American Put Option Pricing for a Stochastic-Volatility, Jump-Diffusion Models, with Log-Uniform Jump-Amplitudes*

Floyd B. Hanson and Guoqing Yan

Department of Mathematics, Statistics, and Computer Science

University of Illinois at Chicago

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1. Introduction

- Classical Black-Scholes (1973) model fails to reflect the three empirical phenomena:
  - Non-normal features: return distribution skewed negative and leptokurtic, with higher peak and heavier tails;
  - Volatility smile: implied volatility not constant as in B-S model;
  - Large, sudden movements in prices: crashes and rallies.
- Recently empirical research (Andersen et al.(2002), Bates (1996) and Bakshi et al.(1997)) imply that most reasonable model of stock prices includes both stochastic volatility and jump diffusions. Stochastic volatility is needed to calibrate the longer maturities and jumps are needed to reflect shorter maturity option pricing.
- Log-uniform jump amplitude distribution is more realistic and accurate to describe high-frequency data; square-root stochastic volatility process allows for systematic volatility risk and generates an analytically tractable method of pricing options.

2. Stochastic-Volatility Jump-Diffusion Model

- **2.1. Stochastic-Volatility Jump-Diffusion (SVJD) SDE:**
  Assume asset price $S(t)$, under a risk-neutral probability measure $\mathcal{M}$, follows a jump-diffusion process and conditional variance $V(t)$ follows Heston's (1993) square-root mean-reverting diffusion process:

$$dS(t) = S(t)\left((r - \lambda\bar{J})dt + \sqrt{V(t)}dW_s(t)\right) + \sum_{k=1}^{dN(t)} S(t_k^-)J(Q_k), \quad (1)$$

$$dV(t) = k_v\left(\theta_v - V(t)\right)dt + \sigma_v\sqrt{V(t)}dW_v(t). \quad (2)$$

  where

  - $r =$ constant risk-free interest rate;
  - $W_s(t)$ and $W_v(t)$ are standard Brownian motions with correlation: $\mathrm{Corr}[dW_s(t), dW_v(t)] = \rho$;
  - $J(Q) =$ Poisson jump-amplitude, $Q =$ underlying Poisson amplitude mark process selected so that $Q = \ln(J(Q) + 1)$;

- $N(t)$ = compound Poisson jump process with intensity λ.

- **2.2. Log-Uniform Jump-Diffusion Model (Hanson et al., 2002):**

$$\phi_Q(q) = \frac{1}{b-a}\left\{\begin{array}{ll} 1, & a \le q \le b \\ 0, & \text{else} \end{array}\right\}, \quad a < 0 < b$$

- Mark Mean: $\mu_j \equiv \mathrm{E}_Q[Q] = 0.5(b+a)$;
- Mark Variance: $\sigma_j^2 \equiv \mathrm{V}ar_Q[Q] = (b-a)^2/12$;
- Jump-Amplitude Mean:

$$\bar{J} \equiv \mathrm{E}[J(Q)] \equiv \mathrm{E}[e^Q - 1] = (e^b - e^a)/(b-a) - 1.$$

- Realism, Jump amplitudes are finite:
  - ⋆ NYSE (1988) uses *circuit breakers* limiting very large jumps;
  - ⋆ In optimal portfolio problem finite distributions allow realistic borrowing and short-selling (Hanson and Zhu 2006).

3. American (Put) Option Pricing:

- Note for American call option on non-dividend stock, it is not optimal to exercise before maturity. So American call price is equal to corresponding European call price, at least in the case of jump-diffusions.

- American Put Option:

$$P^{(A)}(S(t),V(t),t;K,T) = \sup_{\tau\in\mathcal{T}(t,T)} \left[\mathrm{E}\left[e^{-r(\tau-t)}\max[K-S(\tau),0]\middle|\mathcal{F}_t\right]\right]$$

  on the domain $\mathcal{D} = \{(s,t)|[0,\infty)\times[0,T]\}$, where K is the strike price, T is the maturity date, $\mathcal{T}(t,T)$ are a set of stopping times τ satisfying $t<\tau\leq T$.

- Early Exercise Feature: The American option can be exercised at any time $\tau\in[0,T]$, unlike the European option.

