting with the Poisson increment and the square $P^2(t)$, as in the diffusion case since $d(x^2) = 2xdx$ in smooth deterministic calculus,

$$\begin{aligned}\Delta(P^2) &\equiv P^2(t+\Delta t) - P^2(t) = \left((P+\Delta P)^2 - P^2\right)(t) \\ &= \left(2P\Delta P + (\Delta P)^2\right)(t) \; .\end{aligned}$$

Taking the limit $\Delta t \to 0^+$, replacing ΔP by dP, and using the zero-one jump law (1.35) to let $(dP)^2 \stackrel{\text{dt}}{=} dP$ **with probability one** upon neglect of smaller order terms, leads to

$$d(P^2)(t) \stackrel{\text{dt}}{=} (2PdP + dP)(t)$$

in probability. Solving for the integrand-differential while forming an exact differential yields in probability

$$(PdP)(t) \stackrel{\text{dt}}{=} \frac{1}{2} d\left(P^2 - P\right)(t) \; .$$

Therefore, integration by the fundamental theorem of stochastic jump integration (3.3)

$$\int_0^t (PdP)(s) \stackrel{\text{dt}}{=} \frac{1}{2}\int_0^t \left(d\left(P^2 - P\right)(t)\right)(s) = \frac{1}{2}\left(P^2 - P\right)(t) = I^{(\text{dt})}[P](t) \; ,$$

where the initial Poisson condition $P(0) = 0$ **with probability one** has been used to eliminate the initial value of the integral. That takes care of the first part of the proof, but the technique is general enough for other powers.

For the second part, the forward integration approximation can be simplified by the useful finite difference identity (2.14),

$$I_n^{(0)}[P](t) = \sum_{i=0}^{n} P_i \Delta P_i = \frac{1}{2}\left(P^2(t) - \sum_{i=0}^{n} (\Delta P_i)^2\right)$$

for the partition

$$0 = t_0 < t_1 < \cdots < t_{n+1} = t$$

and using the fact (2.13) that

$$P(t) = P_{n+1} = \sum_{i=0}^{n} \Delta P_i \,,$$

the difference between the approximation and the limit reduces to

$$I_n^{(0)}[P](t) - I^{(\mathrm{dt})}[P](t) = \frac{1}{2}\sum_{i=0}^{n}\left(\Delta P_i - (\Delta P_i)^2\right) \,.$$

The mean square again is reduced by splitting up the sums due to the square into independent increments prior to term-wise passing the mean over the sums,

$$\begin{aligned}
\mathrm{E}\left[\left(I_n^{(0)}[P](t) - I^{(\mathrm{dt})}[P](t)\right)^2\right] &= \frac{1}{4}\mathrm{E}\left[\left(\sum_{i=0}^{n}\left(\Delta P_i - (\Delta P_i)^2\right)\right)^2\right] \\
&= \frac{1}{4}\sum_{i=0}^{n}\mathrm{E}\left[\left(\Delta P_i - (\Delta P_i)^2\right)^2\right] \\
&\quad + \frac{1}{4}\sum_{i=0}^{n}\sum_{j\neq i}\mathrm{E}\left[\left(\Delta P_i - (\Delta P_i)^2\right)\cdot\left(\Delta P_j - (\Delta P_j)^2\right)\right] \\
&= \frac{1}{4}\sum_{i=0}^{n}\mathrm{E}\left[(\Delta P_i)^2 - 2(\Delta P_i)^3 + (\Delta P_i)^4\right] \\
&\quad + \frac{1}{4}\sum_{i=0}^{n}\mathrm{E}\left[\Delta P_i - (\Delta P_i)^2\right]\sum_{j\neq i}\mathrm{E}\left[\Delta P_j - (\Delta P_j)^2\right] \\
&= \frac{1}{4}\sum_{i=0}^{n}\left(\lambda\Delta t_i(1+\lambda\Delta t_i) - 2\lambda\Delta t_i(1+3\lambda\Delta t_i + (\lambda\Delta t_i)^2)\right. \\
&\quad \left. + \lambda\Delta t_i(1+7\lambda\Delta t_i + 6(\lambda\Delta t_i)^2 + (\lambda\Delta t_i)^3)\right) \\
&\quad + \frac{1}{4}\sum_{i=0}^{n}\left(\lambda\Delta t_i - \lambda\Delta t_i(1+\lambda\Delta t_i)\right) \\
&\quad \cdot\sum_{j\neq i}\left(\lambda\Delta t_j - \lambda\Delta t_j(1+\lambda\Delta t_j)\right) \\
&\leq \frac{1}{4}\left(\sum_{i=0}^{n}(\lambda\Delta t_i)^2\left(2+4\lambda\Delta t_i\right) + \sum_{i=0}^{n}(\lambda\Delta t_i)^2\sum_{j=0}^{n}(\lambda\Delta t_j)^2\right) \\
&\leq \frac{1}{4}\left(\lambda t(2\lambda\delta t_n + 4(\lambda\delta t_n)^2) + (\lambda t^2)(\lambda\delta t_n)^2\right) \quad\longrightarrow\quad 0
\end{aligned}$$

as $n \to \infty$ and bounded t. For the evaluation of the expectations of powers of Poisson increments, the convenient Table 1.2 has been frequently used. Therefore, the mean square limit has been proven. ▯

Remarks 3.4.

