Chapter 5

Stochastic Calculus for General Markov SDEs: Space-Time Poisson, State-Dependent Noise and Multidimensions

Not everything that counts can be counted,
and not everything that can be counted counts.
—Albert Einstein (1879–1955)

The only reason for time is so that everything doesn't happen at once.
—Albert Einstein at
http://www.brainyquote.com/quotes/authors/a/albert_einstein.html

Time is nature's way of keeping everything from happening at once.
Space is what prevents everything from happening to me.
—attributed to John Archibald Wheeler at
http://en.wikiquote.org/wiki/Time

What about stochastic effects?
—Don Ludwig, University of British Columbia,
printed on his tee-shirt to save having to ask it at each seminar

We are born by accident into a purely random universe. Our lives are determined by entirely fortuitous combinations of genes. Whatever happens happens by chance. The concepts of cause and effect are fallacies. There is only seeming *causes leading to* apparent *effects. Since nothing truly follows from anything else, we swim each day through seas of chaos, and nothing is predictable, not even the events of the very next instant.*

Do you believe that?

If you do, I pity you, because yours must be a bleak and terrifying and comfortless life.
—Robert Silverberg in *The Stochastic Man*, 1975

This chapter completes the generalization of Markov noise in continuous time for this book, by including space-time Poisson noise, state-dependent SDEs and multidimensional SDEs.

5.1 Space-Time Poisson Process

Space-time Poisson processes are also called general compound Poisson processes, marked Poisson point processes and Poisson noise with randomly distributed jump-amplitudes conditioned on a Poisson jump in time. The marked adjective refers to marks which are the underlying stochastic process for the Poisson jump-amplitude or the space component of the space-time Poisson process, whereas the jump-amplitudes of the simple Poisson process are deterministic or fixed with unit magnitude. The space-time Poisson process is a generalization of the Poisson process. The space-time Poisson process formulation helps in understanding the mechanism for applying it to more general jump applications and generalization of the chain rules of stochastic calculus.

Properties 5.1.

- ***Space-time Poisson differential process:*** *The basic space-time or mark-time Poisson differential process denoted as*

$$d\Pi(t) = \int_{\mathcal{Q}} h(t,q)\mathcal{P}(\mathbf{dt},\mathbf{dq}) \tag{5.1}$$

  on the ***Poisson mark space*** $\mathcal{Q}$ *can be defined using the* ***Poisson random measure*** $\mathcal{P}(\mathbf{dt},\mathbf{dq})$*, which is shorthand measure notation for the measure-set equivalence* $\mathcal{P}(\mathbf{dt},\mathbf{dq}) = \mathcal{P}((t,t+dt],(q,q+dq])$*. The jump-amplitude* $h(t,q)$ *is assumed to be continuous and bounded in its arguments.*

- ***Poisson mark Q:*** *The space Poisson mark* Q *is the underlying IID random variable for the mark-dependent jump-amplitude coefficient denoted by* $h(t,Q) = 1$*, i.e., the space part of the space-time Poisson process. The realized variable* $Q = q$ *is used in expectations or conditional expectations, as well as in definition of the type* (5.1)*.*

- ***Time-integrated, space-time Poisson process:***

$$\Pi(t) = \int_0^t \int_{\mathcal{Q}} h(t,q)\mathcal{P}(\mathbf{dt},\mathbf{dq})dt. \tag{5.2}$$

- ***Unit jumps:*** *However, if the jumps have unit amplitudes,* $h(t,Q) \equiv 1$*, then the space time process in* (5.1) *must be the same result as the simple differential Poisson process* $dP(t;Q)$ *modified with a mark parameter argument to allow for generating mark realizations, and we must have the equivalence*

$$\int_{\mathcal{Q}} \mathcal{P}(\mathbf{dt},\mathbf{dq}) \equiv dP(t;Q), \tag{5.3}$$

giving the jump number count on $(t, t+dt]$. *Integrating both sides of* (5.3) *on* $[0, t]$ *gives the jump-count up to time* t,

$$\int_0^t \int_{\mathcal{Q}} \mathcal{P}(\mathbf{dt}, \mathbf{dq}) = \int_0^t dP(s; Q) = P(t; Q). \tag{5.4}$$

Further, in terms of Poisson random measure $\mathcal{P}(\mathbf{dt}, \{1\})$ *on the fixed set* $\mathcal{Q} = \{1\}$, *purely the number of jumps in* $(t, t+dt]$ *is obtained,*

$$\int_{\mathcal{Q}} \mathcal{P}(\mathbf{dt}, \mathbf{dq}) = \mathcal{P}(\mathbf{dt}, \{1\}) = P(\mathbf{dt}) = dP(t; 1) \equiv dP(t)$$

and the marks are irrelevant.

- ***Purely time-dependent jumps:*** *If* $h(t, Q) = h_1(t)$, *then*

$$\int_{\mathcal{Q}} h_1(t)\mathcal{P}(\mathbf{dt}, \mathbf{dq}) \equiv h_1(t)dP(t; Q). \tag{5.5}$$

- ***Compound Poisson process form:*** *An alternate form of the space-time Poisson process* (5.2) *that many may find more comprehensible is the marked generalization of the* ***simple Poisson process*** $P(t; Q)$, ***with IID random mark generation****, that is, the counting sum called the* ***compound Poisson process*** *or* ***marked point process****,*

$$\Pi(t) = \sum_{k=1}^{P(t;Q)} h(T_k^-, Q_k), \tag{5.6}$$

where $h(T_k^-, Q_k)$ *is the* k*th jump-amplitude,* T_k^- *is the prejump value of the* k*th random jump-time,* Q_k *is the corresponding random jump-amplitude mark realization and for the special case that* $P(t; Q)$ *is zero the following reverse-sum convention is used,*

$$\sum_{k=1}^{0} h(T_k^-, Q_k) \equiv 0 \tag{5.7}$$

for any h. *The corresponding differential process has the expectation,*

$$\mathrm{E}[dP(t; Q)] = \lambda(t)dt,$$

although it is possible that the jump-rate is mark-dependent (see [223], *for example) so that*

$$\mathrm{E}[dP(t; Q)] = \mathrm{E}_Q[\lambda(t; Q)]dt.$$

However, it will be assumed here that the jump-rate is mark-independent to avoid complexities with iterated expectations later.

- ***Zero-one law compound Poisson differential process form:*** *Given the Poisson compound process form in* (5.6)*, the corresponding* ***zero-one jump law*** *for the compound Poisson differential process is*

$$d\Pi(t) = h(t, Q)dP(t; Q), \tag{5.8}$$

such that the jump in $\Pi(t)$ *at* $t = T_k$ *is given by*

$$[\Pi](T_k) \equiv \Pi(T_k^+) - \Pi(T_k^-) = h(T_k^-, Q_k). \tag{5.9}$$

For consistency with the Poisson random measure and compound Poisson process forms, it is necessary that

$$\int_0^t h(s, Q)dP(s; Q) = \int_0^t \int_{\mathcal{Q}} h(s, q)\mathcal{P}(\mathbf{ds}, \mathbf{dq}) = \sum_{k=1}^{P(t;Q)} h(T_k^-, Q_k),$$

so

$$\int_0^t dP(s; Q) = \int_0^t \int_{\mathcal{Q}} \mathcal{P}(\mathbf{ds}, \mathbf{dq}) = P(t; Q)$$

and

$$dP(t; Q) = \int_{\mathcal{Q}} \mathcal{P}(\mathbf{dt}, \mathbf{dq}).$$

Note that the selection of the random marks depends on the existence of the Poisson jumps and that the mechanism is embedded in $dP(t; Q)$ *in the formulation of this book.*

- *In the* ***Poisson random measure notation*** $\boldsymbol{\mathcal{P}(\mathbf{dt}, \mathbf{dq})}$*, the arguments* $\mathbf{dt}$ *and* $\mathbf{dq}$ *are semiclosed subintervals when these arguments are expanded,*

$$\mathcal{P}(\mathbf{dt}, \mathbf{dq}) = \mathcal{P}((t, t + dt], (q, q + dq]).$$

These subintervals are closed on the left and open on the right due to the definition of the increment, leaving no overlap between differential increments and correspondings to the simple Poisson right continuity property that

$$\Delta P(t; Q) \to P(t^+; Q) - P(t; Q) \quad as \quad \Delta t \to 0^+,$$

so we can write $\Delta P(t; Q) = P((t, t + \Delta t]; Q)$ *and* $dP(t; Q) = P((t, t + dt]; Q)$*. When* $t_n = t$ *and* $t_{i+1} = t_i + \Delta t_i$*, the covering set of intervals is* $\{[t_i, t_i + \Delta t_i) \text{ for } i = 0 : n\}$ *plus* t*. If the marks* Q *are continuously distributed, then closed subintervals can also be used in the* q *argument. For the one-dimensional mark space* $\mathcal{Q}$*,* $\mathcal{Q}$ *can be a finite interval such as* $\mathcal{Q} = [a, b]$ *or an infinite interval such as* $\mathcal{Q} = (-\infty, +\infty)$*. Also, these subintervals are convenient in partitioning continuous intervals since they avoid overlap at the nodes.*

- ***$\mathcal{P}$* has independent increments** *on nonoverlapping intervals in time t and marks q, i.e., $\mathcal{P}_{i,k} = \mathcal{P}((t_i, t_i + \Delta t_i], (q_k, q_k + \Delta q_k])$ is independent of $\mathcal{P}_{j,\ell} = \mathcal{P}((t_j, t_j + \Delta t_j], (q_\ell, q_\ell + \Delta q_\ell])$, provided that the time interval $(t_j, t_j + \Delta t_j]$ has no overlap with $(t_i, t_i + \Delta t_i]$ and the mark interval $(q_k, q_k + \Delta q_k]$ has no overlap with $(q_\ell, q_\ell + \Delta q_\ell]$. Recall that $\Delta P(t_i; Q) \equiv P(t_i + \Delta t_i; Q) - P(t_i; Q)$ is associated with the time interval $(t_i, t_i + \Delta t_j]$, open on the left since the process $P(t_i; Q)$ has been subtracted to form the increment.*

- *The* ***expectation of $\mathcal{P}(\mathbf{dt}, \mathbf{dq})$*** *is*

$$\mathrm{E}[\mathcal{P}(\mathbf{dt}, \mathbf{dq})] = \Phi_Q(\mathbf{dq})\lambda(t)dt \overset{\text{gen}}{=} \phi_Q(q)dq\lambda(t)dt, \tag{5.10}$$

where, in detail,

$$\begin{aligned} \Phi_Q(\mathbf{dq}) &= \Phi_Q((q, q + dq]) = \Phi_Q(q + dq) - \Phi_Q(q) \\ &= \mathrm{Prob}[Q \le q + dq] - \mathrm{Prob}[Q \le q] = \mathrm{Prob}[q < Q \le q + dq] \\ &\overset{\text{gen}}{=} \phi_Q(q)dq \end{aligned}$$

is the probability distribution measure of the Poisson amplitude mark in measure-theoretic notation corresponding to the mark distribution function $\Phi_Q(q)$ and where $\mathbf{dq}$ *is shorthand for the arguments $(q, q + dq]$, just as the* $\mathbf{dt}$ *in $\mathcal{P}(\mathbf{dt}, \mathbf{dq})$ is shorthand for $(t, t + dt]$. The corresponding mark density will be equal to $\phi_Q(q)$ if Q is continuously distributed with continuously differentiable distribution function and also if the mark density is equal to $\phi_Q(q)$ in the generalized sense (symbol $\overset{\text{gen}}{=}$), for instance, if Q is discretely distributed. Generalized densities will be assumed for almost all distributions encountered in applications. It is also assumed that Φ_Q is a proper distribution,*

$$\int_{\mathcal{Q}} \Phi_Q(\mathbf{dq}) = \int_{\mathcal{Q}} \phi_Q(q)dq = 1.$$

- ***Poisson random measure $\mathcal{P}(\boldsymbol{\Delta}\mathbf{t}_i, \boldsymbol{\Delta}\mathbf{q}_j)$ is Poisson distributed****, i.e.,*

$$\mathrm{Prob}[\mathcal{P}(\boldsymbol{\Delta}\mathbf{t}_i, \boldsymbol{\Delta}\mathbf{q}_j) = k] = e^{-\overline{\mathcal{P}}_{i,j}} \left(\overline{\mathcal{P}}_{i,j}\right)^k / k!, \tag{5.11}$$

where

$$\overline{\mathcal{P}}_{i,j} = \mathrm{E}[\mathcal{P}(\boldsymbol{\Delta}\mathbf{t}_i, \boldsymbol{\Delta}\mathbf{q}_j)] = \Phi_Q(\boldsymbol{\Delta}\mathbf{q}_j) \int_{\boldsymbol{\Delta}\mathbf{t}_i} \lambda(t)dt = \Phi_Q(\boldsymbol{\Delta}\mathbf{q}_j)\Lambda(\boldsymbol{\Delta}\mathbf{t}_i)$$

for sets $\boldsymbol{\Delta}\mathbf{t}_i \equiv [t_i, t_i + \Delta t_i)$ in time and $\boldsymbol{\Delta}\mathbf{q}_j \equiv [q_j, q_j + \Delta q_j)$ in marks.

Thus, as Δt_i and Δq_j approach 0^+, they can be replaced by dt and dq, respectively, so

$$\mathrm{Prob}[\mathcal{P}(\mathbf{dt}, \mathbf{dq}) = k] = e^{-\overline{\mathcal{P}}} \left(\overline{\mathcal{P}}\right)^k / k!, \tag{5.12}$$

where

$$\overline{\mathcal{P}} = \mathrm{E}[\mathcal{P}(\mathbf{dt}, \mathbf{dq})] = \phi_Q(q)dq\lambda(t)dt,$$

so by the zero-one jump law,

$$\text{Prob}[\mathcal{P}(\mathbf{dt},\mathbf{dq})=k] \underset{\text{zol}}{\overset{\text{dt}}{=}} (1-\overline{\mathcal{P}})\delta_{k,0}+\mathcal{P}\delta_{k,1}.$$

- *The* ***expectation of*** $\boldsymbol{dP(t;Q)=\int_{\mathcal{Q}}\mathcal{P}(\mathbf{dt},\mathbf{dq})}$ *is*

$$\mathrm{E}\left[\int_{\mathcal{Q}}\mathcal{P}(\mathbf{dt},\mathbf{dq})\right]=\lambda(t)dt\int_{\mathcal{Q}}\phi_Q(q)dq=\lambda(t)dt=\mathrm{E}[dP(t;Q)], \tag{5.13}$$

corresponding to the earlier Poisson equivalence (5.3) *and using the above proper distribution property. Similarly,*

$$\mathrm{E}\left[\int_0^t\int_{\mathcal{Q}}\mathcal{P}(\mathbf{ds},\mathbf{dq})\right]=\mathrm{E}[P(t;Q)]=\int_0^t\lambda(s)ds=\Lambda(t).$$

- *The* ***variance of*** $\boldsymbol{\int_{\mathcal{Q}}\mathcal{P}(\mathbf{dt},\mathbf{dq})\equiv dP(t;Q)}$ *is by definition*

$$\mathrm{Var}\left[\int_{\mathcal{Q}}\mathcal{P}(\mathbf{dt},\mathbf{dq})\right]=\mathrm{Var}[dP(t;Q)]=\lambda(t)dt. \tag{5.14}$$

Since

$$\mathrm{Var}\left[\int_{\mathcal{Q}}\mathcal{P}(\mathbf{dt},\mathbf{dq})\right]=\int_{\mathcal{Q}}\int_{\mathcal{Q}}\mathrm{Cov}[\mathcal{P}(\mathbf{dt},\mathbf{dq}_1),\mathcal{P}(\mathbf{dt},\mathbf{dq}_2)],$$

then

$$\mathrm{Cov}[\mathcal{P}(\mathbf{dt},\mathbf{dq}_1),\mathcal{P}(\mathbf{dt},\mathbf{dq}_2)]\overset{\text{gen}}{=}\lambda(t)dt\phi_Q(q_1)\delta(q_1-q_2)dq_1dq_2, \tag{5.15}$$

analogous to (1.48) *for* $\mathrm{Cov}[dP(s_1),dP(s_2)]$*. Similarly, since*

$$\mathrm{Var}\left[\int_t^{t+\Delta t}\int_{\mathcal{Q}}\mathcal{P}(\mathbf{ds},\mathbf{dq})\right]=\mathrm{Var}[\Delta P(t;Q)]=\Delta\Lambda(t)$$

and

$$\mathrm{Var}\left[\int_t^{t+\Delta t}\int_{\mathcal{Q}}\mathcal{P}(\mathbf{ds},\mathbf{dq})\right]=\int_t^{t+\Delta t}\int_t^{t+\Delta t}\int_{\mathcal{Q}}\int_{\mathcal{Q}}\mathrm{Cov}[\mathcal{P}(\mathbf{ds}_1,\mathbf{dq}_1),\mathcal{P}(\mathbf{ds}_2,\mathbf{dq}_2)],$$

then

$$\begin{aligned}\mathrm{Cov}[\mathcal{P}(\mathbf{ds}_1,\mathbf{dq}_1),\mathcal{P}(\mathbf{ds}_2,\mathbf{dq}_2)]\overset{\text{gen}}{=}&\,\lambda(s_1)\delta(s_2-s_1)ds_1ds_2\\&\cdot\phi_Q(q_1)\delta(q_1-q_2)dq_1dq_2,\end{aligned} \tag{5.16}$$

embodying the independent increment properties in both time and mark arguments of the space-time or mark-time Poisson process in differential form.

