where only the linear term need be considered for its contribution to the non-negativity of the perturbation since the quadratic term is never negative. When there in minimal bang control, $U^* = -1$*, then the perturbation* δU^* *must necessarily be non-negative, otherwise the control constraints (6.31) would be violated, so for non-negativity of the Hamiltonian perturbation the control perturbation coefficient* $(-1-x)$ *must also be non-negative or that* $x \le -1$*. Similarly, when there is maximal bang control,* $U^* = +1$*, then the perturbation has to be non-positive,* $\delta U^* \le 0$*, to avoid violating the control constraints, so* $\Delta_u \mathcal{H}^* \ge 0$ *(6.40) implies that the coefficient* $(1-x)$ *of* δU^* *must be non-positive or that* $x \ge +1$*.*

- *Similar techniques work with the application of the optimum principles to the case where the Hamiltonian is linear in the control. For example, consider the scalar, linear control Hamiltonian,*

$$\mathcal{H}(x,u,\lambda,t) = C_0(x,t) + C_1(x,t)u + \lambda(F_0(x,t) + F_1(x,t)u)\,,$$

subject to control constraints,

$$U^{(\min)} \le U(t) \le U^{(\max)}\,,$$

and such that

$$\mathcal{H}_u(x,u,\lambda,t) = C_1(x,t) + \lambda F_1(x,t) = \mathcal{H}_u(x,0,\lambda,t)\,,$$

so no regular control exists. However, the perturbed Hamiltonian has the form,

$$\Delta_u \mathcal{H}(X^*,U^*,\lambda^*,t) = \mathcal{H}_u(X^*,0,\lambda^*,t)\delta U^*\,,$$

so optimal control is of the bang-bang form, which for a minimum of $\mathcal{H}$ *using* $\Delta_u \mathcal{H} \ge 0$ *yields the composite form,*

$$U^*(t) = \left\{ \begin{array}{ll} U^{(\min)}, & (\mathcal{H}_u)^* = C_1(X^*(t),t) + \lambda^*(t)F_1(X^*(t),t) > 0 \\ U^{(\max)}, & (\mathcal{H}_u)^* = C_1(X^*(t),t) + \lambda^*(t)F_1(X^*(t),t) < 0 \end{array} \right\}, \quad (6.41)$$

since for $(\mathcal{H}_u)^* > 0$ *then* $\delta U^* \ge 0$ *or equivalently* $U^*(t) = U^{(\min)}$*, and similarly when* $(\mathcal{H}_u)^* < 0$ *then* $\delta U^* \le 0$ *or equivalently* $U^*(t) = U^{(\max)}$*, but if* $(\mathcal{H}_u)^* = 0$ *no information on either* δU^* *or* $U^*(t)$ *can be determined.*

Example 6.8. ***Bang-Bang Control Problem*** *Consider a simple lumped model of a leaky reservoir (after Kirk [109]) given by*

$$\dot{X}(t) = -aX(t) + U(t), \quad X(0) = x_0,$$

where $X(t)$ *is the depth of the reservoir,* $U(t)$ *is the net inflow of water at time* t *constrained and* $a > 0$ *is the rate of leakage. The net inflow is constrained pointwise* $0 \le U(t) \le M$ *for all* $0 < t \le t_f$ *and also cumulatively by*

$$\int_0^{t_f} U(t)dt = K > 0, \quad (6.42)$$

where K, M and t_f are fixed constants, such that $K \leq M \cdot t_f$ for consistency. Find the optimal control law that maximizes the cumulative depth,

$$J[X] = \int_0^{t_f} X(t)dt.$$

Solution: *The extra integral condition (6.42) presents a variation on our standard control problem, but can be treated nicely by extending the state space letting $X_1(t) = X(t)$ and $\dot{X}_2(t) = U(t)$ starting at $X_2(0) = 0$, so that $X_2(t_f) = K$ is precisely the constraint (6.42). Thus, the Hamiltonian is*

$$\mathcal{H}(x_1, x_2, u, \lambda_1, \lambda_2, t) = x_1 + \lambda_1(-ax_1 + u) + \lambda_2 u, \tag{6.43}$$

where λ_1 and λ_2 are Lagrange multipliers. The Hamilton equations for the optimal state and co-state solutions are

$$\begin{aligned} \dot{X}_1^*(t) &= \mathcal{H}_{\lambda_1}^* = -aX_1^*(t) + U^*(t), \quad X_1^*(0) = x_0; \\ \dot{X}_2^*(t) &= \mathcal{H}_{\lambda_2}^* = U^*(t), \quad X_2^*(0) = 0; \\ \dot{\lambda}_1^*(t) &= -\mathcal{H}_{x_1}^* = -1 + a\lambda_1^*(t); \\ \dot{\lambda}_2^*(t) &= -\mathcal{H}_{x_2}^* = 0. \end{aligned}$$