- *The main result (3.5),*

$$\int_0^t P(s)dP(s) \stackrel{\mathrm{dt}}{=} \frac{1}{2}(P(P-1))(t) ,$$

for this basic integral has an interesting mathematical interpretation. Since $P(t)$ is integer valued, the answer is the Pythagorean $(P(t)-1)$th ***triangular number*** *given by the successive sum of $n = P(t) - 1$ integers,*

$$S_n^{(1)} = \sum_{k=0}^{n} k = n(n+1)/2 . \qquad (3.8)$$

The interpretation is not a coincidence, since when $P(t)$ jumps instantaneously by one unit and adds it to its count, $dP(t)$ jumps by one only momentarily so that the integral in (3.5) serves as a ***triangular number counter****. The forward integration approximation serves to keep the count short of the last jump, e.g., the forward approximation is zero when $P(t) = 1$.*

- *The derivation of (3.8) by finite differences gives useful techniques for calculating and interpreting other Poisson jump integrals. The basic lemma for the difference inversion ("discrete integration") is given by*

Lemma 3.5. *If*

$$\Delta[a_n] = \Delta[b_n] ,$$

for two sequences and any integer n, then

$$a_n = b_n + C$$

where C is an arbitrary constant.

The proof is obvious since a constant sequence is the only sequence elements that produces zero difference.

- *Since $\Delta[S_n^{(1)}] = S_{n+1}^{(1)} - S_n^{(1)} = (n+1)$, $\Delta[n] = 1$ and $\Delta[n^2] = 2n+1 = 2n + \Delta[n]$ or $n = \frac{1}{2}\Delta[n^2 - n]$, then $\Delta[S_n^{(1)}] = \Delta[(n^2-n)/2 + n]$ and $S_n^{(1)} = n(n+1)/2$, upon elimination the constant of discrete integration by the initial condition $S_0^{(1)} = 0$. This proves the first triangular number sum (3.8) by finite differences using Lemma 3.5.*

The first few Poisson power integrals are listed with an accuracy with error $\mathrm{o}(dt)$ in the Table 3.1:

Remarks 3.6.

Table 3.1. *Some stochastic jump integrals of powers with an accuracy with error* $\mathrm{o}(dt)$ *as* $dt \to 0^+$.

m	precision-dt: $\int_0^t (P^m dP)(s)$
0	$P(t)$
1	$(P(P-1))(t)/2$
2	$(P(P-1)(2P-1))(t)/6$
3	$(P^2(P-1)^2)(t)/4$

- *The proofs of the formulas for* $m = 2$ *and* $m = 3$ *are left as an exercise for the reader in Exercise 1 on Page 158.*
- *The integral results of Table 3.1 are all in the form of generalized or super-triangular numbers of order* m *when* $n = P(t) - 1$*:*

Definition 3.7. *The* ***super-triangular numbers*** *of order* m *for the first* $n+1$ *non-negative integers are defined as*

$$S_n^{(m)} = \sum_{k=0}^{n} k^m \ ,$$

for integers $m \geq 0$ *and* $n \geq 0$.

The summation form of a pure Poisson integral is generalized in the following theorem:

Theorem 3.8. ***Pure Poisson Integral as Sum Form:*** *Let* $h(p)$ *be a continuous function and let the process* $h(P(t))$ *have a bounded mean integral of squares (3.2). Then,*

$$\int_0^t h(P(s))dP(s) \overset{\mathrm{dt}}{=} \sum_{k=0}^{P(t)-1} h(k) \ , \tag{3.9}$$

with the usual summation convention for irregular forms that

$$\sum_{k=0}^{-1} h(k) \equiv 0 \tag{3.10}$$

for the case that $P(t) = 0$.