- *It is assumed that* ***jump-amplitude function*** h ***has finite second order moments****, i.e.,*

$$\int_{\mathcal{Q}} |h(t,q)|^2 \phi_Q(q)dq < \infty \tag{5.17}$$

for all $t \geq 0$ *and, in particular,*

$$\int_0^t \int_{\mathcal{Q}} |h(s,q)|^2 \phi_Q(q)dq\lambda(s)ds < \infty. \tag{5.18}$$

- *From Theorem* 3.12 *(p.* 73*) and* (3.12)*, a* ***generalization of the standard compound Poisson process*** *is obtained,*

$$\int_0^t \int_{\mathcal{Q}} h(s,q)\mathcal{P}(\mathbf{ds},\mathbf{dq}) = \sum_{k=1}^{P(t;Q)} h(T_k^-, Q_k), \tag{5.19}$$

i.e., the jump-amplitude counting version of the space-time integral, where T_k *is the* k*th jump-time of a Poisson process* $P(t;Q)$ *and provided comparable assumptions are satisfied. This is also consistent for the infinitesimal counting sum form in* (5.6) *and the convention* (5.7) *applies for* (5.19)*. This form is a special case of the filtered compound Poisson process considered in Snyder and Miller* [252, Chapter 5]*. The form* (5.19) *is somewhat awkward due to the presence of three random variables,* $P(t;Q)$*,* T_k *and* Q_k*, requiring multiple iterated expectations.*

- *For a* ***compound Poisson process with time-independent jump-amplitude****,* $h(t,q) = h_2(q)$ *(the simplest case being* $h(t,q) = q$*), i.e., we then have*

$$\Pi_2(t) = \int_0^t \int_{\mathcal{Q}} h_2(q)\mathcal{P}(\mathbf{ds},\mathbf{dq}) = \int_{\mathcal{Q}} h_2(q)\mathcal{P}([0,t),\mathbf{dq}) = \sum_{k=1}^{P(t;Q)} h_2(Q_k), \tag{5.20}$$

where the sum is zero when $P(t;Q) = 0$*, the jump-amplitudes* $h_2(Q_k)$ *form a set of IID random variables independent of the jump-times of the Poisson process* $P(t;Q)$*; see* [56] *and Snyder and Miller* [252, Chapter 4]*. The mean can be computed by double iterated expectations, since the jump-rate is mark-independent,*

$$\begin{aligned} \mathrm{E}[\Pi_2(t)] &= \mathrm{E}_{P(t;Q)}\left[\sum_{k=1}^{P(t;Q)} \mathrm{E}_Q[h_2(Q_k)|P(t;Q)]\right] \\ &= \mathrm{E}_{P(t;Q)}\left[P(t;Q)\mathrm{E}_Q[h_2(Q)]\right] = \mathrm{E}_Q[h_2(Q)]\Lambda(t), \end{aligned}$$

where the IID property and more have been used, e.g., $\Lambda(t) = \int_0^t \lambda(s)ds$*.*

Similarly, the variance is calculated, letting $\overline{h}_2 \equiv \mathrm{E}_Q[h_2(Q)]$,

$$\begin{aligned}
\mathrm{Var}[\Pi_2(t)] &= \mathrm{E}\left[\left(\sum_{k=1}^{P(t;Q)} h_2(Q_k) - \overline{h}_2\Lambda(t)\right)^2\right] \\
&= \mathrm{E}\left[\left(\sum_{k=1}^{P(t;Q)} \left(h_2(Q_k) - \overline{h}_2\right) + \overline{h}_2(P(t;Q) - \Lambda(t))\right)^2\right] \\
&= \mathrm{E}_{P(t;Q)}\Bigg[\sum_{k_1=1}^{P(t;Q)}\sum_{k_2=1}^{P(t;Q)} \mathrm{E}_Q\left[\left(h_2(Q_{k_1}) - \overline{h}_2\right)\left(h_2(Q_{k_2}) - \overline{h}_2\right)\right] \\
&\quad + 2\overline{h}_2(P(t;Q) - \Lambda(t))\sum_{k=1}^{P(t;Q)} \mathrm{E}_Q\left[h_2(Q_k) - \overline{h}_2\right] \\
&\quad + \overline{h}_2^2(P(t;Q) - \Lambda(t))^2\Bigg] \\
&= \mathrm{E}_{P(t;Q)}\Big[P(t;Q)\mathrm{Var}_Q[h_2(Q)] + 2\overline{h}_2(P(t;Q) \\
&\quad -\Lambda(t))P(t;Q)\cdot 0 + \overline{h}_2^2(P(t;Q) - \Lambda(t))^2\Big] \\
&= \left(\mathrm{Var}_Q[h_2(Q)] + \overline{h}_2^2\right)\Lambda(t) = \mathrm{E}_Q\left[h_2^2(Q)\right]\Lambda(t),
\end{aligned}$$

using the IID property, separation into mean-zero forms and the variance-expectation identity (B.186).

- *For* ***compound Poisson process with both time- and mark-dependence****,* $h(t,q)$ *and* $\lambda(t;q)$*, we then have*

$$\Pi(t) = \int_0^t \int_{\mathcal{Q}} h(s,q)\mathcal{P}(\mathbf{ds},\mathbf{dq}) = \sum_{k=1}^{P(t;Q)} h(T_k^-, Q_k); \tag{5.21}$$

however, the iterated expectations technique is not very useful for the compound Poisson form, due to the additional dependence introduced by the jump-time T_k *and the jump-rate* $\lambda(t;q)$*, but the Poisson random measure form is more flexible:*

$$\begin{aligned}
\mathrm{E}[\Pi(t)] &= \mathrm{E}\left[\int_0^t\int_{\mathcal{Q}} h(s,q)\mathcal{P}(\mathbf{ds},\mathbf{dq})\right] = \int_0^t\int_{\mathcal{Q}} \lambda(s,q)h(s,q)\phi_Q(q)dq\,ds \\
&= \int_0^t \mathrm{E}_Q[\lambda(s,Q)h(s,Q)]ds.
\end{aligned}$$

- *Consider the* ***generalization of mean square limits*** *to include mark space integrals. For ease of integration in mean square limits, let the* ***mean-zero Poisson random measure*** *be denoted by*

$$\widetilde{\mathcal{P}}(\mathbf{dt},\mathbf{dq}) \equiv \mathcal{P}(\mathbf{dt},\mathbf{dq}) - \mathrm{E}[\mathcal{P}(\mathbf{dt},\mathbf{dq})] = \mathcal{P}(\mathbf{dt},\mathbf{dq}) - \phi_Q(q)dq\lambda(t)dt \tag{5.22}$$

and let the corresponding space-time integral be

$$\widetilde{I} \equiv \int_{\mathcal{Q}} h(t,q)\widetilde{\mathcal{P}}(\mathbf{dt},\mathbf{dq}). \tag{5.23}$$

Let $\mathcal{T}_n = \{t_i | t_{i+1} = t_i + \Delta t_i$ *for* $i = 0:n$, $t_0 = 0$, $t_{n+1} = t$, $\max_i[\Delta t_i] \to 0$ *as* $n \to +\infty\}$ *be a proper partition of* $[0,t)$. *Let* $\mathcal{Q}_m = \{\Delta\mathcal{Q}_j$ *for* $j = 1 : m | \cup_{j=1}^{m} \Delta\mathcal{Q}_j = \mathcal{Q}\}$ *be a proper partition of the mark space* $\mathcal{Q}$, *noting that it is implicit that the subsets* $\Delta\mathcal{Q}_j$ *are disjoint. Let* $h(t,q)$ *be a continuous function in time and marks. Let the corresponding partially discrete approximation*

$$\widetilde{I}_{m,n} \equiv \sum_{i=0}^{n}\sum_{j=1}^{m} h(t_i, q_j^*) \int_{\mathcal{Q}_j} \widetilde{\mathcal{P}}([t_i, t_i + \Delta T), dq_j) \tag{5.24}$$

for some $q_j^* \in \Delta\mathcal{Q}_j$. *Note that if* $\mathcal{Q}$ *is a finite interval* $[a,b]$, *then* $\mathcal{Q}_j = [q_j, q_j + \Delta q]$ *using even spacing with* $q_1 = a$, $q_{m+1} = b$ *and* $\Delta q = (b-a)/m$. *Then* $\widetilde{I}_{m,n}$ *converges in the mean square limit to* $\widetilde{I}$ *if*

$$\mathrm{E}[(\widetilde{I} - \widetilde{I}_{m,n})^2] \to 0 \tag{5.25}$$

as m *and* $n \to +\infty$.

For more advanced and abstract treatments of the Poisson random measure, see Gihman and Skorohod [95, Part 2, Chapter 2], Snyder and Miller [252, Chapters 4 and 5], Cont and Tankov [60] and Øksendal and Sulem [223] or the *applied to abstract bridge* Chapter 12.

Theorem 5.2. ***Basic Infinitesimal Moments of the Space-Time Poisson Process.***

$$\mathrm{E}[d\Pi(t)] = \lambda(t)dt \int_{\mathcal{Q}} h(t,q)\phi_Q(q)dq \equiv \lambda(t)dt\mathrm{E}_Q[h(t,Q)] \equiv \lambda(t)dt\overline{h}(t) \tag{5.26}$$

and

$$\mathrm{Var}[d\Pi(t)] = \lambda(t)dt \int_{\mathcal{Q}} h^2(t,q)\phi_Q(q)dq = \lambda(t)dt\mathrm{E}_Q[h^2(t;Q)] \equiv \lambda(t)dt\overline{h^2}(t). \tag{5.27}$$

Proof. The jump-amplitude function $h(t,Q)$ is independently distributed, through the mark process Q, from the underlying Poisson counting process here, except that this jump in space is conditional on the occurrence of the jump-time or -count of the underlying Poisson process. However, the function $h(t,q)$ is deterministic since it depends on the realization q in the space-time Poisson definition, rather than the random variable Q. The infinitesimal mean (5.26) is straightforward:

$$\begin{aligned}\mathrm{E}[d\Pi(t)] &= \mathrm{E}\left[\int_{\mathcal{Q}} h(t,q)\mathcal{P}(\mathbf{dt},\mathbf{dq})\right] = \int_{\mathcal{Q}} h(t,q)\mathrm{E}[\mathcal{P}(\mathbf{dt},\mathbf{dq})] \\ &= \lambda(t)dt \int_{\mathcal{Q}} h(t,q)\phi_Q(q)dq = \lambda(t)dt\mathrm{E}_Q[h(t,Q)] \equiv \lambda(t)dt\overline{h}(t);\end{aligned}$$

note that the expectation operator applied to the mark integral can be moved to apply just to the Poisson random measure $\mathcal{P}(\mathbf{dt}, \mathbf{dq})$.

However, the result for the variance in (5.27) is not so obvious, but the covariance formula for two Poisson random measures with differing mark variables $\text{Cov}[\mathcal{P}(\mathbf{dt}, \mathbf{dq}_1), \mathcal{P}(\mathbf{dt}, \mathbf{dq}_2)]$ in (5.15) will be made useful by converting it to the mean-zero Poisson random measure $\widetilde{\mathcal{P}}(\mathbf{dt}, \mathbf{dq})$ in (5.22),

$$\begin{aligned}
\text{Var}[d\Pi(t)] &= \text{E}\left[\left(\int_{\mathcal{Q}} h(t,q)\mathcal{P}(\mathbf{dt}, \mathbf{dq}) - \overline{h}(t)\lambda(t)dt\right)^2\right] \\
&= \text{E}\left[\left(\int_{\mathcal{Q}} (h(t,q)\mathcal{P}(\mathbf{dt}, \mathbf{dq}) - h(t,q)\phi_Q(q)\lambda(t)dt)\right)^2\right] \\
&= \text{E}\left[\left(\int_{\mathcal{Q}} h(t,q)\widetilde{\mathcal{P}}(\mathbf{dt}, \mathbf{dq})\right)^2\right] \\
&= \text{E}\left[\int_{\mathcal{Q}} h(t,q_1)\int_{\mathcal{Q}} h(t,q_2)\widetilde{\mathcal{P}}(\mathbf{dt}, \mathbf{dq}_1)\widetilde{\mathcal{P}}(\mathbf{dt}, \mathbf{dq}_1)\right] \\
&= \int_{\mathcal{Q}} h(t,q_1)\int_{\mathcal{Q}} h(t,q_2)\text{Cov}\left[\widetilde{\mathcal{P}}(\mathbf{dt}, \mathbf{dq}_1), \widetilde{\mathcal{P}}(\mathbf{dt}, \mathbf{dq}_1)\right] \\
&= \lambda(t)dt\int_{\mathcal{Q}} h^2(t,q_1)\phi_Q(q_1)dq_1 = \lambda(t)dt\text{E}_Q\left[h^2(t,Q)\right] \equiv \lambda(t)dt\overline{h^2}(t).
\end{aligned}$$

□

Examples 5.3.

- ***Uniformly distributed jump-amplitudes:***
  As an example of a continuous distribution, consider the uniform density for the jump-amplitude mark Q given by

$$\phi_Q(q) = \frac{1}{b-a}U(q;a,b),\ \ a < b, \tag{5.28}$$

  where $U(q;a,b) = \mathbf{1}_{q\in[a,b]}$ is the step or indicator function for the interval $[a,b]$, i.e., $U(q;a,b)$ is one when $a \le q \le b$ and zero otherwise. The first few moments are

$$\begin{aligned}
\text{E}_Q[1] &= \frac{1}{b-a}\int_a^b dq = 1, \\
\text{E}_Q[Q] &= \frac{1}{b-a}\int_a^b q dq = \frac{b+a}{2}, \\
\text{Var}_Q[Q] &= \frac{1}{b-a}\int_a^b \left(q - \frac{b+a}{2}\right)^2 dq = \frac{(b-a)^2}{12}.
\end{aligned}$$

  In the case of the log-uniform amplitude letting $Q = \ln(1+h(Q))$ be the mark-amplitude relationship using the log-transformation form from the linear SDE *problem* (4.76)*, we then then*

$$h(Q) = e^Q - 1$$

and the expected jump-amplitude is

$$\mathrm{E}_Q[h(Q)] = \frac{1}{b-a}\int_a^b (e^q - 1)dq = \frac{e^b - e^a}{b-a} - 1.$$

- ***Poisson distributed jump-amplitudes:***
  As an example of a discrete distribution of jump-amplitudes, consider

$$\Phi_Q(k) = p_k(u) = e^{-u}\frac{u^k}{k!}$$

for $k = 0 : \infty$*. Thus, the jump process is a Poisson–Poisson process or a Poisson-mark Poisson process. The mean and variance are*

$$\mathrm{E}_Q[Q] = u,$$
$$\mathrm{Var}_Q[Q] = u.$$

Remark *5.4. For the general discrete distribution,*

$$\Phi_Q(k) = p_k, \quad \sum_{k=0}^{\infty} p_k = 1,$$

the comparable continuous form is

$$\Phi_Q(q) \stackrel{\text{gen}}{=} \sum_{k=0}^{\infty} H_R(q-k)p_k = \sum_{k=0}^{\lfloor q \rfloor} p_k,$$

where $H_R(q)$ *is again the right-continuous Heaviside step function and* $\lfloor q \rfloor$ *is the maximum integer not exceeding* q*. The corresponding generalized density is*

$$\phi_Q(q) \stackrel{\text{gen}}{=} \sum_{k=0}^{\infty} \delta_R(q-k)p_k.$$

The reader should verify that this density yields the correct expectation and variance forms.