Consequently, $\lambda_2^*(t) = C_2$, a constant, and $X_2^*(t_f) = K$ is fixed. Also, $\lambda_1^*(t) = C_1 \exp(at) + 1/a$ with the constant determined from the transversality condition $\lambda_1^*(t_f) = 0$ of Table 6.1 with $X_1^*(t_f)$ free and no terminal cost, i.e., $S(x) \equiv 0$, so $C_1 = -\exp(-at_f)/a$ and

$$\lambda_1^*(t) = \frac{1}{a}\left(1 - e^{-a(t_f - t)}\right). \tag{6.44}$$

Since

$$\mathcal{H}_u^* = \lambda_1^*(t) + \lambda_2^*(t) \neq 0$$

in general, so the usual critical point condition will not directly produce an optimal control $U^*(t)$, but a bang-bang control will work. By applying the maximum principle in the form (6.38-6.39) with $\delta U(t) = U(t) - U^*(t)$,

$$\Delta_u \mathcal{H}(\mathbf{X}^*(t), \mathbf{U}^*(t), \boldsymbol{\lambda}^*(t), t) = (\lambda_1^*(t) + \lambda_2^*(t))(U(t) - U^*(t)) \leq 0,$$

so if $(\lambda_1^*(t) + \lambda_2^*(t)) > 0$ then $U(t) - U^*(t) \leq 0$ and $U^*(t) = \max[U(t)] = M$, but if $(\lambda_1^*(t) + \lambda_2^*(t)) < 0$ then $U(t) - U^*(t) \geq 0$ and $U^*(t) = \min[U(t)] = 0$. Now, $U^*(t)$ can not be zero on all of $[0, t_f]$ or be M on all of $[0, t_f]$, because both options would violate the constraint (6.42) in the strict case $K < M \cdot t_f$. In this case and noting that $\lambda_1^*(t)$ is decreasing of time, there must be a switch time t_s on $[0, t_f]$ such that $\lambda_1^*(t_s) + \lambda_2^*(t_s) = 0$, $C_2 = \lambda_2^*(t_s) = -\lambda_1^*(t_s) = -(1 - \exp(-a(t_f - t_s))/a < 0$ and

$$X_2^*(t_f) = K = \int_0^{t_s} M dt + \int_{t_s}^{t_f} 0 dt = M t_s,$$

so $t_s = K/M$. The composite bang-bang control law is then

$$U^*(t) = \left\{ \begin{array}{ll} M, & 0 \le t \le t_s \\ 0, & t_s < t \le t_f \end{array} \right\}, \tag{6.45}$$

and the corresponding state trajectory is given by

$$X_1^*(t) = X^*(t) = x_0\ e^{-at} + \frac{M}{a} \left\{ \begin{array}{ll} \left(1 - e^{-at}\right), & 0 \le t \le t_s \\ e^{-at}\left(e^{+at_s} - 1\right), & t_s < t \le t_f \end{array} \right\}. \tag{6.46}$$

The optimal control (6.45), state (6.46) and the switch time indicator multiplier sum (6.44), $\lambda_1^*(t) + \lambda_2^*(t)$, are plotted together in Fig. 6.4 with sample numerical parameter values.

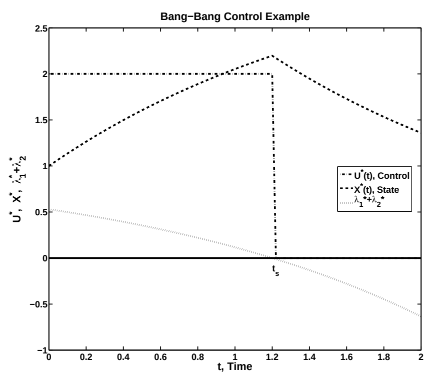

Figure 6.4. *Optimal control, state and switch time multiplier sum are shown for bang-bang control example with sample parameter values $t_0 = 0$, $t_f = 2.0$, $a = 0.6$, $M = 2$, $K = 2.4$ and $x_0 = 1.0$. The computed switch time t_s is also indicated.*

Example 6.9. ***Singular Control Problem*** *Consider the scalar dynamical system for a natural resource with state or mass $X(t)$*

$$\dot{X}(t) \equiv \frac{dX}{dt}(t) = (\mu_0 - U(t))X(t)\,, \quad X(t_0) = x_0 > 0,\,, \quad t_0 \le t \le t_f\,, \tag{6.47}$$

where μ_0 is the natural growth rate and $U(t)$ is the harvest rate or effort that will be taken as the control variable. Thus, (6.47) represents exponential growth of the resource whose growth rate is modified by the control. Let the running "cost" for the objective functional be

$$C(x, u, t) = e^{-\delta_0 t} \max\left[p_0 x - c_0, 0\right] u(t)\,, \tag{6.48}$$

where $p_0 > 0$ is the fixed price per unit effort per unit mass and $c_0 > 0$ is the fixed cost per unit effort, so $p_0 X(t) - c_0$ is the net instantaneous profit at time t.