Proof. It is only necessary to confirm that both sides of Eq. (3.9) satisfy the same differential. The tools used will be the Fundamental Theorem of Stochastic Calculus (3.3) and the idea of Zero-One Jump Power Law (1.36). By the fundamental theorem, the differential of the left hand side of (3.9),

$$d\left(\int_0^t h(P(s))dP(s)\right) \stackrel{\mathrm{dt}}{=} h(P(t))dP(t)\ ,$$

where we are using the symbol $\stackrel{\mathrm{dt}}{=}$ as a substitution for $\stackrel{ims}{=}$. Then, by using the incremental definition of the differential for the right hand side of (3.9),

$$d\left(\sum_{k=0}^{P(t)-1} h(k)\right) = \sum_{k=0}^{P(t)+dP(t)-1} h(k) - \sum_{k=0}^{P(t)-1} h(k) \stackrel{\mathrm{dt}}{=} h(P(t))dP(t)\ ,$$

where the last step is due to the zero-one jump law since the difference in the two sums in the first line is zero if $dP(t) = 0$, else there is only one extra term in the first sum in the alternate case $dP(t) = 1$. Also $dP(t) = 1$ is used in the argument of h. Hence, the differential of both sides of (3.9) are the same. The final result then follows for the reasons:

1. both sides satisfy the same initial condition,
2. the vanishing of the jump integral in the limit,
$$\lim_{t\to 0^+} \int_0^t h(P(s))dP(s) = 0,$$
3. the vanishing of the Poisson sum in the limit.
$$\lim_{t\to 0^+} \sum_{k=0}^{P(t)-1} h(k) = \sum_{k=0}^{-1} h(k) \equiv 0\ ,$$
4. $P(0^+) = 0$ and
5. the irregular summation convention (3.10).

The argument is analogous to that of mathematical induction, since we have shown that both sides of (3.9) satisfy the same initial condition and the same changes so lead to the same result hypothesized in the theorem. □

Remarks 3.9.

- *Note that in this theorem the sum is over all $P(t)$ jump amplitudes for $k+1 = 1 : P(t)$ jumps, but that the jump amplitude h is evaluated at the pre-jump value $h(k)$ for $k = 0 : P(t) - 1$ by the definition of the Poisson jump with amplitude determined by the function h. This jump amplitude evaluation is consistent with the Itô forward integral approximation,*

$$\Delta \int_0^t h(P(s))dP(s) \simeq h(P(t))\Delta P(t)$$

for a single, sufficiently small time step Δt, picking the prior value of h at $P(t)$ in the case $\Delta P(t) > 0$, though it is not that obvious for the simple jump amplitude dependence $h(P(t))$, the picking of the pre-jump value is also a consequence of the right continuity property of the Poisson process (1.20).

Corollary 3.10.

$$\int_{t=0}^{t} P^m(s)dP(s) \stackrel{\text{dt}}{=} S^{(m)}_{P(t)-1} = \sum_{k=0}^{P(t)-1} k^m \,, \tag{3.11}$$

for $m \geq 0$ and the irregular summation convention (3.10) is applicable.

Remark 3.11. *A simple consistency check on (3.11) is to verify the simplest case when $m = 0$ and the integral of $(P^m dP)(t) = dP(t)$ on $[0, t]$ must be $P(t)$ by the fundamental theorem. The right hand side of (3.11), with $k^m = 1$, is*

$$\sum_{k=0}^{P(t)-1} 1 = (P(t) - 1 + 1) \cdot 1 = P(t) \,.$$

Theorem 3.12. ***General Poisson Stochastic Integral:***
Let $h(x, t)$ be a continuous function and let the process $h(X(t), t)$ have a bounded mean integral of squares (3.2) and satisfy the i-PWCA Mean Square Limits Assumption 2.15 for $Y(t) = h(X(t), t)$. Then,

$$\int_0^t h(X(s), s)dP(s) \stackrel{\text{dt}}{=} \sum_{k=1}^{P(t)} h(X(T_k^-), T_k^-) \,, \tag{3.12}$$

where T_k is the kth jump of Poisson process $P(t)$.

Proof. Here, we rely explicitly on both the Itô forward integration rule ($\theta = 0$) and the right-continuity property of $P(t)$. It is sufficient to examine the processes $P(t)$, $\Delta P(t)$ and $h(X(t), t)$ in the very neighborhood of the kth jump at time T_k, such that Δt is small enough that we can exclude the prior jump at T_{k-1} and the next jump at T_{k+1} with $T_{k-1} < t < T_{k+1}$. After all, the Poisson process is a rare event process. Thus, the Poisson process has the simple, right-continuous form

$$P(t) = \left\{ \begin{array}{ll} k-1, & T_{k-1} < t \leq T_k^- \\ k, & T_k = T_k^+ \leq t < T_{k+1} \end{array} \right\} ,$$

where $1 \leq k \leq P(t)$. However, the increment $\Delta P(t_i) = P(t_i + \Delta t) - P(t_i)$ is a function of both t_i and Δt for $i = 1 : n$, but we are interested in the limit as $\Delta t \to 0^+$ with t_i fixed in (T_{k-1}, T_{k+1}), so there are three case, both t_i and $t_i + \Delta t$