5.2 State-Dependent Generalization of Jump-Diffusion SDEs

5.2.1 State-Dependent Generalization for Space-Time Poisson Processes

The space-time Poisson process is generalized to include state-dependence with $X(t)$ in both the jump-amplitude and the Poisson measure, such that

$$d\Pi(t; X(t), t) = \int_{\mathcal{Q}} h(X(t), t, q)\mathcal{P}(\mathbf{dt}, \mathbf{dq}; X(t), t) \tag{5.29}$$

on the Poisson mark space $\mathcal{Q}$ with Poisson random measure $\mathcal{P}(\mathbf{dt}, \mathbf{dq}; X(t), t)$, which helps to describe the space-time Poisson mechanism and related calculus. The space-time state-dependent Poisson mark, $Q = q$, is again the underlying random variable for the state-dependent and mark-dependent jump-amplitude coefficient $h(x, t, q)$. The double time t arguments of $d\Pi$, dP and $\mathcal{P}$ are not considered redundant for applications, since the first time t or time set dt is the usual Poisson jump process implicit time dependence, while the second to the right of the semicolon denotes explicit or parametric time dependence paired with explicit state dependence that is known in advance and is appropriate for the application model.

Alternatively, the Poisson zero-one law form may be used, i.e.,

$$d\Pi(t; X(t), t) \overset{\mathrm{dt}}{\underset{\mathrm{zol}}{=}} h(X(t), t, Q)dP(t; Q, X(t), t) \tag{5.30}$$

with the jump of $\Pi(t; X(t), t)$ being

$$[\Pi](T_k) = h(X(T_k^-), T_k^-, Q_k)$$

at jump-time T_k and jump-mark Q_k. The multitude of random variables in this sum means that expectations or other Poisson integrals will be very difficult to calculate even by conditional expectation iterations.

Definition 5.5. *The* ***conditional expectation of*** $\mathcal{P}(\mathbf{dt}, \mathbf{dq}; X(t), t)$ *is*

$$\mathrm{E}[\mathcal{P}(\mathbf{dt}, \mathbf{dq}; X(t), t)|X(t) = x] = \phi_Q(q; x, t)dq\lambda(t; x, t)dt, \tag{5.31}$$

where $\phi_Q(q; x, t)dq$ is the probability density of the now state-dependent Poisson amplitude mark and the jump rate $\lambda(t; x, t)$ now has state-time dependence. In this notation, the relationship to the simple counting process is given by

$$\int_{\mathcal{Q}} \mathcal{P}(\mathbf{dt}, \mathbf{dq}; X(t), t) = dP(t; Q, X(t), t).$$

Hence, when $h(x, t, q) = \widetilde{h}(x, t)$, i.e., independent of the mark q, the space-time Poisson is the simple jump process with mark-independent amplitude,

$$d\Pi(t; X(t), t) = \widetilde{h}(X(t), t)dP(t; Q, X(t), t),$$

but with nonunit jumps in general. Effectively the same form is obtained when there is a single discrete mark, e.g., $\phi_Q(q) = \delta(q-1)$, so $h(x,t,q) = h(x,t,1)$ always.

Theorem 5.6. ***Basic Conditional Infinitesimal Moments of the State-Dependent Poisson Process.***

$$\begin{aligned}\mathrm{E}[d\Pi(t;X(t),t)|X(t)=x] &= \int_{\mathcal{Q}} h(x,t,q)\phi_Q(q;x,t)dq\lambda(t;x,t)dt \\ &\equiv \mathrm{E}_Q[h(x,t,Q)]\lambda(t;x,t)dt \end{aligned} \tag{5.32}$$

and

$$\begin{aligned}\mathrm{Var}[d\Pi(t;X(t),t)|X(t)=x] &= \int_{\mathcal{Q}} h^2(x,t,q)\phi_Q(q;x,t)dq\lambda(t;x,t)dt \\ &\equiv \mathrm{E}_Q[h^2(x,t;Q)]\lambda(t;x,t)dt. \end{aligned} \tag{5.33}$$

Proof. The justification is the same justification as for (5.26)–(refVarSTPoisson). It is assumed that the jump-amplitude $h(x,t,Q)$ is independently distributed due to Q from the underlying Poisson counting process here, except that this jump in space is conditional on the occurrence of the jump-time of the underlying Poisson process. □

5.2.2 State-Dependent Jump-Diffusion SDEs

The general, scalar SDE takes the form

$$\begin{aligned} dX(t) = &\ f(X(t),t)dt + g(X(t),t)dW(t) + \int_{\mathcal{Q}} h(X(t),t,q)\mathcal{P}(\mathbf{dt},\mathbf{dq};X(t),t) \\ \stackrel{\mathrm{dt}}{=} &\ f(X(t),t)dt + g(X(t),t)dW(t) + h(X(t),t,Q)dP(t;Q,X(t),t) \end{aligned} \tag{5.34}$$

for the state process $X(t)$ with a set of continuous coefficient functions $\{f,g,h\}$. However, the SDE model is just a useful symbolic model for many applied situations, but the more basic model relies on specifying the method of integration. So

$$\begin{aligned} X(t) = &\ X(t_0) + \int_{t_0}^{t} (f(X(s),s)ds + g(X(s),s)dW(s) \\ &+ h(X(t),s,Q)dP(s;Q,X(s),s)) \\ \stackrel{\mathrm{ims}}{=} &\ X(t_0) + \operatorname*{\overset{ms}{lim}}_{n\to\infty} \left[\sum_{i=0}^{n}\left(f_i\Delta t_i + g_i\Delta W_i + \sum_{k=P_i+1}^{P_i+\Delta P_i} h_{i,k}\right)\right], \end{aligned} \tag{5.35}$$

where $f_i = f(X_i,t_i)$, $g_i = g(X_i,t_i)$, $h_{i,k} = h(X_i,T_k,Q_k)$, $\Delta t_i = t_{i+1}-t_i$, $\Delta P_i = \Delta P(t_i;Q,X_i,t_i)$ and $\Delta W_i = \Delta W(t_i)$. Here, T_k is the kth jump-time and $\{Q,Q_k\}$ are the corresponding random marks.

The **conditional infinitesimal moments for the state process** are

$$\mathrm{E}[dX(t)|X(t)=x] = f(x,t)dt + \overline{h}(x,t)\lambda(t;x,t)dt, \tag{5.36}$$

$$\overline{h}(x,t)\lambda(t;x,t)dt \equiv \mathrm{E}_Q[h(x,t,Q)]\lambda(t;x,t)dt, \tag{5.37}$$

and

$$\mathrm{Var}[dX(t)|X(t)=x] = g^2(x,t)dt + \overline{h^2}(x,t)\lambda(t;x,t)dt, \tag{5.38}$$

$$\overline{h^2}(x,t)\lambda(t;x,t)dt \equiv \mathrm{E}_Q[h^2(x,t,Q)]\lambda(t;x,t)dt \tag{5.39}$$

using (1.1), (5.32), (5.33), (5.34) and assuming that the Poisson process is independent of the Wiener process. The jump in the state at jump time T_k in the underlying Poisson process is

$$[X](T_k) \equiv X(T_k^+) - X(T_k^-) = h(X(T_k^-), T_k^-, Q_k) \tag{5.40}$$

for $k = 1, 2, \ldots$, now depending on the kth mark Q_k at the prejump-time T_k^- at the kth jump.

Rule 5.7. ***Stochastic Chain Rule for State-Dependent SDEs.***
*The **stochastic chain rule** for a sufficiently differentiable function $Y(t) = F(X(t),t)$ has the form*

$$\begin{aligned} dY(t) = dF(X(t),t) &\overset{\text{sym}}{=} F(X(t)+dX(t), t+dt) - F(X(t),t) \\ &= d_{(\text{cont})}F(X(t),t) + d_{(\text{jump})}F(X(t),t) \\ &\overset{\text{dt}}{=} F_t(X(t),t)dt + F_x(X(t),t)(f(X(t),t)dt + g(X(t),t)dW(t)) \\ &\quad + \frac{1}{2}F_{xx}(X(t),t)g^2(X(t),t)dt \\ &\quad + \int_{\mathcal{Q}} (F(X(t)+h(X(t),t,q),t) - F(X(t),t))\mathcal{P}(\mathbf{dt},\mathbf{dq};X(t),t) \end{aligned} \tag{5.41}$$

to precision-dt. It is sufficient that F be twice continuously differentiable in x and once in t.

5.2.3 Linear State-Dependent SDEs

Let the state-dependent jump-diffusion process satisfy an SDE linear in the state process $X(t)$ with time-dependent rate coefficients

$$dX(t) \overset{\text{dt}}{=} X(t)\left(\mu_d(t)dt + \sigma_d(t)dW(t) + \nu(t,Q)dP(t;Q)\right) \tag{5.42}$$

for $t > t_0$ with $X(t_0) = X_0$ and $\mathrm{E}[dP(t;Q)] = \lambda(t)dt$, where $\mu_d(t)$ denotes the mean and $\sigma_d^2(t)$ denotes the variance of the diffusion process, while Q_k denotes the kth mark and T_k denotes the kth time of the jump process.

Again, using the log-transformation $Y(t) = \ln(X(t))$ and the stochastic chain rule (5.41),

$$dY(t) \stackrel{\rm dt}{=} (\mu_d(t) - \sigma_d^2(t)/2)dt + \sigma_d(t)dW(t) + \ln\left(1 + \nu(t,Q)\right) dP(t;Q) \qquad (5.43)$$

with immediate integralsvspace*-0.5em

$$Y(t) = \ln(x_0) + \int_{t_0}^{t} dY(s) \qquad (5.44)$$

and

$$X(t) = x_0 \exp\left(\int_{t_0}^{t} dY(s)\right), \qquad (5.45)$$

or in recursive form,

$$X(t + \Delta t) = X(t) \exp\left(\int_{t}^{t+\Delta t} dY(s)\right). \qquad (5.46)$$

Linear Mark-Jump-Diffusion Simulation Forms

For simulations, a small time-step, $\Delta t_i \ll 1$, approximation of the recursive form (5.46) would be more useful with $X_i = X(t_i)$, $\mu_i = \mu_d(t_i)$, $\sigma_i = \sigma_d(t_i)$, $\Delta W_i = \Delta W(t_i)$, $\Delta P_i = \Delta P(t_i;Q)$ and the convenient jump-amplitude coefficient approximaton, $\nu(t,Q) \simeq \nu_0(Q) \equiv \exp(Q) - 1$, i.e.,

$$X_{i+1} \simeq X_i \exp\left((\mu_i - \sigma_i^2/2)\Delta t_i + \sigma_i \Delta W_i\right)(1 + \nu_0(Q))^{\Delta P_i} \qquad (5.47)$$

for $i = 1 : N$ time-steps, where a zero-one jump law approximation has been used.

For the diffusion part, it has been shown that

$$\mathrm{E}\left[e^{\sigma_i \Delta W_i}\right] = e^{\sigma_i^2 \Delta t_i/2},$$

using the completing the square technique. In addition, there is the following lemma for the jump part of (5.47).

Lemma 5.8. ***Jump Term Expectation.***

$$\mathrm{E}\left[(1 + \nu_0(Q))^{\Delta P_i}\right] = e^{\lambda_i \Delta t_i \mathrm{E}[\nu_0(Q)]}, \qquad (5.48)$$

where $\mathrm{E}[\Delta P_i] = \lambda_i \Delta t_i$ *and* $\nu_0(Q) = \exp(Q) - 1$.

Proof. Using given forms, iterated expectations, the Poisson distribution and the IID property of the marks Q_k, we then have

$$\begin{aligned}
\mathrm{E}\left[(1 + \nu_0(Q))^{\Delta P_i}\right] &= \mathrm{E}\left[e^{Q \Delta P_i}\right] \\
&= e^{-\lambda_i \Delta t_i} \sum_{k=0}^{\infty} (\lambda_i \Delta t_i)^k \mathrm{E}_Q\left[e^{kQ}\right] \\
&= e^{-\lambda_i \Delta t_i} \sum_{k=0}^{\infty} (\lambda_i \Delta t_i)^k \left(\mathrm{E}_Q\left[e^{Q}\right]\right)^k \\
&= e^{-\lambda_i \Delta t_i} e^{\lambda_i \Delta t_i \mathrm{E}_Q\left[e^{Q}\right]} \\
&= e^{\lambda_i \Delta t_i \mathrm{E}_Q[\nu_0(Q)]}.
\end{aligned}$$

□

An immediate consequence of this result is the following corollary.

Corollary 5.9. ***Discrete State Expectations.***

$$\mathrm{E}[X_{i+1}|X_i] \simeq X_i \exp((\mu_i + \lambda_i \mathrm{E}_Q[\nu_0(Q)])\Delta t_i) \tag{5.49}$$

and

$$\mathrm{E}[X_{i+1}] \simeq x_0 \exp\left(\sum_{j=0}^{i}(\mu_j + \lambda_j \mathrm{E}_Q[\nu_0(Q)])\Delta t_j\right). \tag{5.50}$$

Further, as Δt_i and $\delta t_n \to 0^+$, the continuous form of the expectation follows and is given later in Corollary 5.13 on p. 149 using other justification.

Example 5.10. ***Linear, Time-Independent, Constant-Rate Coefficient Case.***

In the linear, time-independent, constant-rate coefficient case with $\mu_d(t) = \mu_0$, $\sigma_d(t) = \sigma_0$, $\lambda(t) = \lambda_0$ *and* $\nu(t, Q) = \nu_0(Q) = e^Q - 1$,

$$X(t) = x_0 \exp\left((\mu_0 - \sigma_0^2/2)(t - t_0) + \sigma_0(W(t) - W(t_0)) + \sum_{k=1}^{P(t;Q)-P(t_0;Q)} \nu_0 Q_k\right), \tag{5.51}$$

where the Poisson counting sum form is now more manageable since the marks do not depend on the prejump-times T_k^-.

Using the independence of the three underlying stochastic processes, $(W(t) - W(t_0))$, $(P(t;Q) - P(t_0;Q))$ *and* Q_i, *as well as the stationarity of the first two and the law of exponents to separate exponents, leads to partial reduction of the expected state process:*

$$\begin{aligned}
\mathrm{E}[X(t)] &= x_0 e^{(\mu_0 - \sigma_0^2/2)(t-t_0)} \mathrm{E}_W\left[e^{\sigma_0 W(t-t_0)}\right] \sum_{k=0}^{\infty} \mathrm{E}[P(t-t_0;Q) = k]\mathrm{E}\left[e^{\sum_{\ell=1}^{k} Q_\ell}\right] \\
&= x_0 e^{(\mu_0 - \sigma_0^2/2)(t-t_0)} \int_{-\infty}^{+\infty} \frac{e^{-w^2/(2(t-t_0))}}{\sqrt{2\pi(t-t_0)}} e^{\sigma_0 w} dw \\
&\quad \cdot e^{-\lambda_0(t-t_0)} \sum_{k=0}^{\infty} \frac{(\lambda_0(t-t_0))^k}{k!} \prod_{i=1}^{k} \mathrm{E}_Q\left[e^Q\right] \\
&= x_0 e^{\mu_0(t-t_0)} e^{-\lambda_0(t-t_0)} \sum_{k=0}^{\infty} \frac{(\lambda_0(t-t_0))^k}{k!} \mathrm{E}_Q^k\left[e^Q\right] \\
&= x_0 e^{\left(\mu_0 + \lambda_0\left(\mathrm{E}_Q\left[e^Q\right]-1\right)\right)(t-t_0)},
\end{aligned} \tag{5.52}$$

where $\lambda_0(t - t_0)$ *is the Poisson parameter and* $\mathcal{Q} = (-\infty, +\infty)$ *is taken as the mark space for specificity with*

$$\mathrm{E}_Q\left[e^Q\right] = \int_{\mathcal{Q}} e^q \phi_Q(q) dq.$$

Little more useful simplification can be obtained analytically, except for infinite expansions or equivalent special functions, when the mark density $\phi_Q(q)$ *is specified. Numerical procedures may be more useful for practical purposes. The state expectation in this distributed mark case* (5.52) *should be compared with the pure constant linear coefficient case* (4.81) *of Chapter* 4.

Exponential Expectations

Sometimes it is necessary to get the expectation of an exponential of the integral of a jump-diffusion process. The procedure is much more complicated for distributed amplitude Poisson jump processes than for diffusions since the mark-time process is a product process, i.e., the product of the mark process and the Poisson process. For the time-independent coefficient case, as in a prior example, the exponential processes are easily separable by the law of exponents. However, for the time-dependent case, it is necessary to return to using the space-time process $\mathcal{P}$ and the decomposition approximation used in the mean square limit. The h in the following theorem might be the amplitude coefficient in (5.43) or $h(s,q) = q = \ln(1+\nu(s,q))$.

Theorem 5.11. ***Expectation for the Exponential of Space-Time Counting Integrals.***
Assuming finite second order moments for $h(t,q)$ *and convergence in the mean square limit,*

$$\mathrm{E}\left[\exp\left(\int_{t_0}^{t}\int_{\mathcal{Q}} h(s,q)\mathcal{P}(\mathbf{ds},\mathbf{dq})\right)\right] = \exp\left(\int_{t_0}^{t}\int_{\mathcal{Q}}\left(e^{h(s,q)}-1\right)\phi_Q(q,s)dq\lambda(s)ds\right)$$
$$\equiv \exp\left(\int_{t_0}^{t}\overline{(e^h-1)}(s)\lambda(s)ds\right). \tag{5.53}$$

Proof. Let the proper partition of the mark space over disjoint subsets be

$$\mathcal{Q}_m = \{\Delta\mathcal{Q}_j \text{ for } j = 1\!:\!m \,|\, \cup_{j=1}^{m}\Delta\mathcal{Q}_j = \mathcal{Q}\}.$$

Since Poisson measure is Poisson distributed,

$$\Phi_{\mathcal{P}_j}(k) = \mathrm{Prob}[\mathcal{P}(\mathbf{dt},\Delta\mathcal{Q}_j) = k] = e^{-\overline{\mathcal{P}_j}}\frac{(\overline{\mathcal{P}_j})^k}{k!}$$

with Poisson parameter

$$\overline{\mathcal{P}_j} \equiv \mathrm{E}[\mathcal{P}(\mathbf{dt},\Delta\mathcal{Q}_j)] = \lambda(t)dt\Phi_Q(\Delta\mathcal{Q}_j,t_i)$$

for each subset $\{\Delta\mathcal{Q}_j\}$.

Similarly, let the proper partition over the time interval be

$$\mathcal{T}_n = \{t_i | t_{i+1} = t_i + \Delta t_i \text{ for } i = 0\!:\!n, t_0 = 0, t_{n+1} = t, \max_i[\Delta t_i] \to 0 \quad as \quad n \to +\infty\}.$$

The disjoint property over subsets and time intervals means $\mathcal{P}([t_i,t_i+\Delta t_i),\Delta\mathcal{Q}_j)$ and $\mathcal{P}([t_i,t_i+\Delta t_i),\Delta\mathcal{Q}'_j)$ will be pairwise independent provided $j' \neq j$ for fixed

i corresponding to property (5.15) for infinitesimals, while $\mathcal{P}([t_i, t_i+\Delta t_i), \Delta\mathcal{Q}_j)$ and $\mathcal{P}([t_i, t_i+\Delta t'_i), \Delta\mathcal{Q}'_j)$ will be pairwise independent provided $i' \neq i$ and $j' \neq j$, corresponding to property (5.16) for infinitesimals.

For brevity, let $h_{i,j} \equiv h(t_i, q^*_j)$, where $q^*_j \in \Delta\mathcal{Q}_j$, $\mathcal{P}_{i,j} \equiv \mathcal{P}_i([t_i, t_i+\Delta t_i), \Delta\mathcal{Q}_j)$ and $\overline{\mathcal{P}}_{i,j} \equiv \lambda_i \Delta t_i \Phi_Q(\Delta\mathcal{Q}_j)$.

Using mean square limits, with $\mathcal{P}_{i,j}$ playing the dual roles of the two increments $(\Delta t_i, \Delta\mathcal{Q}_j)$, the law of exponents and the independence denoted by $\overset{\text{ind}}{\underset{\text{inc}}{=}}$, we have

$$\begin{aligned}
\mathrm{E}\left[\exp\left(\int_{t_0}^t \int_{\mathcal{Q}} h\mathcal{P}\right)\right] &\overset{\text{ims}}{=} \overset{\text{ms}}{\lim_{m,n\to\infty}} \mathrm{E}\left[\exp\left(\sum_{i=0}^n \sum_{j=1}^m h_{i,j}\mathcal{P}_{i,j}\right)\right] \\
&\overset{\text{ind}}{\underset{\text{inc}}{=}} \overset{\text{ms}}{\lim_{m,n\to\infty}} \Pi_{i=0}^n \Pi_{j=1}^m \mathrm{E}\left[\exp\left(h_{i,j}\mathcal{P}_{i,j}\right)\right] \\
&= \overset{\text{ms}}{\lim_{m,n\to\infty}} \Pi_{i=0}^n \Pi_{j=1}^m \exp\left(-\overline{\mathcal{P}_{i,j}}\right) \sum_{k_{i,j}=0}^{\infty} \frac{\overline{\mathcal{P}_{i,j}}^{k_{i,j}}}{k_{i,j}!} \exp\left(h_{i,j}k_{i,j}\right) \\
&= \overset{\text{ms}}{\lim_{m,n\to\infty}} \Pi_{i=0}^n \Pi_{j=1}^m \exp\left(\overline{\mathcal{P}_{i,j}}\left(\exp(h_{i,j})-1\right)\right) \\
&= \overset{\text{ms}}{\lim_{m,n\to\infty}} \exp\left(\sum_{i=0}^n \sum_{j=1}^m \left(\exp(h_{i,j})-1\right)\lambda_i \Delta t_i \Phi_Q(\Delta Q_i, t_i)\right) \\
&\overset{\text{ims}}{=} \exp\left(\int_{t_0}^t \int_{\mathcal{Q}} \left(\exp(h(s,q))-1\right)\phi_Q(q,s)dq\lambda(s)ds\right) \\
&\equiv \exp\left(\int_{t_0}^t \overline{(\exp(h(s,Q))-1)}\lambda(s)ds\right).
\end{aligned}$$

Thus, the main technique is to unassemble the mean square limit discrete approximation to get at the independent random part, take its expectation and then reassemble the mean square limit, justifying the interchange of expectation and exponent-integration. □

Remarks 5.12.

- *Note that the mark space subset $\Delta\mathcal{Q}_j$ is never used directly as a discrete element of integration, since the subset would be infinite if the mark space were infinite. The mark space element is used only through the distribution which would be bounded. This is quite unlike the time domain, where we can select t to be finite. If the mark space were finite, say, $\mathcal{Q} = [a,b]$, then a concrete partition of $[a,b]$ similar to the time-partition can be used.*

- *Also note that the dependence on $(X(t), t)$ was not used, but could be considered suppressed but absorbed into the existing t dependence of h and $\mathcal{P}$.*

Corollary 5.13. ***Expectation of X(t) for Linear SDE.***
Let $X(t)$ be the solution (5.45) with $\overline{\nu}(t) \equiv \mathrm{E}[\nu(t,Q)]$ of (5.42). Then

$$\begin{aligned}\mathrm{E}[X(t)] &= x_0 \exp\left(\int_{t_0}^{t} (\mu_d(s) + \lambda(s)\overline{\nu}(s))\, ds\right) \\ &= x_0 \exp\left(\int_{t_0}^{t} \mathrm{E}[dX(s)/X(s)] ds\right). \end{aligned} \tag{5.54}$$

Proof. The jump part, i.e., the main part, follows from exponential Theorem 5.11, (5.53) and the lesser part for the diffusion is left as an exercise for the reader.

However, note that the exponent is the time integral of $\mathrm{E}[dX(t)/X(t)]$, the relative conditional infinitesimal mean, which is independent of $X(s)$ and is valid only for the linear mark-jump-diffusion SDE. ∎

Remark 5.14. *The relationship in* (5.54) *is a* ***quasi-deterministic equivalence*** *for linear mark-jump-diffusion SDEs and was shown by Hanson and Ryan* [115] *in* 1989*. They also produced a nonlinear jump counterexample that has a formal closed-form solution in terms of the gamma function, for which the result does not hold and a very similar example is given in Exercise* 9 *in Chapter* 4*.*

Moments of Log-Jump-Diffusion Process

For the log-jump-diffusion process $dY(t)$ in (5.43), suppose that the jump-amplitude is time-independent and that the mark variable was conveniently chosen as

$$Q = \ln(1 + \nu(t,Q))$$

so that the SDE has the form

$$dY(t) \stackrel{\mathrm{dt}}{=} \mu_{\mathrm{ld}}(t)dt + \sigma_d(t)dW(t) + QdP(t;Q), \tag{5.55}$$

or in the case of applications for which the time-step Δt is an increment that is not infinitesimal like dt, there is some probability of more than one jump,

$$\Delta Y(t) = \mu_{\mathrm{ld}}(t)\Delta t + \sigma_d(t)\Delta W(t) + \sum_{k=P(t;Q)+1}^{P(t;Q)+\Delta P(t;Q)} Q_k. \tag{5.56}$$

The results for the infinitesimal case (5.55) are contained in the incremental case (5.56).

The first few moments can be found in general for (5.56), and if up to the fourth moment, then the skew and kurtosis coefficients can be calculated. These calculations can be expedited by the following lemma, concerning sums of zero-mean IID random variables.

Lemma 5.15. ***Zero-Mean IID Random Variable Sums.***
Let $\{X_i | i = 1\!:\!n\}$ *be a set of zero-mean IID random variables, i.e.,* $\mathrm{E}[X_i] = 0$. *Let* $M^{(m)} \equiv E[X_i^m]$ *be the mth moment and*

$$S_n^{(m)} \equiv \sum_{i=1}^{n} X_i^m$$

with $S_n^{(1)} = S_n$ *the usual partial sum over the set and*

$$\mathrm{E}[S_n^{(m)}] = nM^{(m)}; \tag{5.57}$$

then the expectation of powers of S_n *for* $m = 1\!:\!4$ *is*

$$\mathrm{E}\left[(S_n)^m\right] = \left\{ \begin{array}{ll} 0, & m = 1 \\ nM^{(2)}, & m = 2 \\ nM^{(3)}, & m = 3 \\ nM^{(4)} + 3n(n-1)\left(M^{(2)}\right)^2, & m = 4 \end{array} \right\}. \tag{5.58}$$

Proof. The proof is done first by the linear property of the expectation and the IID properties of the X_i,

$$\mathrm{E}\left[S_n^{(m)}\right] = \sum_{i=1}^{n} \mathrm{E}[X_i^m] = \sum_{i=1}^{n} M^{(m)} = nM^{(m)}. \tag{5.59}$$

The $m = 1$ case is trivial due to the zero-mean property of the X_i's and the linearity of the expectation operator, $\mathrm{E}[S_n] = \sum_{i=1}^{n} \mathrm{E}[X_i] = 0$.

For $m = 2$, the induction hypothesis from (5.58) is

$$\mathrm{E}\left[S_n^2\right] \equiv \mathrm{E}\left[\left(\sum_{i=1}^{n} X_i^2\right)\right] = nM^{(2)},$$

where the initial condition at $n = 1$ is $\mathrm{E}[S_1^2] = \mathrm{E}[X_1^2] = M^{(2)}$ by definition. The hypothesis can be proved easily by partial sum recursion $S_{n+1} = S_n + X_{n+1}$, application of the binomial theorem, expectation linearity and the zero-mean IID property:

$$\begin{aligned} \mathrm{E}\left[S_{n+1}^2\right] &= \mathrm{E}\left[(S_n + X_{n+1})^2\right] = \mathrm{E}\left[S_n^2 + 2X_{n+1}S_n + X_{n+1}^2\right] \\ &= nM^{(2)} + 2 \cdot 0 \cdot 0 + M^{(2)} = (n+1)M^{(2)}. \end{aligned} \tag{5.60}$$

QED for $m = 2$.

This is similar for the power $m = 3$, again beginning with the induction hypothesis

$$\mathrm{E}\left[S_n^3\right] \equiv \mathrm{E}\left[\left(\sum_{i=1}^{n} X_i\right)^3\right] = nM^{(3)}.$$

where the initial condition at $n = 1$ is $\mathrm{E}[S_1^3] = \mathrm{E}[X_1^3] = M^{(3)}$ by definition. Using the same techniques as in (5.60),

$$\begin{aligned}\mathrm{E}\left[S_{n+1}^3\right] &= \mathrm{E}\left[(S_n + X_{n+1})^3\right] = \mathrm{E}\left[S_n^3 + 3X_{n+1}S_n^2 + 3X_{n+1}^2 S_n^2 + X_{n+1}^3\right] \\ &= nM^{(3)} + 3\cdot 0\cdot nM^{(2)} + 3\cdot M^{(2)}\cdot 0 + M^{(3)} = (n+1)M^{(3)}. \end{aligned} \quad (5.61)$$

QED for $m = 3$.

Finally, the case for the power $m = 4$ is a little different since an additional nontrivial term arises from the product of the squares of two independent variables. The induction hypothesis is

$$\mathrm{E}\left[S_n^4\right] \equiv \mathrm{E}\left[\left(\sum_{i=1}^{n} X_i\right)^4\right] = nM^{(4)} + 3n(n-1)(M^{(2)})^2,$$

where the initial condition at $n = 1$ is $\mathrm{E}[S_1^4] = \mathrm{E}[X_1^4] = M^{(4)}$ by definition. Using the same techniques as in (5.60),

$$\begin{aligned}\mathrm{E}\left[S_{n+1}^4\right] &= \mathrm{E}\left[(S_n + X_{n+1})^4\right] = \mathrm{E}\left[S_n^4 + 4X_{n+1}S_n^3 + 6X_{n+1}^2 S_n^2 + 4X_{n+1}^3 S_n^1 + X_{n+1}^4\right] \\ &= nM^{(4)} + 3n(n-1)(M^{(2)})^2 + 4\cdot 0\cdot nM^{(3)} + 6\cdot M^{(2)}\cdot nM(2) \\ &\quad +4\cdot M^{(3)}\cdot 0 + M^{(4)} \\ &= (n+1)M^{(4)} + 3(n+1)((n+1)-1)(M^{(2)})^2. \end{aligned} \quad (5.62)$$

QED for $m = 4$. □

Remark 5.16. *The results here depend on the IID and zero-mean properties, but do not otherwise depend on the particular distribution of the random variables. The results are used in the following theorem.*

Theorem 5.17. ***Some Moments of the Log-jump-Diffusion (LJD) Process $\Delta Y(t)$.***
Let $\Delta Y(t)$ satisfy the stochastic difference equation (5.56)and let the marks Q_k be IID with mean $\mu_j \equiv \mathrm{E}_Q[Q_k]$ and variance $\sigma_j^2 \equiv \mathrm{Var}_Q[Q_k]$. Then the first four moments, $m = 1{:}4$, are

$$\mu_{\mathrm{ljd}}(t) \equiv \mathrm{E}[\Delta Y(t)] = (\mu_{\mathrm{ld}}(t) + \lambda(t)\mu_j)\Delta t; \quad (5.63)$$

$$\sigma_{\mathrm{ljd}}(t) \equiv \mathrm{Var}[\Delta Y(t)] = \left(\sigma_d^2(t) + \left(\sigma_j^2 + \mu_j^2\right)\lambda(t)\right)\Delta t; \quad (5.64)$$

$$M_{\mathrm{ljd}}^{(3)}(t) \equiv \mathrm{E}\left[(\Delta Y(t) - \mathrm{E}[\Delta Y(t)])^3\right] = \left(M_j^{(3)} + \mu_j\left(3\sigma_j^2 + \mu_j^2\right)\right)\lambda(t)\Delta t, \quad (5.65)$$

where $M_j^{(3)} \equiv \mathrm{E}_Q[(Q_i - \mu_j)^3]$;

$$\begin{aligned}M_{\mathrm{ljd}}^{(4)}(t) &\equiv \mathrm{E}\left[(\Delta Y(t) - \mathrm{E}[\Delta Y(t)])^4\right] \\ &= \left(M_j^{(4)} + 4\mu_j M_j^{(3)} + 6\mu_j^2\sigma_j^2 + \mu_j^4\right)\lambda(t)\Delta t \\ &\quad +3\left(\sigma_d^2(t) + \left(\sigma_j^2 + \mu_j^2\right)\lambda(t)\right)^2(\Delta t)^2, \end{aligned} \quad (5.66)$$

where $M_j^{(4)} \equiv \mathrm{E}_Q[(Q_i - \mu_j)^4]$.