to the left of T_k, T_k between t_i and $t_i + \Delta t$ and both on the right of T_k, i.e.,

$$
\begin{aligned}
&h(X(t_i),t_i)\Delta P(t_i)\\
&\qquad = \left\{ \begin{array}{ll} 0, & T_{k-1} < t_i < t_i + \Delta t \le T_k^- \\ h(X(t_i),t_i), & T_{k-1} < t_i \le T_k^- < T_k = T_k^+ \le t_i + \Delta t < T_{k+1} \\ 0, & T_k = T_k^+ \le t_i < t_i + \Delta t < T_{k+1} \end{array} \right\} \\
&\qquad \to \left\{ \begin{array}{ll} 0, & T_{k-1} < t_i < T_k^- \\ h(X(T_k^-),T_k^-), & T_{k-1} < t_i = T_k^- \\ 0, & T_k = T_k^+ \le t_i < T_{k+1} \end{array} \right\},
\end{aligned}
$$

as $\Delta t \to 0^+$ with t_i fixed in (T_{k-1}, T_{k+1}) and this is valid for $1 \le k \le P(t)$. Thus, the Itô approximate sum is

$$
\begin{aligned}
\int_0^t h(X(s),s)dP(s) &\simeq \sum_{i=0}^{n} h(X(t_i),t_i)\Delta P(t_i) \\
&\to \sum_{k=1}^{P(t)} h(X(T_k^-),T_k^-)\,,
\end{aligned}
$$

as $n \to +\infty$ and $\delta t_n = \max_j[\Delta t_j] \to 0^+$, since for large n the $\Delta P(t_i)$ will be mostly zero and only the time intervals that straddle a jump T_k^- will be selected. The state process, different from the simple jump Poisson process, will in general undergo continuous changes between jumps of $P(t)$, but the right-continuity causes the immediate pre-jump value of the jump-amplitude at T_k^- to be chosen for each jump time T_k. □

Remark 3.13. *Obviously, if $h(X(t),t) = 1$, then $\sum_{k=1}^{P(t)} 1 = P(t)$. Another simple consistency check on (3.12) is to verify the case when $h(X(t),t) = P(t)$ and the integral of $(PdP)(t)$ on $[0,t]$ must be $(P(P-1))(t)/2$ by (3.5). The right hand side of (3.11), with $h(X(t),t) = P(t)$, $P(T_k^-) = k-1$, is*

$$
\sum_{k=1}^{P(t)} P(T_k^-) = \sum_{k=1}^{P(t)} (k-1) = P(t)(P(t)-1)/2\,,
$$

using the standard triangular number summation. Hence, Eq. (3.12) is consistent with Eq. (3.9).

Definition 3.14. ***Jump Function*** $[X](t)$***:***
The jump value of the state X at the pre-jump time T_k^- is defined as

$$
[X](T_k) \equiv X(T_k^+) - X(T_k^-) = h(X(T_k^-),T_k^-)\,, \tag{3.13}
$$

when the kth jump is at time T_k. For finite discontinuities, the jump function includes all the change of the function, the ***zeroth change or discrete derivative*** *of the state $X(t)$.*

Example 3.15. *Let*

$$Y(t) = \int_0^t h(X(s), s) dP(s)$$

and

$$\Delta Y(t) = \int_t^{t+\Delta t} h(X(s), s) dP(s) \simeq h(X(t), t)\Delta P(t)$$

for $0 < \Delta t \ll 1$*, so*

$$[Y](t) \equiv Y(t^+) - Y(t^-) = \int_{t^-}^{t^+} h(X(s), s) dP(s) = h(X(t^-), t^-) dP(t) \,, \quad (3.14)$$

since $dP(t) = dP(t^-)$ *with both being one when* $t = T_k^-$ *or* $t^- = T_k^-$ *but otherwise zero when* $T_{k-1} < t < T_k^-$ *or* $T_{k-1} < t^- < T_k^-$*.*

In the non-Itô integration approximation, $0 < \theta \le 1$,

$$\Delta \int_0^t h(P(s)) dP(s) \simeq h(P(t + \theta\Delta t))\Delta P(t) \,,$$

so if the last jump is T_k and the next one is T_{k+1}, such that $T_k < t < T_{k+1} < t + \Delta t$, i.e., within the single time step, then $P(t) = k$ and we get the jump amplitude is $h(k)$ if the jump is late, $t + \theta\Delta t < T_{k+1} < t + \Delta t$, since $P(t + \theta\Delta t) = k$, but we get the amplitude $h(k + 1)$ if the jump is early, $t < T_{k+1} < t + \theta\Delta t$, since $P(t + \theta\Delta t) = k + 1$. Thus, the Itô formulation has much less complexity and is more straight-forward to implement.