Proof. One general technique for calculating moments of the log-jump-diffusion process is **iterated expectations**. Thus

$$\begin{aligned}\mu_{\mathrm{ljd}}(t) = \mathrm{E}[\Delta Y(t)] &= \mu_{\mathrm{ld}}(t)\Delta t + \sigma_d(t)\cdot 0 + \mathrm{E}_{\Delta P(t;Q)}\left[\mathrm{E}_Q\left[\sum_{i=1}^{\Delta P(t;Q)} Q_i \middle| \Delta P(t;Q)\right]\right]\\ &= \mu_{\mathrm{ld}}(t)\Delta t + \mathrm{E}_{\Delta P(t;Q)}\left[\sum_{i=1}^{\Delta P(t;Q)} \mathrm{E}_Q[Q_i]\right]\\ &= \mu_{\mathrm{ld}}(t)\Delta t + \mathrm{E}_{\Delta P(t;Q)}[\Delta P(t;Q)\mathrm{E}_Q[Q_i]] = (\mu_{\mathrm{ld}}(t) + \mu_j\lambda(t))\,\Delta t,\end{aligned}$$

proving the first moment formula, using the increment jump-count.

For the higher moments, the main key technique for efficient calculation of the moments is decomposing the log-jump-diffusion process deviation into zero-mean deviation factors, i.e.,

$$\Delta Y(t) - \mu_{\mathrm{ljd}}(t) = \sigma_d(t)\Delta W(t) + \sum_{i=1}^{\Delta P(t;Q)} (Q_i - \mu_j) + \mu_j(\Delta P(t;Q) - \lambda(t)\Delta t).$$

In addition, the multiple applications of the binomial theorem and the convenient increment power Table 1.1 for $\Delta W(t)$ and Table 1.2 for $\Delta P(t;Q)$ are used.

The incremental process variance is found by

$$\begin{aligned}\sigma_{\mathrm{ljd}}(t) \equiv\ & \mathrm{Var}[\Delta Y(t)]\\ =\ & \mathrm{E}\left[\left(\sigma_d(t)\Delta W(t) + \sum_{i=1}^{\Delta P(t;Q)} (Q_i - \mu_j) + \mu_j(\Delta P(t;Q) - \lambda(t)\Delta t)\right)^2\right]\\ =\ & \sigma_d^2(t)\mathrm{E}_{\Delta W(t)}[(\Delta W)^2(t)] + 2\sigma_d\cdot 0\\ & + \mathrm{E}\left[\left(\sum_{i=1}^{\Delta P(t;Q)} (Q_i - \mu_j) + \mu_j(\Delta P(t;Q) - \lambda(t)\Delta t)\right)^2\right]\\ =\ & \sigma_d^2(t)\Delta t + \mathrm{E}_{\Delta P(t;Q)}\left[\sum_{i=1}^{\Delta P(t;Q)}\sum_{k=1}^{\Delta P(t;Q)} \mathrm{E}_Q[(Q_i - \mu_j)(Q_k - \mu_j)]\right.\\ & \left. + 2\mu_j(\Delta P(t;Q) - \lambda(t)\Delta t)\sum_{i=1}^{\Delta P(t;Q)} \mathrm{E}_Q[(Q_i - \mu_j)] + \mu_j^2(\Delta P(t;Q) - \lambda(t)\Delta t)^2\right]\\ =\ & \sigma_d^2(t)\Delta t + \mathrm{E}_{\Delta P(t;Q)}\left[\Delta P(t;Q)\sigma_j^2 + 0 + \mu_j^2(\Delta P(t;Q) - \lambda(t)\Delta t)^2\right]\\ =\ & \left(\sigma_d^2(t) + \left(\sigma_j^2 + \mu_j^2\right)\lambda(t)\right)\Delta t.\end{aligned}$$

The case of the third central moment is similarly calculated,

$$
\begin{aligned}
M^{(3)}_{\rm ljd}(t) \equiv\ & \mathrm{E}\left[(\Delta Y(t)-\mu_{\rm ljd}(t))^3\right] \\
=\ & \mathrm{E}\left[\left(\sigma_d(t)\Delta W(t)+\sum_{i=1}^{\Delta P(t;Q)}(Q_i-\mu_j)+\mu_j(\Delta P(t;Q)-\lambda(t)\Delta t)\right)^3\right] \\
=\ & \sigma_d^3(t)\mathrm{E}_{\Delta W(t)}\left[(\Delta W)^3(t)\right] \\
& +3\sigma_d^2\mathrm{E}_{\Delta W(t)}\left[(\Delta W)^2(t)\right]\mathrm{E}\left[\sum_{i=1}^{\Delta P(t;Q)}(Q_i-\mu_j)+\mu_j(\Delta P(t;Q)-\lambda(t)\Delta t)\right] \\
& +3\sigma_d\cdot 0+\mathrm{E}\left[\left(\sum_{i=1}^{\Delta P(t;Q)}(Q_i-\mu_j)+\mu_j(\Delta P(t;Q)-\lambda(t)\Delta t)\right)^3\right] \\
=\ & \sigma_d^3(t)\cdot 0+3\sigma_d^2(t)\Delta t\cdot 0 \\
& +\mathrm{E}_{\Delta P(t;Q)}\left[\sum_{i=1}^{\Delta P(t;Q)}\sum_{k=1}^{\Delta P(t;Q)}\sum_{\ell=1}^{\Delta P(t;Q)}\mathrm{E}_Q[(Q_i-\mu_j)(Q_k-\mu_j)(Q_\ell-\mu_j)]\right. \\
& +3\mu_j(\Delta P(t;Q)-\lambda(t)\Delta t)\sum_{i=1}^{\Delta P(t;Q)}\sum_{k=1}^{\Delta P(t;Q)}\mathrm{E}_Q[(Q_i-\mu_j)(Q_k-\mu_j)] \\
& \left.+3\mu_j^2(\Delta P(t;Q)-\lambda(t)\Delta t)^2\cdot 0+\mu_j^3(\Delta P(t;Q)-\lambda(t)\Delta t)^3\right] \\
=\ & \mathrm{E}_{\Delta P(t;Q)}\left[\Delta P(t;Q)M_j^{(3)}+3\mu_j(\Delta P(t;Q)-\lambda(t)\Delta t)\Delta P(t;Q)\sigma_j^2\right. \\
& \left.+\mu_j^3(\Delta P(t;Q)-\lambda(t)\Delta t)^3\right] \\
=\ & \left(M_j^{(3)}+\mu_j\left(3\sigma_j^2+\mu_j^2\right)\right)\lambda(t)\Delta t,
\end{aligned}
$$

depending only on the jump component of the jump-diffusion.

The case of the fourth central moment is similarly calculated,

$$
\begin{aligned}
M^{(4)}_{\mathrm{ljd}}(t) \equiv\ & \mathrm{E}\left[(\Delta Y(t) - \mu_{\mathrm{ljd}}(t))^4\right] \\
=\ & \mathrm{E}\left[\left(\sigma_d(t)\Delta W(t) + \sum_{i=1}^{\Delta P(t;Q)} (Q_i - \mu_j) + \mu_j(\Delta P(t;Q) - \lambda(t)\Delta t)\right)^4\right] \\
=\ & \sigma_d^4(t)\mathrm{E}_{\Delta W(t)}\left[(\Delta W)^4(t)\right] + 4\sigma_d^3 \cdot 0 + 6\sigma_d^2 \mathrm{E}_{\Delta W(t)}\left[(\Delta W)^2(t)\right] \\
& \mathrm{E}\left[\left(\sum_{i=1}^{\Delta P(t;Q)} (Q_i - \mu_j) + \mu_j(\Delta P(t;Q) - \lambda(t)\Delta t)\right)^2\right] \\
& + 4\sigma_d \cdot 0 + \mathrm{E}\left[\left(\sum_{i=1}^{\Delta P(t;Q)} (Q_i - \mu_j) + \mu_j(\Delta P(t;Q) - \lambda(t)\Delta t)\right)^4\right] \\
=\ & 3\sigma_d^4(t)(\Delta t)^2 + 6\sigma_d^2(t)\Delta t \mathrm{E}_{\Delta P(t;Q)}\left[\sum_{i=1}^{\Delta P(t;Q)} \sum_{k=1}^{\Delta P(t;Q)} \mathrm{E}_Q[(Q_i - \mu_j)(Q_k - \mu_j)]\right. \\
& \left. + 2\mu_j(\Delta P(t;Q) - \lambda(t)\Delta t) \cdot 0 + \mu_j^2(\Delta P(t;Q) - \lambda(t)\Delta t)^2\right] \\
& + \mathrm{E}_{\Delta P(t;Q)}\left[\sum_{i=1}^{\Delta P(t;Q)} \sum_{k=1}^{\Delta P(t;Q)} \sum_{\ell=1}^{\Delta P(t;Q)} \sum_{m=1}^{\Delta P(t;Q)}\right. \\
& \mathrm{E}_Q[(Q_i - \mu_j)(Q_k - \mu_j)(Q_\ell - \mu_j)(Q_m - \mu_j)] \\
& + 4\mu_j(\Delta P(t;Q) - \lambda(t)\Delta t) \sum_{i=1}^{\Delta P(t;Q)} \sum_{k=1}^{\Delta P(t;Q)} \sum_{\ell=1}^{\Delta P(t;Q)} \\
& \mathrm{E}_Q[(Q_i - \mu_j)(Q_k - \mu_j)(Q_\ell - \mu_j)] \\
& + 6\mu_j^2(\Delta P(t;Q) - \lambda(t)\Delta t)^2 \sum_{i=1}^{\Delta P(t;Q)} \sum_{k=1}^{\Delta P(t;Q)} \mathrm{E}_Q[(Q_i - \mu_j)(Q_k - \mu_j)] \\
& \left. + 4\mu_j^3(\Delta P(t;Q) - \lambda(t)\Delta t)^3 \cdot 0 + \mu_j^4(\Delta P(t;Q) - \lambda(t)\Delta t)^4\right] \\
=\ & 3\sigma_d^4(t)(\Delta t)^2 + 6\sigma_d^2(t)\Delta t \mathrm{E}_{\Delta P(t;Q)}\left[\Delta P(t;Q)\sigma_j^2 + \mu_j^2(\Delta P(t;Q) - \lambda(t)\Delta t)^2\right] \\
& + \mathrm{E}_{\Delta P(t;Q)}\left[\Delta P(t;Q) M_j^{(4)} + 3\Delta P(t;Q)(\Delta P(t;Q) - 1)\sigma_j^4\right. \\
& + 4\mu_j(\Delta P(t;Q) - \lambda(t)\Delta t)\Delta P(t;Q) M_j^{(3)} \\
& \left. + 6\mu_j^2(\Delta P(t;Q) - \lambda(t)\Delta t)^2 \Delta P(t;Q)\sigma_j^2 + \mu_j^4(\Delta P(t;Q) - \lambda(t)\Delta t)^4\right] \\
=\ & \left(M_j^{(4)} + 4\mu_j M_j^{(3)} + 6\mu_j^2\sigma_j^2 + \mu_j^4\right)\lambda(t)\Delta t + 3\left(\sigma_d^2(t) + \left(\sigma_j^2 + \mu_j^2\right)\lambda(t)\right)^2(\Delta t)^2,
\end{aligned}
$$

completing the proofs for moments $m = 1{:}4$.

Also, as used throughout, the expectations of odd powers of $\Delta W(t)$, the single powers of $(Q_i - \mu_j)$ and the single powers of $(\Delta P(t;Q) - \lambda(t)\Delta t)$ were immediately set to zero. In addition, the evaluation of the mark deviation sums of the form $\mathrm{E}[\sum_{i=1}^{k}(Q_i - \mu_j)^m]$ for $m = 1 : 4$ is based upon general formulas of Lemma 5.15. ∎

Remarks 5.18.

- *Recall that the third and fourth moments are measures of skewness and peakedness (kurtosis), respectively. The normalized representations in the current notation are the coefficient of skewness,*

$$\eta_3[\Delta Y(t)] \equiv M^{(3)}_{\text{ljd}}(t)/\sigma^3_{\text{ljd}}(t), \tag{5.67}$$

from (B.11), *and the coefficient of kurtosis,*

$$\eta_4[\Delta Y(t)] \equiv M^{(4)}_{\text{ljd}}(t)/\sigma^4_{\text{ljd}}(t), \tag{5.68}$$

from (B.12).

- *For example, if the marks are normally or uniformly distributed, then*

$$M^{(3)}_j = 0,$$

since the normal and uniform distributions are both symmetric about the mean, so they lack skew and thus we have

$$\eta_3[\Delta Y(t)] = \frac{\mu_j\left(3\sigma_j^2+\mu_j^2\right)\lambda(t)\Delta t}{\sigma^3_{\text{ljd}}(t)} = \frac{\mu_j\left(3\sigma_j^2+\mu_j^2\right)\lambda(t)}{\left(\sigma_d^2(t)+\left(\sigma_j^2+\mu_j^2\right)\lambda(t)\right)^3(\Delta t)^2},$$

using $\sigma_{\text{ljd}}(t)$ given by (5.64). *For the uniform distribution, the mean μ_j is given explicitly in terms of the uniform interval $[a,b]$ by* (B.15) *and the variance σ_j^2 by* (B.16), *while for the normal distribution, μ_j and σ_j^2 are the normal model parameters. In general, the normal and unform distribution versions of the log-jump-diffusion process will have skew, although the component incremental diffusion and mark processes are skewless.*

In the normal and uniform mark cases, the fourth moment of the jump marks are

$$M^{(4)}_j/\sigma_j^4 = \left\{\begin{array}{ll} 3, & \textit{normal } Q_i \\ 1.8, & \textit{uniform } Q_i \end{array}\right\},$$

which are in fact the coefficients of kurtosis for the normal and uniform distributions, respectively, so

$$\begin{aligned}\eta_4[\Delta Y(t)] = {} & \left(\left\{\begin{array}{ll} 3, & \textit{normal } Q_i \\ 1.8, & \textit{uniform } Q_i \end{array}\right\}\sigma_j^4 + 6\mu_j^2\sigma_j^2 + \mu_j^4\right)\lambda(t)\Delta t/\sigma^4_{\text{ljd}}(t) \\ & + 3\left(\sigma_d^2(t)+\left(\sigma_j^2+\mu_j^2\right)\lambda(t)\right)^2(\Delta t)^2/\sigma^4_{\text{ljd}}(t).\end{aligned}$$

- *The moment formulas for the differential log-jump-diffusion process $dY(t)$ follow immediately from Theorem* 5.17 *by dropping terms* $\mathrm{O}((\Delta t)^2)$ *and replacing Δt by dt.*

Distribution of Increment Log-Process

Theorem 5.19. ***Distribution of the State Increment Logarithm Process for Linear Mark-Jump-Diffusion SDE.***
Let the logarithm-transform jump-amplitude be $\ln(1+\nu(t,q)) = q$. *Then the increment of the logarithm process* $Y(t) = \ln(X(t))$, *assuming* $X(t_0) = x_0 > 0$ *and the jump-count increment, approximately satisfies*

$$\Delta Y(t) \simeq \mu_{\rm ld}(t)\Delta t + \sigma_d(t)\Delta W(t) + \sum_{j}^{\Delta P(t;Q)} \widehat{Q}_j \tag{5.69}$$

for sufficiently small Δt, *where* $\mu_{\rm ld}(t) \equiv \mu_d(t) - \sigma_d^2(t)/2$ *is the log-diffusion drift,* $\sigma_d > 0$ *and the* $\widehat{Q}_j$ *are pairwise IID jump marks for* $P(s;Q)$ *for* $s \in (t, t+\Delta t]$, *counting only jumps associated with* $\Delta P(t;Q)$ *given* $P(t;Q)$, *with common density* $\phi_Q(q)$. *The* $\widehat{Q}_j$ *are independent of both* $\Delta P(t;Q)$ *and* $\Delta W(t)$.

Then the distribution of the log-process $Y(t)$ *is the Poisson sum of nested convolutions*

$$\Phi_{\Delta Y(t)}(x) \simeq \sum_{k=1}^{\infty} p_k(\lambda(t)\Delta t)\left(\Phi_{\Delta G(t)}\,(*\phi_Q)^k\right)(x), \tag{5.70}$$

where $\Delta G(t) \equiv \mu_{\rm ld}(t)\Delta t + \sigma_d(t)\Delta W(t)$ *is the infinitesimal Gaussian process and* $(\Phi_{\Delta G(t)}(*\phi_Q)^k)(x)$ *denotes a convolution of one distribution with* k *identical densities* ϕ_Q. *The corresponding log-process density is*

$$\phi_{\Delta Y(t)}(x) \simeq \sum_{k=1}^{\infty} p_k(\lambda(t)\Delta t)\left(\phi_{\Delta G(t)}\,(*\phi_Q)^k\right)(x). \tag{5.71}$$

Proof. By the law of total probability (B.92), the distribution of the log-jump-diffusion $\Delta Y(t) \simeq \Delta G(t) + \sum_j^{\Delta P(t;Q)} \widehat{Q}_j$ is

$$\begin{aligned}\Phi_{\Delta Y(t)}(x) &= \text{Prob}[\Delta Y(t) \le x] = \text{Prob}\left[\Delta G(t) + \sum_{j=1}^{\Delta P(t;Q)} \widehat{Q}_j \le x\right] \\ &= \sum_{k=0}^{\infty} \text{Prob}\left[\Delta G(t) + \sum_{j=1}^{\Delta P(t;Q)} \widehat{Q}_j \le x | \Delta P(t;Q) = k\right] \text{Prob}[\Delta P(t;Q) = k] \\ &= \sum_{k=0}^{\infty} p_k(\lambda(t)\Delta t)\Phi^{(k)}(x),\end{aligned} \tag{5.72}$$

where $p_k(\lambda(t)\Delta t)$ is the Poisson distribution with parameter $\lambda(t)\Delta t$ and

$$\Phi^{(k)}(x) \equiv \text{Prob}\left[\Delta G(t) + \sum_{j=1}^{k} \widehat{Q}_j \le x\right].$$

For each discrete condition $\Delta P(t;Q) = k$, $\Delta Y(t)$ is the sum of $k+1$ terms, the normally distributed Gaussian diffusion part $\Delta G(t) = \mu_{\rm ld}(t)\Delta t + \sigma_d(t)\Delta W(t)$ and the Poisson counting sum $\sum_{j=1}^{k} \widehat{Q}_j$, where the marks $\widehat{Q}_j$ are assumed to be IID but otherwise distributed with density $\phi_Q(q)$, while independent of the diffusion and the Poisson counting differential process $\Delta P(t;Q)$. Using the fact that $\Delta W(t)$ is normally distributed with zero-mean and Δt-variance,

$$\begin{aligned}\Phi_{\Delta G(t)}(x) &= \text{Prob}[\Delta G(t) \le x] = \text{Prob}[\mu_{\rm ld}(t)\Delta t + \sigma_d(t)\Delta W(t) \le x] \\ &= \text{Prob}[\Delta W(t) \le (x - \mu_{\rm ld}(t)\Delta t)/\sigma_d(t)] = \Phi_{\Delta W(t)}((x - \mu_{\rm ld}(t)\Delta t)/\sigma_d(t)) \\ &= \Phi_n((x - \mu_{\rm ld}(t)\Delta t)/\sigma_d(t); 0, \Delta t) = \Phi_n(x; \mu_{\rm ld}(t)\Delta t, \sigma_d^2(t)\Delta t),\end{aligned}$$

provided $\sigma_d(t) > 0$, while also using identities for normal distributions, where $\Phi_n(x;\mu,\sigma^2)$ denotes the normal distribution with mean μ and variance σ^2.

Since $\Phi^{(k)}$ is the distribution for the sum of $k+1$ independent random variables, with one normally distributed random variable and k IID jump marks $\widehat{Q}_j$ for each k, $\Phi^{(k)}$ will be the nested convolutions as given in (B.100). Upon expanding in convolutions starting from the distribution for the random variable $\Delta G(t)$ and the kth Poisson counting sum

$$J_k \equiv \sum_{j=1}^{k} \widehat{Q}_j,$$

we get

$$\Phi^{(k)}(x) = \left(\Phi_{\Delta G(t)} * \phi_{J_k}\right)(x) = \left(\Phi_{\Delta G(t)} \prod_{i=1}^{k} (*\phi_{Q_i})\right)(x) = \left(\Phi_{\Delta G(t)} (*\phi_Q)^k\right)(x),$$

using the identically distributed property of the Q_i's and the compact convolution operator notation

$$\left(\Phi_{\Delta G(t)} \prod_{i=1}^{k} (*\phi_{Q_i})\right)(x) = ((\cdots((\Phi_{\Delta G(t)} * \phi_{Q_1}) * \phi_{Q_2}) \cdots * \phi_{Q_{k-1}}) * \phi_{Q_k})(x),$$

which collapses to the operator power form for IID marks since $\prod_{i=1}^{k} c = c^k$ for some constant c. Substituting the distribution into the law of total probability form (5.72), the desired result is (5.70), which when differentiated with respect to x yields the kth density $\phi_{\Delta Y(t)}(x)$ in (5.71). □

Remark 5.20. *Several specialized variations of this theorem are found in Hanson and Westman [124, 126], but corrections to these papers are made here.*

Corollary 5.21. ***Density of Linear Jump-Diffusion with Log-Normally Distributed Jump-Amplitudes.***
Let $X(t)$ be a linear jump-diffusion satisfying SDE (5.69) and let the jump-amplitude mark Q be normally distributed such that

$$\phi_Q(x;t) = \phi_n(x; \mu_j(t), \sigma_j^2(t)) \tag{5.73}$$

with jump mean $\mu_j(t) = \mathrm{E}[Q]$ and jump variance $\sigma_j^2(t) = \mathrm{Var}[Q]$. Then the jump-diffusion density of the log-process $Y(t)$ is

$$\phi_{\Delta Y(t)}(x) = \sum_{k=1}^{\infty} p_k(\lambda(t)\Delta t)\phi_n(x; \mu_{\rm ld}(t)\Delta t + k\mu_j(t), \sigma_d^2(t)\Delta t + k\sigma_j^2(t)). \quad (5.74)$$

Proof. By (B.101) the convolution of two normal densities is a normal distribution with a mean that is the sum of the means and a variance that is the sum of the variances. Similarly, by the induction exercise result in (B.196), the pairwise convolution of one normally distributed diffusion process $\Delta G(t) = \mu_{\rm ld}(t)\Delta t + \sigma_d(t)\Delta W(t)$ density and k random mark Q_i densities ϕ_Q for $i = 1\!:\!k$ will be a normal density whose mean is the sum of the $k+1$ means and whose variance is the sum of the $k+1$ variances. Thus starting with the result (5.72) and then applying (B.196),

$$\begin{aligned}\phi_{\Delta Y(t)}(x) &= \sum_{k=1}^{\infty} p_k(\lambda(t)\Delta t)\left(\phi_{\Delta G(t)}\,(*\phi_Q)^k\right)(x)\\ &= \sum_{k=1}^{\infty} p_k(\lambda(t)\Delta t)\phi_n\left(x; \mu_{\rm ld}(t)\Delta t + \sum_{i=1}^{k}\mu_j(t), \sigma_d^2(t)\Delta t + \sum_{i=1}^{k}\sigma_j^2(t)\right)\\ &= \sum_{k=1}^{\infty} p_k(\lambda(t)\Delta t)\phi_n(x; \mu_{\rm ld}(t)\Delta t + k\mu_j(t), \sigma_d^2(t)\Delta t + k\sigma_j^2(t)).\end{aligned}$$

□

Remark 5.22. *The normal jump-amplitude jump-diffusion distribution has been used in financial applications, initially by Merton* [202] *and then by others such as Düvelmeyer* [76], *Andersen et al.* [6] *and Hanson and Westman* [124].

Corollary 5.23. ***Density of Linear Jump-Diffusion with Log-Uniformly Distributed Jump-Amplitudes.***
Let $X(t)$ be a linear jump-diffusion satisfying SDE (5.69), *and let the jump-amplitude mark Q be uniformly distributed as in* (5.28), *i.e.,*

$$\phi_Q(q) = \frac{1}{b-a}U(q; a, b),$$

where $U(q; a, b)$ is the unit step function on $[a, b]$ with $a < b$. The jump-mean is $\mu_j(t) = (b+a)/2$ and jump-variance is $\sigma_j^2(t) = (b-a)^2/12$.

Then the jump-diffusion density of the increment log-process $\Delta Y(t)$ satisfies the general convolution form (5.71), *i.e.,*

$$\phi_{\Delta Y(t)}(x) = \sum_{k=1}^{\infty} p_k(\lambda(t)\Delta t)\left(\phi_{\Delta G(t)}\,(*\phi_Q)^k\right)(x) = \sum_{k=1}^{\infty} p_k(\lambda(t)\Delta t)\phi_{\rm ujd}^{(k)}(x), \quad (5.75)$$

where $p_k(\lambda(t)\Delta t)$ is the Poisson distribution with parameter $\lambda(t)$. The $\Delta G(t) = \mu_{\rm ld}(t)\Delta t + \sigma_d(t)\Delta W(t)$ is the diffusion term and Q is the uniformly distributed *jump-amplitude mark. The first few coefficients of* $p_k(\lambda(t)\Delta t)$ *for the uniform jump-distribution (ujd) are*

$$\phi^{(0)}_{\text{ujd}}(x) = \phi_{\Delta G(t)}(x) = \phi_n(x;\mu_{\text{ld}}(t)\Delta t, \sigma^2_d(t)\Delta t), \tag{5.76}$$

where $\phi_n(x;\mu_{\text{ld}}(t)\Delta t, \sigma^2_d(t)\Delta t)$ *denotes the normal density with mean* $\mu_{\text{ld}}(t)$ *and variance* $\sigma_d(t)\Delta t$,

$$\phi^{(1)}_{\text{ujd}}(x) = \left(\phi_{\Delta G(t)} * \phi_Q\right)(x) = \phi_{sn}(x-b, x-a;\mu_{\text{ld}}(t)\Delta t, \sigma^2_d(t)\Delta t), \tag{5.77}$$

where ϕ_{sn} *is the* ***secant-normal density***

$$\begin{aligned}\phi_{sn}(x_1,x_2;\mu,\sigma^2) &\equiv \frac{1}{(x_2-x_1)}\Phi_n(x_1,x_2;\mu,\sigma^2) \\ &\equiv \frac{\Phi_n(x_2;\mu,\sigma^2)-\Phi_n(x_1;\mu,\sigma^2)}{x_2-x_1}\end{aligned} \tag{5.78}$$

with normal distribution $\Phi_n(x_1,x_2;\mu,\sigma^2)$ *such that*

$$\Phi_n(x_i;\mu,\sigma^2) \equiv \Phi_n(-\infty,x_i;\mu,\sigma^2)$$

for $i = 1\!:\!2$, *and*

$$\begin{aligned}\phi^{(2)}_{\text{ujd}}(x) &= \left(\phi_{\Delta G(t)}(*\phi_Q)^2\right)(x) \\ &= \frac{2b-x+\mu_{\text{ld}}(t)\Delta t}{b-a}\phi_{sn}(x-2b, x-a-b;\mu_{\text{ld}}(t)\Delta t, \sigma^2(t)\Delta t) \\ &\quad + \frac{x-2a-\mu_{\text{ld}}(t)\Delta t}{b-a}\phi_{sn}(x-a-b, x-2a;\mu_{\text{ld}}(t)\Delta t, \sigma^2_d(t)\Delta t) \\ &\quad + \frac{\sigma^2_d(t)\Delta t}{(b-a)^2}\big(\phi_n(x-2b;\mu_{\text{ld}}(t)\Delta t, \sigma^2_d(t)\Delta t) \\ &\quad - 2\phi_n(x-a-b;\mu_{\text{ld}}(t)\Delta t, \sigma^2_d(t)\Delta t) + \phi_n(x-2a;\mu_{\text{ld}}(t)\Delta t, \sigma^2_d(t)\Delta t)\big)\,.\end{aligned} \tag{5.79}$$

Proof. First the finite range of the jump-amplitude uniform density is used to truncate the convolution integrals for each k using existing results for the mark convolutions, such as $\phi^{(2)}_{\text{uq}}(x) = (\phi_Q * \phi_Q)(x) = \phi_{Q_1+Q_2}(x)$ for IID marks when $k = 2$.

The case for $k = 0$ is trivial since it is given in the theorem equations (5.76).

For a $k = 1$ jump,

$$\begin{aligned}\phi^{(1)}_{\text{ujd}}(x) = (\phi_{\Delta G(t)} * \phi_Q)(x) &= \int_{-\infty}^{+\infty} \phi_{\Delta G(t)}(x-y)\phi_Q(y)dy \\ &= \frac{1}{b-a}\int_a^b \phi_n(x-y;\mu_{\text{ld}}(t)\Delta t, \sigma^2_d(t)\Delta t)dy \\ &= \frac{1}{b-a}\int_{x-b}^{x-a} \phi_n(z;\mu_{\text{ld}}(t)\Delta t, \sigma^2_d(t)\Delta t)dz \\ &= \frac{1}{b-a}\Phi_n(x-b, x-a;\mu_{\text{ld}}(t)\Delta t, \sigma^2_d(t)\Delta t) \\ &= \phi_{sn}(x-b, x-a;\mu_{\text{ld}}(t)\Delta t, \sigma^2_d(t)\Delta t),\end{aligned}$$

where $-\infty < x < +\infty$, upon change of variables and use of identities.

For $k = 2$ jumps, the triangular distribution exercise result (B.197) is

$$\phi_{\mathrm{uq}}^{(2)}(x) = (\phi_Q * \phi_Q)(x) = \frac{1}{(b-a)^2}\left\{\begin{array}{ll} x-2a, & 2a \le x < a+b \\ 2b-x, & a+b \le x \le 2b \\ 0, & \text{otherwise} \end{array}\right\}. \tag{5.80}$$

Hence,

$$\begin{aligned}
\phi_{\mathrm{ujd}}^{(2)}(x) &= (\phi_{\Delta G(t)} * (\phi_Q * \phi_Q))(x) = \int_{-\infty}^{+\infty} \phi_{\Delta G(t)}(x-y)(\phi_Q * \phi_Q)(y)dy \\
&= \frac{1}{(b-a)^2}\left(\int_{2a}^{a+b}(y-2a)\phi_{\Delta G(t)}(x-y)dy + \int_{a+b}^{2b}(2b-y)\phi_{\Delta G(t)}(x-y)dy\right) \\
&= \frac{1}{(b-a)^2}\left(\int_{x-a-b}^{x-2a}(x-z-2a)\phi_{\Delta G(t)}(z)dz\right. \\
&\quad \left. + \int_{x-2b}^{x-a-b}(2b-x+z)\phi_{\Delta G(t)}(z)dz\right) \\
&= \frac{2b-x+\mu_{\mathrm{ld}}(t)\Delta t}{b-a}\phi_{sn}(x-2b, x-a-b; \mu_{\mathrm{ld}}(t)\Delta t, \sigma_d^2(t)\Delta t) \\
&\quad + \frac{x-2a-\mu_{\mathrm{ld}}(t)\Delta t}{b-a}\phi_{sn}(x-a-b, x-2a; \mu_{\mathrm{ld}}(t)\Delta t, \sigma_d^2(t)\Delta t) \\
&\quad + \frac{\sigma_d^2(t)\Delta t}{(b-a)^2}\left(\phi_n(x-2b; \mu_{\mathrm{ld}}(t)\Delta t, \sigma_d^2(t)\Delta t)\right. \\
&\quad \left. - 2\phi_n(x-a-b; \mu_{\mathrm{ld}}(t)\Delta t, \sigma_d^2(t)\Delta t) + \phi_n(x-2a; \mu_{\mathrm{ld}}(t)\Delta t, \sigma_d^2(t)\Delta t)\right),
\end{aligned}$$

where the exact integral for the normal density has been used . □

Remarks 5.24.