Some other jump differential products whose mean square limits will be useful are $dt\ dP(t)$ and $dP(t)\ dW(t)$, since they arise in the expansions of functions of stochastic differentials:

Lemma 3.16. ***Differentials Product*** $dt\ dP(t)$ ***and*** $dP(t)\ dW(t)$***:***

$$\int_0^t ds\ dP(s) \overset{ims}{=} 0, \quad (3.15)$$

or in symbolic notation

$$dt\ dP(t) \underset{sym}{\overset{ims}{=}} 0 \,, \quad (3.16)$$

and

$$\int_0^t dP(s)\ dW(s) \overset{ims}{=} 0, \quad (3.17)$$

or in symbolic notation

$$dP(t)\ dW(t) \underset{sym}{\overset{ims}{=}} 0 \,. \quad (3.18)$$

Proof. The proofs are similar to the proof for $dt\; dW(t)$, with a minor change in argument due to the non-zero incremental mean

$$\mathrm{E}[\Delta P(t_i)] = \lambda \Delta t_i .$$

Let

$$I[dt](t) = \int_0^t ds\; dP(s) \simeq I_n[dt](t) \equiv \sum_{i=0}^{n} \Delta t_i \Delta P(t_i). \tag{3.19}$$

The expectation of the sum $I_n[dt](t)$ yields

$$\begin{aligned}\mathrm{E}[I_n[dt](t)] &= \sum_{i=0}^{n} \mathrm{E}\left[\Delta t_i \Delta P(t_i)\right] = \sum_{i=0}^{n} \lambda (\Delta t_i)^2 \\ &\le \lambda t \delta t_n \to 0^+ \;,\end{aligned}$$

as $n \to +\infty$. The result suggests that the Itô mean square value is given by

$$I[dt](t) \overset{ims}{=} \underset{n\to\infty}{\overset{\mathrm{ms}}{\lim}} I_n[dt](t) = 0 \;.$$

This can be verified in the mean square limit by showing that the mean square limit is zero, while the splitting into independent increments is employed,

$$\begin{aligned}\mathrm{E}\left[\left(\sum_{i=0}^{n} \Delta t_i \Delta P_i - 0\right)^2\right] &= \sum_{i=0}^{n}\left((\Delta t_i)^2 \mathrm{E}[(\Delta P_i)^2] + \sum_{j\ne i} \Delta t_i \Delta t_j \mathrm{E}[\Delta P_i]\mathrm{E}[\Delta P_i]\right) \\ &= \sum_{i=0}^{n}\left(\lambda (\Delta t_i)^3 (1 + \lambda \Delta t_i) + \sum_{j\ne i} \lambda^2 (\Delta t_i \Delta t_j)^2\right) \\ &= \mathrm{O}^2(\delta t_n) \to 0 \;,\end{aligned}$$

as $n \to +\infty$. So,

$$dt\; dP(t) \underset{sym}{\overset{ims}{=}} 0 \;.$$

The cross product of differentials $dP(t)dW(t)$ works out similarly, except here we have the benefit of independence of processes as well as independence of respective process increments. Let

$$J(t) = \int_0^t dP(s)\; dW(s) \simeq J_n(t) \equiv \sum_{i=0}^{n} \Delta P(t_i) \Delta W(t_i). \tag{3.20}$$

The expectation of the sum $J_n(t)$ yields

$$\mathrm{E}[J_n(t)] = \sum_{i=0}^{n} \mathrm{E}\left[\Delta P(t_i) \Delta W(t_i)\right] = \sum_{i=0}^{n} \lambda (\Delta t_i) \cdot 0 = 0.$$

This result suggests that the Itô mean square value is given by

$$J(t) \overset{ims}{=} \underset{n\to\infty}{\overset{\text{ms}}{\lim}} [J_n(t)] = 0 ,$$

so that it is intuitively clear that the mean square limit will also behave like the cases $dt\ dW(t)$ and $dt\ dP(t)$, but the verification of the mean square limit is still needed and is left as Exercise 3 for the reader. □

The Itô mean square limits to an accuracy with error $\mathrm{o}(dt)$ in the case of the Poisson jump process are summarized in the Table 3.2:

Table 3.2. *Some Itô stochastic jump differentials with an accuracy with error* $\mathrm{o}(dt)$ *as* $dt \to 0^+$.

Differential Jump Form	Itô Mean Square Limit
$dP(t)$	$dP(t)$
dt	dt
$dt\ dP(t)$	0
$(dP)^m(t)$	$dP(t),\ m \ge 1$
$dP(t)\ dW(t)$	0
$(dt)^k(dP)^m(t)$	$0,\ k \ge 1,\ \ m \ge 1$
$(dt)^k(dP)^m(t)(dW)^n(t)$	$0,\ \ k \ge 1,\ \ m \ge 1,\ \ n \ge 1$

Remarks 3.17.