- *This density form ϕ_{sn} in (5.78) is called a* ***secant-normal density*** *since the numerator is an increment of the normal distribution and the denominator is the corresponding increment in its state arguments, i.e., a secant approximation, which here has the form $\Delta\Phi_n/\Delta x$.*
- *The uniform jump-amplitude jump-diffusion distribution has been used in financial applications, initially by the authors in [126] as a simple, but appropriate, representation of a jump component of market distributions, and some errors have been corrected here.*

Example 5.25. ***Linear SDE Simulator for Log-Uniformly Distributed Jump-Amplitudes.***

The linear SDE jump-diffusion simulator MATLAB code C.14 in Online Appendix C can be converted from the simple discrete jump process to the distributed jump process here. The primary change is the generation of another set of random numbers for the mark process Q, e.g.,

$$Q = a + (b-a) * \texttt{rand}(1, n+1)$$

for a set of $n+1$ *uniformly distributed marks on* (a, b) *so that the jump-amplitudes of* $X(t)$ *are log-uniformly distributed.*

An example is demonstrated in Figure 5.1 *for uniformly distributed marks* Q *on* $(a, b) = (-2, +1)$ *and time-dependent coefficients* $\{\mu_d(t), \sigma_d(t), \lambda(t)\}$. *The MATLAB linear mark-jump-diffusion code* C.15, *called* `linmarkjumpdiff06fig1.m` *in Online Appendix* C, *is a modification of the linear jump-diffusion SDE simulator code* C.14 *illustrated in Figure* 4.3 *for constant coefficients and discrete mark-independent jumps. The state exponent* $Y(t)$ *is simulated as*

$$YS(i+1) = YS(i) + (\mu_d(i) - \sigma_d^2(i)/2) * \Delta t + \sigma_d(i) * DW(i) + Q(i) * DP(i)$$

with $t(i+1) = t0 + i * \Delta t$ *for* $i = 0 : n$ *with* $n = 1{,}000$, $t0 = 0$, $0 \le t(i) \le 2$. *The incremental Poisson jump term* $\Delta P(i) = P(t_i + \Delta t) - P(t_i)$ *is simulated by a uniform random number generator on* $(0, 1)$ *using the acceptance-rejection technique* [230, 97] *to implement the zero-one jump law to obtain the probability of* $\lambda(i)\Delta t$ *that a jump is accepted there. The same random state is used to obtain the simulations of uniformly distributed* Q *on* (a, b) *conditional on a jump event.*

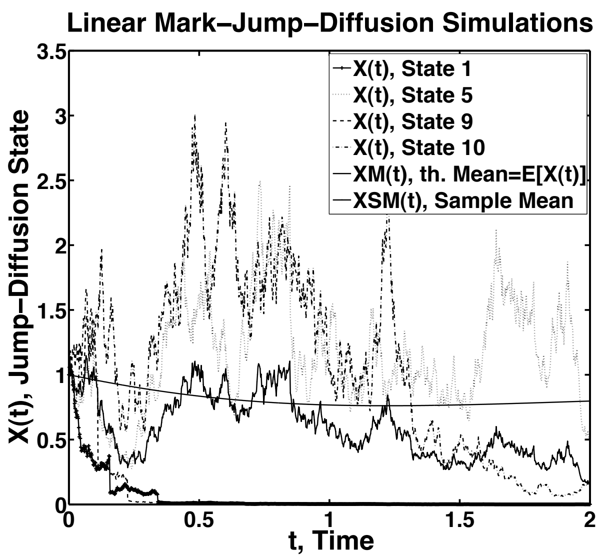

Figure 5.1. *Four linear mark-jump-diffusion sample paths for time-dependent coefficients are simulated using MATLAB* [210] *with* $N = 1{,}000$ *time-steps, maximum time* $T = 2.0$ *and four* **randn** *and four* **rand** *states. Initially,* $x_0 = 1.0$. *Parameter values are given in vectorized functions using vector functions and dot-element operations,* $\mu_d(t) = 0.1 * \sin(t)$, $\sigma_d(t) = 1.5 * \exp(-0.01 * t)$ *and* $\lambda = 3.0 * \exp(-t. * t)$. *The marks are uniformly distributed on* $[-2.0, +1.0]$. *In addition to the four simulated states, the expected state* $\mathrm{E}[X(t)]$ *is presented using the quasi-deterministic equivalence* (5.54) *of Hanson and Ryan* [115], *but also presented are the sample mean of the four sample paths.*

5.3 Multidimensional Markov SDE

The general, multidimensional Markov SDE is presented here, along with the corresponding chain rule, establishing proper matrix-vector notation, or extensions where the standard linear algebra is inadequate, for what follows. In the case of the vector[1] state process $\mathbf{X}(t) = [X_i(t)]_{n_x \times 1}$ on some n_x-dimensional state space $\mathcal{D}_x$, the multidimensional SDE can be of the form

$$d\mathbf{X}(t) \overset{\text{sym}}{=} \mathbf{f}(\mathbf{X}(t),t)dt + g(\mathbf{X}(t),t)d\mathbf{W}(t) + h(\mathbf{X}(t),t,\mathbf{Q})d\mathbf{P}(t;Q,\mathbf{X}(t),t), \quad (5.81)$$

where also

$$\int_{\mathcal{Q}} h(\mathbf{X}(t),t,\mathbf{q})\boldsymbol{\mathcal{P}}(\mathbf{dt},\mathbf{dq};\mathbf{X}(t),t) \underset{\text{zol}}{\overset{\text{dt}}{=}} h(\mathbf{X}(t),t,\mathbf{Q})d\mathbf{P}(t;Q,\mathbf{X}(t),t) \quad (5.82)$$

is the notation for the space-time Poisson terms, $\mathbf{W}(t) = [W_i(t)]_{n_w \times 1}$ is an n_w-dimensional vector Wiener process, $\mathbf{P}(t;Q,\mathbf{X}(t),t) = [P_i(t;\mathbf{X}(t),t)]_{n_p \times 1}$ is an n_p-dimensional vector state-dependent Poisson process, the coefficient $\mathbf{f}$ has the same dimension as $\mathbf{X}$, and the coefficients in the set $\{g,h\}$ have dimensions commensurate in multiplication with the set of vectors $\{\mathbf{W},\mathbf{P}\}$, respectively. Here, $\boldsymbol{\mathcal{P}} = [\mathcal{P}_i]_{n_p \times 1}$ is a vector form of the Poisson random measure with mark random vector $\mathbf{Q} = [Q_i]_{n_p \times 1}$ and $\mathbf{dq} = [(q_i, q_i + dq_i]]_{n_p \times 1}$ is the symbolic vector version of the mark measure notation. The $d\mathbf{P}(t;\mathbf{X}(t),t)$ jump-amplitude coefficient has the component form

$$h(\mathbf{X}(t),t;\mathbf{Q}) = [h_{i,j}(\mathbf{X}(t),t;Q_j)]_{n_x \times n_p},$$

such that the jth Poisson component depends on only the jth mark Q_j since simultaneous jumps are unlikely.

In component and jump counter form, the SDE is

$$dX_i(t) \overset{\text{dt}}{=} f_i(\mathbf{X}(t),t)dt + \sum_{j=1}^{n_w} g_{i,j}(\mathbf{X}(t),t)dW_j(t) + \sum_{j=1}^{n_p} h_{i,j}(\mathbf{X}(t),t,\mathbf{Q})dP_j(t;Q,\mathbf{X}(t),t) \quad (5.83)$$

for $i = 1\!:\!n_x$ state components. The jump of the ith state due to the jth Poisson process

$$[X_i](T_{j,k}) = h_{i,j}(\mathbf{X}(T_{j,k}^-),T_{j,k}^-,Q_{j,k}),$$

where $T_{j,k}^-$ is the prejump-time and its k realization with jump-amplitude mark $Q_{j,k}$. The diffusion noise components have zero-mean,

$$\mathrm{E}[dW_i(t)] = 0 \quad (5.84)$$

[1] Boldface variables or processes denote column vector variables or processes, respectively. The subscript i usually denotes a row index in this notation, while j denotes a column index. For example, $\mathbf{X}(t) = [X_i(t)]_{n_x \times 1}$ denotes that X_i is the ith component for $i = 1 : n_x$ of the single-column vector $\mathbf{X}(t)$.

for $i = 1\!:\!n_w$, while **correlations** are allowed between components,

$$\mathrm{Cov}[dW_i(t), dW_j(t)] = \rho_{i,j}dt = [\delta_{i,j} + \rho_{i,j}(1-\delta_{i,j})]dt, \tag{5.85}$$

for $i, j = 1\!:\!n_x$, where $\rho_{i,j}$ is the correlation coefficient between i and j components.

The jump-noise components, conditioned on $\mathbf{X}(t) = \mathbf{x}$, are Poisson distributed with $\mathcal{P}$ mean assumed to be of the form

$$\mathrm{E}[\mathcal{P}_j(\mathbf{dt}, \mathbf{dq}_j; \mathbf{X}(t), t)|\mathbf{X}(t) = \mathbf{x}] = \phi^{(j)}_{\mathcal{Q}_j}(q_j; \mathbf{x}, t)dq_j\lambda_j(t; \mathbf{x}, t)dt, \tag{5.86}$$

for each jump component $j = 1\!:\!n_p$ with jth density $\phi^{(j)}_Q(q_j; \mathbf{x}, t)$ depending only on q_j assuming independence of the marks for different Poisson components but IID for the same component, so that the Poisson mark integral is

$$\begin{aligned}
\mathrm{E}[dP_j(t; Q, \mathbf{X}(t), t)|\mathbf{X}(t) = \mathbf{x}] &= \mathrm{E}\left[\int_{\mathcal{Q}_j} \mathcal{P}_j(\mathbf{dt}, \mathbf{dq}_j; \mathbf{x}(t), t)\right] \\
&= \int_{\mathcal{Q}_j} \mathrm{E}\left[\mathcal{P}_j(\mathbf{dt}, \mathbf{dq}_j; \mathbf{x}(t), t)\right] \\
&= \int_{\mathcal{Q}_j} \phi^{(j)}_Q(q_j; \mathbf{x}, t)dq_i\lambda_j(t; \mathbf{x}, t)dt \\
&= \lambda_j(t; \mathbf{x}, t)dt
\end{aligned} \tag{5.87}$$

for $i = 1\!:\!n_p$, while the components are assumed to be uncorrelated, with conditioning $\mathbf{X}(t) = \mathbf{x}$ preassumed for brevity,

$$\mathrm{Cov}[\mathcal{P}_j(\mathbf{dt}, \mathbf{dq}_j; \mathbf{x}, t)\mathcal{P}_k(\mathbf{dt}, \mathbf{dq}_k; \mathbf{x}, t)] = \phi^{(j)}_Q(q_j; \mathbf{x}, t)\delta(q_k - q_j)dq_k dq_j\lambda_j(t; \mathbf{x}, t)dt, \tag{5.88}$$

generalizing the scalar form (5.15) to vector form, and

$$\begin{aligned}
\mathrm{Cov}[dP_j(t; Q_j, \mathbf{x}, t), dP_k(t; Q_k, \mathbf{x}, t)] &= \int_{\mathcal{Q}_j}\int_{\mathcal{Q}_k} \mathrm{Cov}[\mathcal{P}_j(\mathbf{dt}, \mathbf{dq}_j; \mathbf{x}, t)\mathcal{P}_k(\mathbf{dt}, \mathbf{dq}_k; \mathbf{x}, t)] \\
&= \lambda_j(t; \mathbf{x}, t)dt\ \delta_{j,k}
\end{aligned} \tag{5.89}$$

for $j, k = 1\!:\!n_p$, there being enough complexity for most applications. In addition, it is assumed that, as vectors, the diffusion noise $d\mathbf{W}$, Poisson noise $d\mathbf{P}$ and mark random variable $\mathbf{Q}$ are pairwise independent, but the mark random variable depends on the existence of a jump.

This Poisson formulation is somewhat different from others, such as [95, Part 2, Chapter 2]. The linear combination form has been found to be convenient for both jumps and diffusion when there are several sources of noise in the application.

5.3.1 Conditional Infinitesimal Moments in Multidimensions

The conditional infinitesimal moments for the vector state process $\mathbf{X}(t)$ are more easily calculated by component first, using the noise infinitesimal moments (5.84)–

(5.89). The conditional infinitesimal mean is

$$\begin{aligned}
\mathrm{E}[dX_i(t)|\mathbf{X}(t)=\mathbf{x}] &= f_i(\mathbf{x},t)dt + \sum_{j=1}^{n_w} g_{i,j}(\mathbf{x},t)\mathrm{E}[dW_j(t)] \\
&\quad + \sum_{j=1}^{n_p}\int_{\mathcal{Q}_j} h_{i,j}(\mathbf{x},t,q_j)E[\mathcal{P}_j(\mathbf{dt},\mathbf{dq}_j;\mathbf{x},t)] \\
&= f_i(\mathbf{x},t)dt + \sum_{j=1}^{n_p}\int_{\mathcal{Q}_j} h_{i,j}(\mathbf{x},t,q_j)\phi_Q^{(j)}(q_j;\mathbf{x},t)dq_j\lambda_j(t;\mathbf{x},t)dt \\
&= \left[f_i(\mathbf{x},t) + \sum_{j=1}^{n_p}\overline{h}_{i,j}(\mathbf{x},t)\lambda_j(t;\mathbf{x},t)\right]dt, \qquad (5.90)
\end{aligned}$$

where $\overline{h}_{i,j}(\mathbf{x},t) \equiv E_Q[h_{i,j}(\mathbf{x},t,Q_j)]$. Thus, in vector form

$$\mathrm{E}[d\mathbf{X}(t)|\mathbf{X}(t)=\mathbf{x}] = \left[\mathbf{f}(\mathbf{x},t)dt + \overline{\boldsymbol{h}}(\mathbf{x},t)\boldsymbol{\lambda}(t;\mathbf{x},t)\right]dt, \qquad (5.91)$$

where $\boldsymbol{\lambda}(t;\mathbf{x},t) = [\lambda_i(t;\mathbf{x},t)]_{n_p\times 1}$.

For the conditional infinitesimal covariance, again with preassuming conditioning on $\mathbf{X}(t)=\mathbf{x}$ for brevity,

$$\begin{aligned}
\mathrm{Cov}[dX_i(t),dX_j(t)] &= \sum_{k=1}^{n_w}\sum_{\ell=1}^{n_w} g_{i,k}(\mathbf{x},t)g_{j,\ell}(\mathbf{x},t)\mathrm{Cov}[dW_k(t),dW_\ell(t)] \\
&\quad + \sum_{k=1}^{n_p}\sum_{\ell=1}^{n_p}\int_{\mathcal{Q}_k}\int_{\mathcal{Q}_\ell} h_{i,k}(\mathbf{x},t;q_k)h_{j,\ell}(\mathbf{x},t;q_\ell) \\
&\quad \mathrm{Cov}[\mathcal{P}_k(\mathbf{dt},\mathbf{dq}_k;\mathbf{x},t),\mathcal{P}_\ell(\mathbf{dt},\mathbf{dq}_\ell;\mathbf{x},t)] \\
&= \sum_{k=1}^{n_w}\left(g_{i,k}(\mathbf{x},t)g_{j,k}(\mathbf{x},t) + \sum_{\ell\neq k}\rho_{k,\ell}g_{i,k}(\mathbf{x},t)g_{j,\ell}(\mathbf{x},t)\right)dt \\
&\quad + \sum_{k=1}^{n_p}(h_{i,k}h_{j,k})(\mathbf{x},t)\phi_Q^{(k)}(q_k;\mathbf{x},t)\lambda_k(t;\mathbf{x},t)dt \\
&= \sum_{k=1}^{n_w}\left(g_{i,k}(\mathbf{x},t)g_{j,k}(\mathbf{x},t) + \sum_{\ell\neq k}\rho_{k,\ell}g_{i,k}(\mathbf{x},t)g_{j,\ell}(\mathbf{x},t)\right)dt \\
&\quad + \sum_{k=1}^{n_p}\overline{(h_{i,k}h_{j,k})}(\mathbf{x},t)\lambda_k(t;\mathbf{x},t)dt \qquad (5.92)
\end{aligned}$$

for $i=1:n_x$ and $j=1:n_x$ in precision-dt, where the infinitesimal jump-diffusion covariance formulas (5.85) and (5.88) have been used. Hence, the matrix-vector form of this covariance is

$$\mathrm{Cov}[d\mathbf{X}(t),d\mathbf{X}^\top(t)|\mathbf{X}(t)=\mathbf{x}] \stackrel{\mathrm{dt}}{=} \left[g(\mathbf{x},t)R'g^\top(\mathbf{x},t) + \overline{\mathbf{h}\Lambda\mathbf{h}^\top}(\mathbf{x},t)\right]dt, \qquad (5.93)$$

where

$$R' \equiv [\rho_{i,j}]_{n_w \times n_w} = [\delta_{i,j} + \rho_{i,j}(1-\delta_{i,j})]_{n_w \times n_w}\,, \tag{5.94}$$

$$\Lambda = \Lambda(t;\mathbf{x},t) = [\lambda_i(t;\mathbf{x},t)\delta_{i,j}]_{n_p \times n_p}\,. \tag{5.95}$$

The jump in the ith component of the state at jump-time $T_{j,k}$ in the underlying jth component of the vector Poisson process is

$$[X_i](T_{j,k}) \equiv X_i(T_{j,k}^+) - X_i(T_{j,k}^-) = h_{i,j}(X(T_{j,k}^-), T_{j,k}^-; Q_{j,k}) \tag{5.96}$$

for $k = 1:\infty$ jumps and $i = 1:n_x$ state components, now depending on the jth mark's kth realization $Q_{j,k}$ at the prejump-time $T_{j,k}^-$ at the kth jump of the jth component Poisson process.