- *In the use of Table 3.2, the differential entries are just symbols of the underlying integral basis and care should be taken when applying them to find the mean square representation of differentials, especially when they appear in multiplicative combinations.*

- *The mean square limit justification of the power rule* $(dP)^m(t) \overset{\text{dt}}{=} dP(t)$ *is left as Exercise 4, along with Exercise 3 previously mentioned for* $dP(t)dW(t)$.

3.2 Stochastic Jump Integration Rules and Expectations:

Theorem 3.18. ***Itô Stochastic Jump Integral Simple Rules:***
Let h, h_1 *and* h_2 *satisfy the mean square integrability condition (2.44) on* $0 \le t_0 \le t$, *while letting* c_1 *and* c_2 *be constants.*

- ***Operator Linearity:***

$$\int_{t_0}^{t}[c_1 h_1(P(s),s) \;+\; c_2 h_2(P(s),s)]dP(s)$$
$$\stackrel{ims}{=} c_1\int_{t_0}^{t} h_1(P(s),s)dP(s) + c_2\int_{t_0}^{t} h_2(P(s),s)dP(s)\;.$$

- ***Additivity over Subintervals:***

$$\int_{t_0}^{t} h(P(s),s)dP(s) \stackrel{ims}{=} \int_{t_0}^{r} h(P(s),s)dP(s) + \int_{r}^{t} h(P(s),s)dP(s) \tag{3.21}$$

for $0 \le t_0 \le r \le t$.

Proof. These are clearly true by examining the forward integration approximation. □

Poisson jump processes may seem easier in terms of differentials, but they can lead to more difficulties when more complicated integral properties are considered.

Theorem 3.19. ***Some Mean Stochastic Jump Integrals:***
Let $h(P(t),t)$ *satisfy the mean square integrability condition on* $0 \le t_0 \le t$,

$$\mathrm{E}\left[\int_{t_0}^{t} h^2(P(s),s)ds\right] < \infty \tag{3.22}$$

and the i-PWCA Mean Square Limits Assumption 2.15 for $Y(t) = h(P(t),t)$, *where* $\mathrm{E}[dP(t)] = \lambda(t)dt$, *then*

1. $\boldsymbol{E[\int h(P(s),s)dP(s)]}$*:*

$$\mathbf{E}\left[\int_{t_0}^{t} \boldsymbol{h(P(s),s)dP(s)}\right] \stackrel{ims}{=} \int_{t_0}^{t} \mathbf{E}[\boldsymbol{h(P(s),s)}]\boldsymbol{\lambda(s)ds}. \tag{3.23}$$

2. $\mathbf{E}[\int \boldsymbol{h(P(s),s)d\widehat{P}(s)}]$*: Letting*

$$\boldsymbol{d\widehat{P}(t) \equiv dP(t) - \lambda(t)dt} \tag{3.24}$$

be the simple ***mean-zero Poisson process****,*

$$\mathbf{E}\left[\int_{t_0}^{t} \boldsymbol{h(P(s),s)d\widehat{P}(s)}\right] \stackrel{ims}{=} \mathbf{0}\;. \tag{3.25}$$

3. $\mathbf{E}[|\int \boldsymbol{h(P(s),s)dP(s)}|]$ ***Estimate:***

$$\mathbf{E}\left[\left|\int_{t_0}^{t} \boldsymbol{h(P(s),s)dP(s)}\right|\right] \le \int_{t_0}^{t} \mathbf{E}\left[|\boldsymbol{h(P(s),s)}|\right]\boldsymbol{\lambda(s)ds}\;, \tag{3.26}$$

where the inequality is in the mean square sense.

4. $\mathbf{E}[\int \boldsymbol{h_1(P(s),s)d\widehat{P}(s)} \int \boldsymbol{h_2(P(r),r)d\widehat{P}(r)}]$*: Let* $\boldsymbol{h_1(P(t),t)}$ *and* $\boldsymbol{h_2(P(t),t)}$ *satisfy the same mean square integrability condition (2.44) as* $\boldsymbol{h(P(t),t)}$ *on* $\boldsymbol{0 \le t_0 \le t}$*, then the Itô-Covariance for jump stochastic integrals is*

$$\begin{aligned} &\mathbf{E}\left[\int_{t_0}^{t} \boldsymbol{h_1(P(s),s)d\widehat{P}(s)} \int_{t_0}^{t} \boldsymbol{h_2(P(r),r)d\widehat{P}(r)}\right] \\ &\stackrel{ims}{=} \int_{t_0}^{t} \mathbf{E}\left[\boldsymbol{h_1(P(s),s)h_2(P(s),s)}\right] \boldsymbol{\lambda(s)ds} \ . \end{aligned} \tag{3.27}$$