5.3.2 Stochastic Chain Rule in Multidimensions

The stochastic chain rule for a scalar function $\mathbf{Y}(t) = \mathbf{F}(\mathbf{X}(t),t)$, twice continuously differentiable in $\mathbf{x}$ and once in t, comes from the expansion

$$\begin{aligned}
d\mathbf{Y}(t) &= d\mathbf{F}(\mathbf{X}(t),t) = \mathbf{F}(\mathbf{X}(t)+d\mathbf{X}(t), t+dt) - \mathbf{F}(\mathbf{X}(t),t) \qquad (5.97)\\
&= \mathbf{F}_t(\mathbf{X}(t),t) + \sum_{i=1}^{n_x} \frac{\partial \mathbf{F}}{\partial x_i}(\mathbf{X}(t),t)\left(f_i(\mathbf{X}(t),t)dt + \sum_{k=1}^{n_w} g_{i,k}(\mathbf{X}(t),t)dW_k(t)\right)\\
&\quad + \frac{1}{2}\sum_{i=1}^{n_x}\sum_{j=1}^{n_x}\sum_{k=1}^{n_w}\sum_{\ell=1}^{n_w}\left(\frac{\partial^2 \mathbf{F}}{\partial x_i \partial x_j} g_{i,k} g_{j,\ell}\right)(\mathbf{X}(t),t)dW_k(t)dW_\ell(t)\\
&\quad + \sum_{j=1}^{n_p}\int_{\mathcal{Q}}\left(\mathbf{F}\left(\mathbf{X}(t) + \widehat{\boldsymbol{h}}_j(\mathbf{X}(t),t,q_j),t\right) - \mathbf{F}(\mathbf{X}(t),t)\right)\\
&\quad \cdot \mathcal{P}_j(\mathbf{dt}, \mathbf{dq}_j; \mathbf{X}(t),t),\\
&\stackrel{\text{dt}}{=} \left(\mathbf{F}_t(\mathbf{X}(t),t) + \mathbf{f}^\top(\mathbf{X}(t),t)\nabla_x[\mathbf{F}](\mathbf{X}(t),t)\right)dt\\
&\quad + \frac{1}{2}\sum_{i=1}^{n_x}\sum_{j=1}^{n_x}\frac{\partial^2 \mathbf{F}}{\partial x_i \partial x_j}\sum_{k=1}^{n_w}\left(g_{i,k}g_{j,k} + \sum_{\ell \neq k}^{n_w}\rho_{k,\ell}g_{i,k}g_{j,\ell}\right)(\mathbf{X}(t),t)dt\\
&\quad + \sum_{j=1}^{n_p}\int_{\mathcal{Q}_j}\Delta_j[\mathbf{F}]\mathcal{P}_j\\
&= \left[\mathbf{F}_t + \mathbf{f}^\top\nabla_x[\mathbf{F}] + \frac{1}{2}\left(gR'g^\top\right) : \nabla_x\left[\nabla_x^\top[\mathbf{F}]\right]\right](\mathbf{X}(t),t)dt\\
&\quad + \int_{\mathcal{Q}}\boldsymbol{\Delta}^\top[\mathbf{F}]\boldsymbol{\mathcal{P}}
\end{aligned}$$

to precision-dt. Here, the

$$\nabla_x[\mathbf{F}] \equiv \left[\frac{\partial \mathbf{F}}{\partial x_i}(\mathbf{x},t)\right]_{n_x \times 1}$$

is the state space gradient (a column n_x-vector),

$$\nabla_x^\top[\mathbf{F}] \equiv \left[\frac{\partial \mathbf{F}}{\partial x_j}(\mathbf{x},t)\right]_{1\times n_x}$$

is the transpose of the state space gradient (a row n_x-vector),

$$\nabla_x\left[\nabla_x^\top[\mathbf{F}]\right] \equiv \left[\frac{\partial^2 \mathbf{F}}{\partial x_i \partial x_j}(\mathbf{x},t)\right]_{n_x\times n_x}$$

is the Hessian matrix for $\mathbf{F}$, R' is a correlation matrix defined in (5.94),

$$A : B \equiv \sum_{i=1}^{n}\sum_{j=1}^{n} A_{i,j}B_{i,j} = \mathrm{Trace}[AB^\top] \tag{5.98}$$

is the **double-dot product** of two $n \times n$ matrices, related to the trace,

$$\widehat{\boldsymbol{h}}_j(\mathbf{x},t,q_j) \equiv [h_{i,j}(\mathbf{x},t,q_j)]_{n_x\times 1} \tag{5.99}$$

is the jth jump-amplitude vector corresponding to the jth Poisson process,

$$\begin{aligned}\boldsymbol{\Delta}^\top[\mathbf{F}] = &\ [\Delta_j[\mathbf{F}](\mathbf{X}(t),t,q_j)]_{1\times n_p}\\ \equiv &\ \left[\mathbf{F}(\mathbf{X}(t)+\widehat{\boldsymbol{h}}_j(\mathbf{X}(t),t,q_j),t) - \mathbf{F}(\mathbf{X}(t),t)\right]_{1\times n_p}\end{aligned} \tag{5.100}$$

is the general jump-amplitude change vector for any t and

$$\boldsymbol{\mathcal{P}} = [\mathcal{P}_i(\mathbf{dt},\mathbf{dq}_i;\mathbf{X}(t),t)]_{n_p\times 1}$$

is the Poisson random measure vector condition. The corresponding jump in $\mathbf{Y}(t)$ due to the jth Poisson component and its kth realization is

$$[\mathbf{Y}]\left(T_{j,k}^-\right) = \mathbf{F}\left(\mathbf{X}\left(T_{j,k}^-\right) + \widehat{\boldsymbol{h}}_j\left(\mathbf{X}\left(T_{j,k}^-\right), T_{j,k}^-, Q_{j,k}\right), T_{j,k}^-\right) - \mathbf{F}\left(\mathbf{X}\left(T_{j,k}^-\right), T_{j,k}^-\right).$$

Example 5.26. ***Merton's Analysis of the Black–Scholes Option Pricing Model.***
A good application of multidimensional SDEs in finance is the survey of Merton's [201]*,* [203, Chapter 8] *analysis of the Black–Scholes [34] financial options pricing model in Section* 10.2 *of Chapter* 10*. This treatment will serve as motivation for the study of SDEs and contains details not in Merton's paper.*

5.4 Distributed Jump SDE Models Exactly Transformable

Here, exactly transformable distributed jump-diffusion SDE models are listed, in the scalar and the vector cases, where conditions are applicable.

5.4.1 Distributed Jump SDE Models Exactly Transformable

- **Distributed scalar jump SDE:**

$$dX(t) = f(X(t),t)dt + g(X(t),t)dW(t) + \int_{\mathcal{Q}} h(X(t),t,q)\mathcal{P}(\mathbf{dt},\mathbf{dq}).$$

- **Transformed scalar process:** $Y(t) = F(X(t),t)$.
- **Transformed scalar SDE:**

$$dY(t) = \left(F_t + F_x f + \frac{1}{2}F_{xx}g^2\right)dt + F_x g dW(t) + \int_{\mathcal{Q}} (F(X(t)+h(X(t),t,q),t) - F(X(t),t))\mathcal{P}(\mathbf{dt},\mathbf{dq}).$$

- **Target explicit scalar SDE:**

$$dY(t) = C_1(t)dt + C_2(t)dW(t) + \int_{\mathcal{Q}} C_3(t,q)\mathcal{P}(\mathbf{dt},\mathbf{dq}).$$

5.4.2 Vector Distributed Jump SDE Models Exactly Transformable

- **Vector distributed jump SDE:**

$$d\mathbf{X}(t) = \mathbf{f}(\mathbf{X}(t),t)dt + g(\mathbf{X}(t),t)d\mathbf{W}(t) + \int_{\mathcal{Q}} h(\mathbf{X}(t),t,\mathbf{q})\boldsymbol{\mathcal{P}}(\mathbf{dt},\mathbf{dq}).$$

- **Vector transformed process:** $\mathbf{Y}(t) = \mathbf{F}(\mathbf{X}(t),t)$.
- **Transformed component SDE:**

$$dY_i(t) = \left(F_{i,t} + \sum_j F_{i,j}f_j + \frac{1}{2}\sum_j\sum_k\sum_l F_{i,jk}g_{jl}g_{kl}\right)dt + \sum_j F_{i,j}\sum_l g_{jl}dW_l(t) + \sum_\ell \int_{\mathcal{Q}} (y_i(\mathbf{X}+\mathbf{h}_\ell,t) - F_i(\mathbf{X},t))\mathcal{P}_\ell(\mathbf{dt},\mathbf{dq}_\ell),$$

$$\mathbf{h}_\ell(\mathbf{x},t,\mathbf{q}_\ell) \equiv [h_{i,\ell}(\mathbf{x},t,q_\ell)]_{m\times 1}.$$

- **Transformed vector SDE:**

$$d\mathbf{Y}(t) = \left(\mathbf{F}_t + (\mathbf{f}^T\nabla_x)\mathbf{F} + \frac{1}{2}(gg^T : \nabla_x\nabla_x)\mathbf{F}\right)dt + \left((gd\mathbf{W}(t))^T\nabla_x\right)\mathbf{F} + \sum_\ell \int_{\mathcal{Q}} (\mathbf{F}(\mathbf{X}+\mathbf{h}_\ell,t) - \mathbf{F}(\mathbf{X},t))\mathcal{P}_\ell(\mathbf{dt},\mathbf{dq}_\ell).$$

- **Vector target explicit SDE:**

$$d\mathbf{Y}(t) = \mathbf{C}_1(t)dt + C_2(t)d\mathbf{W}(t) + \sum_\ell \int_{\mathcal{Q}} \mathbf{C}_{3,\ell}(t,q_\ell)\mathcal{P}_\ell(\mathbf{dt},\mathbf{dq}_\ell).$$

- **Original coefficients:**

$$\begin{aligned}\mathbf{f}(\mathbf{x},t) &= \left(\nabla_x \mathbf{F}^T\right)^{-T}\Big(\mathbf{C}_1(t) - y_t \\ &\qquad -\frac{1}{2}(\nabla_x\mathbf{F}^T)^{-T}C_2C_2^T(\nabla_x\mathbf{F}^T)^{-1} : \nabla_x\nabla_x^T\mathbf{F}\Big); \\ g(\mathbf{x},t) &= (\nabla_x\mathbf{F}^T)^{-T}C_2(t), \\ \mathbf{F}(\mathbf{x}+\mathbf{h}_\ell,t) &= \mathbf{F}(\mathbf{x},t) + \mathbf{C}_{3,\ell}(t,q_\ell). \quad \textit{(Note: left in implicit form.)}\end{aligned}$$

- **Vector affine transformation example:**

$$\begin{aligned}\mathbf{F} &= A(t)\mathbf{x} + \mathbf{B}(t), \\ \mathbf{F}_t &= A'\mathbf{x} + \mathbf{B}', \\ (\nabla_x\mathbf{F}^T)^T &= A, \\ \mathbf{f}(\mathbf{x},t) &= A^{-1}(\mathbf{C}_1(t) - A'\mathbf{x} - \mathbf{B}'), \\ g(\mathbf{x},t) &= A^{-1}C_2(t), \\ \mathbf{h}_\ell(\mathbf{x},t,q_\ell) &= A^{-1}\mathbf{C}_{3,\ell}(t,q_\ell).\end{aligned}$$

5.5 Exercises

1. Simulate $X(t)$ for the log-normally distributed jump-amplitude case with mean $\mu_j = \mathrm{E}[Q] = 0.28$ and variance $\sigma_j^2 = \mathrm{Var}[Q] = 0.15$ for the linear jump-diffusion SDE model (5.42) using $\mu_d(t) = 0.82\sin(2\pi t - 0.75\pi)$, $\sigma_d(t) = 0.88 - 0.44\sin(2\pi t - 0.75\pi)$ and $\lambda(t) = 8.0 - 1.82\sin(2\pi t - 0.75\pi)$, $N = 10,000$ time-steps, $t_0 = 0$, $t_f = 1.0$, $X(0) - x_0$, for $k = 4$ random states, i.e., $\nu(t,Q) = \nu_0(Q) = \exp(Q) - 1$ with Q normally distributed. Plot the k sample states $X_j(t_i)$ for $j = 1 : k$, along with theoretical mean state path, $\mathrm{E}[X(t_i)]$ from (5.49), and the sample mean state path, i.e., $M_x(t_i) = \sum_{j=1}^k X_j(t_i)/k$, all for $i = 1 : N+1$.

   (*Hint: Modify the linear mark-jump-diffusion SDE simulator of Example* 5.25 *with MATLAB code* C.15 *from Online Appendix* C *and Corollary* 5.9 *for the discrete exponential expectation.*)

2. For the log-double-uniform jump distribution,

$$\phi_Q(q;t) \equiv \left\{ \begin{array}{ll} 0, & -\infty < q < a(t) \\ p_1(t)/|a|(t), & a(t) \le q < 0 \\ p_2(t)/b(t), & 0 \le q \le b(t) \\ 0, & b(t) < q < +\infty \end{array} \right\}, \quad (5.101)$$

where $p_1(t)$ is the probability of a negative jump and $p_2(t)$ is the probability of a positive jump on $a(t) < 0 \le b(t)$, show that

(a) $\mathrm{E}_Q[Q] = \mu_j(t) = (p_1(t)a(t) + p_2(t)b(t))/2$;
(b) $\mathrm{Var}_Q[Q] = \sigma_j^2(t) = (p_1(t)a^2(t) + p_2(t)b^2(t))/3 - \mu_j^2(t)$;
(c) $E_Q\big[(Q-\mu_j(t))^3\big] = (p_1(t)a^3(t)+p_2(t)b^3(t))/4 - \mu_j(t)(3\sigma_j^2(t)+\mu_j^2(t))$;
(d) $\mathrm{E}[\nu(Q)] = \mathrm{E}[\exp(Q)-1]$, where the answer needs to be derived.

3. Show that the Itô mean square limit for the integral of the product of two correlated mean-zero, dt-variance, differential diffusion processes, $dW_1(t)$ and $dW_2(t)$, symbolically satisfy the SDE,

$$dW_1(t)dW_2(t) \stackrel{\mathrm{dt}}{=} \rho(t)dt, \quad (5.102)$$

where

$$\mathrm{Cov}[\Delta W_1(t_i), \Delta W_2(t_i)] \simeq \rho(t_i)\Delta t_i$$

for sufficiently small Δt_i. Are any modified considerations required if $\rho = 0$ or $\rho = \pm 1$? You may use the bivariate normal density in (B.144), boundedness Theorem B.59, Table B.1 of selected moments and other material in Online Appendix B of preliminaries.

4. Finish the proof of Corollary 5.13 by showing the diffusion part using the techniques of Theorem 5.11, (5.53).

5. Prove the corresponding corollary for the variance of $X(t)$ from the solution of the linear SDE:

Corollary 5.27. ***Variance of X(t) for Linear SDE.***
Let $X(t)$ be the solution (5.45) with $\overline{\nu^2}(t) \equiv \mathrm{E}[\nu^2(t,Q)]$ of (5.42). Then

$$\mathrm{Var}[dX(t)/X(t)] \stackrel{\mathrm{dt}}{=} \sigma_d^2(t) + \overline{\nu^2}(t)$$

and

$$\mathrm{Var}[X(t)] = \mathrm{E}^2[X(t)]\left(\exp\left(\int_{t_0}^{t} \mathrm{Var}[dX(s)/X(s)]ds\right) - 1\right). \quad (5.103)$$

Be sure to state what extra conditions on processes and precision are needed that were not needed for proving Corollary 5.13 on $\mathrm{E}[X(t)]$.

6. Justify (5.93) for the covariance in multidimensions by giving the reasons for each step in the derivation. See the proof for (5.27).

Suggested References for Further Reading

- Çinlar, 1975 [56]
- Cont and Tankov, 2004 [60]
- Gihman and Skorohod, 1972 [95, Part 2, Chapter 2]
- Hanson, 1996 [109]
- Itô, 1951 [150]
- Kushner and Dupuis, 2001 [179]
- Øksendal and Sulem, 2005 [223]
- Snyder and Miller, 1991 [252, Chapter 4 and 5]
- Westman and Hanson, 1999 [276]
- Westman and Hanson, 2000 [277]
- Zhu and Hanson, 2006 [293]