5. $\mathbf{E}[(\int \boldsymbol{h(P(s),s)d\widehat{P}(s)})^2]$*: Also the Itô-Variance for jump stochastic integrals,*

$$\mathbf{E}\left[\left(\int_{t_0}^{t} \boldsymbol{h(P(s),s)d\widehat{P}(s)}\right)^2\right] \stackrel{ims}{=} \int_{t_0}^{t} \mathbf{E}[\boldsymbol{h^2(P(s),s)}]\boldsymbol{\lambda(s)ds} \ . \tag{3.28}$$

Sketch of Proof. Only fast heuristic or formal justification will be given here to keep this presentation simple, since many of the techniques have been given earlier for diffusion $W(t)$ and our interests are in applications.

1. Using the Itô mean square limit (2.27), we have the formal finite sum approximation using partition (2.28) with $h_i = h(P(t_i), t_i)$ for the expectation,

$$\begin{aligned} \mathrm{E}\left[\int_{t_0}^{t} h(P(s),s)dP(s)\right] &\simeq \sum_{i=0}^{n} \mathrm{E}[h_i \Delta P_i] = \sum_{i=0}^{n} \mathrm{E}[h_i]\mathrm{E}[\Delta P_i] \\ &= \sum_{i=0}^{n} \mathrm{E}[h_i]\lambda_i \Delta t_i \ , \end{aligned}$$

the last line using the independent increments and mean properties. Hence (3.23) is formally justified.

2. The form (3.25) follows immediately by combining both sides of the mean square equation in part (a).

3. Again using the forward integration approximation, but with the triangular inequality, the expectation of the absolute value of the stochastic jump integral formally follows,

$$\begin{aligned} \mathrm{E}\left[\left|\int_{t_0}^{t} h(P(s),s)dP(s)\right|\right] &\simeq \mathrm{E}\left[\left|\sum_{i=0}^{n} h_i \Delta P_i\right|\right] \le \sum_{i=0}^{n} \mathrm{E}[|h_i|\Delta P_i] \\ &= \sum_{i=0}^{n} \mathrm{E}[|h_i|]\mathrm{E}[\Delta P_i] = \sum_{i=0}^{n} \mathrm{E}[|h_i|]\lambda_i \Delta t_i \\ &\xrightarrow{ims} \int_{t_0}^{t} \mathrm{E}[|h(P(s),s)|]\lambda(s)ds \ , \end{aligned}$$

as $n \to +\infty$, using the means square limit in the last step to get the desired limiting estimate.

4. Due to the mean zero property (3.25) of the stochastic jump integral with respect to the mean zero process $d\widehat{P}(t)$ (3.24), the Itô forward integration approximation to the covariance of the stochastic jump integral follows. However, the use of the mean zero process is critical, otherwise the independent increment property is not very helpful. As in the $W(t)$ diffusion case, the approximate finite difference double sum is split up into three parts, the diagonal $(j = i)$, lower diagonal $(j < i)$ and upper diagonal $(j > i)$ parts,

$$\begin{aligned}
\mathrm{E}\left[\int_{t_0}^{t} h_1(P(s),s)d\widehat{P}(s)\int_{t_0}^{t} h_2(P(r),r)d\widehat{P}(r)\right] &\simeq \sum_{i=0}^{n}\sum_{j=0}^{n}\mathrm{E}[h_{1,i}\Delta\widehat{P}_i h_{2,i}\Delta\widehat{P}_j] \\
&\simeq \sum_{i=0}^{n}\mathrm{E}[h_{1,i}h_{2,i}]\mathrm{E}[(\Delta\widehat{P}_i)^2] \\
&\quad + \sum_{i=0}^{n}\sum_{j=0}^{i-1}\mathrm{E}[h_{1,i}h_{2,j}\Delta\widehat{P}_j]\mathrm{E}[\Delta\widehat{P}_i] \\
&\quad + \sum_{i=0}^{n}\sum_{j=i+1}^{n}\mathrm{E}[h_{1,i}h_{2,j}\Delta\widehat{P}_i]\mathrm{E}[\Delta\widehat{P}_j] \\
&= \sum_{i=0}^{n}\mathrm{E}[h_{1,i}h_{2,i}]\lambda_i\Delta t_i \\
&\xrightarrow{ims} \int_{t_0}^{t}\mathrm{E}\left[h_1(P(s),s)h_2(P(s),s)\right]\lambda(s)ds\,,
\end{aligned}$$

giving the desired conclusion except for replacing the approximately equals $(\simeq)$ by the mean square limit as $n \to \infty$

5. The Itô-variance stochastic jump integral follows immediately from part (d) for the Itô-covariance stochastic jump integral by replacing the functions h_1 and h_2 by h. This result (3.28) is also called *Itô isometry* or *martingale isometry* since $\widehat{P}(t)$ is a martingale.

□

3.3 Conclusion

In this chapter, the foundations have been laid for the integrals of the third type in the integrated SDE (2.2), i.e., using the stochastic jump integral of Itô of Definition 2.8 extended to the more general case and defined in Definition 3.1 at the beginning of this chapter:

$$\begin{aligned}
\int_0^t h(X(s),s)dP(s) &\overset{ims}{=} \underset{n\to\infty}{\overset{\text{ms}}{\lim}}\left[\sum_{i=0}^{n} h(X(t_i),t_i)dP(t_i)\right]\,, \\
&= \sum_{k=1}^{P(t)} h(X(T_k^-),T_k^-)
\end{aligned} \tag{3.29}$$

where $X(t)$ in the integrand function h has an implied dependence on the simple Poisson jump process $P(t)$, but also depends on the diffusion process $W(t)$. The integrand process $h(X(t), t)$ is also assumed to have a bounded mean integral of squares (3.2),

$$\mathrm{E}\left[\int_0^t h^2(X(s), s)ds\right] < \infty \,,$$

with the usual grid partition specifications on $[0, t]$. However, as previously explained, the Poisson jump process fits within the framework of Itô stochastic integration since it is piece-wise continuous. The stochastic jump integration rule (3.29) has been motivated and illustrated by a number of examples using functions and powers of the jump process $P(t)$.

3.4 Exercises

1. Show that the power rules for stochastic integration for Poisson noise can be written as the recursions,

$$\int_0^t P^m(s)dP(s) = \frac{1}{m+1}P^{m+1}(t) - \sum_{k=2}^{m+1}\binom{m+1}{k}\int_0^t P^{m+1-k}(s)dP(s) \,,$$

using the jump form of the stochastic chain rule and mathematical induction.

   (a) Illustrate the application of the formulae for $P(t)$ to find the results for the cases $m = 2$ and $m = 3$, to get explicit formulas that do not have an integral. See Table 3.1.

   (b) Alternatively, show the general result for $m \geq 1$.

2. Show that the partial sums of the **geometric series** can be summed as

$$S_n \equiv \sum_{k=0}^{n} x^k = \left\{ \begin{array}{cc} \frac{1-x^{n+1}}{1-x}, & x \neq 1 \\ n+1, & x = 1 \end{array} \right\} , \tag{3.30}$$

for integers $n \geq 0$ by showing that the difference of the defined summation (ΔS_n) and the difference of the summed answer to the far right are the same and that the discrete initial conditions are the same at $n = 0$.

3. Show the mean square limit for the product of $dP(t)$ and $dW(t)$ in (3.17-3.18) by proving that

$$\mathrm{Var}\left[\sum_{i=0}^{n} \Delta P_i \Delta W_i\right] \to 0 \,, \tag{3.31}$$

as $n \to +\infty$ and $\delta t_n \to 0^+$.

4. Show the mean square limit for the Poisson increment power $(dP)^m(t)$ version of the Zero-One-Jump law in Table 3.2 by proving that

$$\mathrm{E}\left[\left(\sum_{i=0}^{n}((\Delta P_i)^m - \Delta P_i)\right)^2\right] \to 0\,, \tag{3.32}$$

as $n \to +\infty$ and $\delta t_n \to 0^+$ for $m \geq 1$. *Note, you are asked to prove the mean square limit for this particular law,*

$$(dP)^m(t) \underset{sym}{\overset{ims}{=}} dP(t),$$

not the Zero-One-Jump law itself.

5. Show that

$$\int_0^t e^{aP(s)}dP(s) = \left\{ \begin{array}{cc} \dfrac{e^{aP(t)}-1}{e^a-1}, & e^a \neq 1 \text{ or } a \neq 0 \\ P(t), & e^a = 1 \text{ or } a = 0 \end{array} \right\}, \tag{3.33}$$

for real constant a, in two ways, showing that they give the same answers,

(a) Using the Poisson sum form $\sum_{k=0}^{P(t)-1} h(k)$ of Theorem 3.8 and the geometric series partial sum results in (3.30) of this Exercise section.

(b) Using the Zero-One Jump Law and the Fundamental Theorem of Jump Calculus applied to $d\exp(aP(t))$ to evaluate the integral.

Suggested References for Further Reading

- Çinlar, 1975 [55].
- Protter 1990, [228].
- Snyder and Miller, 1991 [247].
- Tuckwell, 1995 [265].