- Hence, there exists a **Critical Curve** $s = S^*(t)$, a free boundary, in the (s,t)-plane, separating the domain $\mathcal{D}$ into two regions:
  - Continuation Region $\mathcal{C}$, where it is optimal to hold the option, i.e., if $s > S^*(t)$, then $P^{(A)}(s,v,t;K,T) > \max[K-s,0]$. Here, $P^{(A)}$ will have the same description as the European price $P^{(E)}$.
  - Exercise Region $\mathcal{E}$, where it is optimal to exercise the option, i.e., if $s \le S^*(t)$, then $P^{(A)}(s,v,t;K,T) = \max[K-s,0]$.
- The American put option satisfies a PIDE similar to that of the European option, letting $s = S(t)$ and $v = V(t)$,

$$\begin{aligned} 0 = \ & \frac{\partial P^{(A)}}{\partial t}(s,v,t;K,T) + \mathcal{A}\left[P^{(A)}\right](s,v,t;K,T) \\ \equiv \ & \frac{\partial P^{(A)}}{\partial t} + \left(r - \lambda \bar{J}\right) s \frac{\partial P^{(A)}}{\partial s} + k_v(\theta_v - v)\frac{\partial P^{(A)}}{\partial v} - rP^{(A)} \\ & + \frac{1}{2} v s^2 \frac{\partial^2 P^{(A)}}{\partial s^2} + \rho \sigma_v v s \frac{\partial^2 P^{(A)}}{\partial s \partial v} + \frac{1}{2}\sigma_v^2 v \frac{\partial^2 P^{(A)}}{\partial v^2} \\ & + \lambda \int_{-\infty}^{\infty} \left(P^{(A)}(se^q, v, t; K,T) - P^{(A)}(s,v,t;K,T)\right) \phi_Q(q) dq, \end{aligned} \tag{3}$$

for $(s,t) \in \mathcal{C}$ and defining the backward operator $\mathcal{A}$.

- American put option pricing problem as free boundary problem:

$$0 = \frac{\partial P^{(A)}}{\partial t}(s,v,t;K,T) + \mathcal{A}\left[P^{(A)}\right](s,v,t;K,T) \quad (4)$$

for $(s,t) \in \mathcal{C} \equiv [S^*(t),\infty) \times [0,T]$;

$$0 > \frac{\partial P^{(A)}}{\partial t}(s,v,t;K,T) + \mathcal{A}\left[P^{(A)}\right](s,v,t;K,T) \quad (5)$$

for $(s,t) \in \mathcal{E} \equiv [0,S^*(t)] \times [0,T]$. where critical stock price $S^*(t)$ is not known *a priori* as a function of time, called the free boundary.

Conditions in the Continuation Region $\mathcal{C}$:

- European put terminal condition limit:

$$\lim_{t \to T} P^{(A)}(s,v,t;K,T) = \max[K-s,0],$$

- Zero stock price limit of option:

$$\lim_{s \to 0} P^{(A)}(s,v,t;K,T) = K,$$

- Infinite stock price limit of option:

$$\lim_{s \to \infty} P^{(A)}(s,v,t;K,T) = 0,$$

- Critical option value limit:

$$\lim_{s \to S^*(t)} P^{(A)}(s,v,t;K,T) = K - S^*(t),$$

- Critical tangency/contact limit in addition:

$$\lim_{s \to S^*(t)} \left(\partial P^{(A)} \Big/ \partial s\right)(s,v,t;K,T) = -1.$$

4. *Quadratic Approximation* for American Put Option:

- The heuristic quadratic approximation (MacMillan, 1986) key insight: if the PIDE applies to American options $P^{(A)}$ as well as European options $P^{(E)}$ in the continuation region, it also applies to the American option optimal exercise premium,
$$\epsilon^{(P)}(s,v,t;K,T) \equiv P^{(A)}(s,v,t;K,T) - P^{(E)}(s,v,t;K,T),$$
where $P^{(E)}$ is given by Fourier inverse in Yan and Hanson (2006).
- Change in Time: Assuming $\epsilon^{(P)}(s,v,t;K,T) \simeq G(t)Y(s,v,G(t))$ and choosing $G(t) = 1 - e^{-r(T-t)}$ as a new time variable such that $\epsilon^{(P)} = 0$ when $G = 0$ at $t = T$.
- After dropping the term $rG(1-G)\partial Y/\partial G$ since the quadratic $g(1-g) \leq 0.25$ on [0,1], making $G(t)$ a parameter instead of variable, then the *quadratic approximation* of the PIDE is
$$\begin{aligned} 0 \;=\; & +\left(r - \lambda\bar{J}\right) s\frac{\partial Y}{\partial s} - \frac{r}{G}Y + k_v(\theta_v - v)\frac{\partial Y}{\partial v} + \frac{1}{2}vs^2\frac{\partial^2 Y}{\partial s^2} + \rho\sigma_v vs\frac{\partial^2 Y}{\partial s\partial v} \\ & + \frac{1}{2}\sigma_v^2 v\frac{\partial^2 Y}{\partial v^2} + \lambda\int_{-\infty}^{\infty}\left(Y(se^q,v,t) - Y(s,v,t)\right)\phi_Q(q)dq, \end{aligned} \tag{6}$$

with quadratic approximation boundary conditions:

$$\lim_{s\to\infty} Y(s,v,G(t)) = 0,$$
$$\lim_{s\to S^*} Y(s,v,G(t)) = \left(K - S^* - P^{(E)}(S^*,v,t)\right)/\,G, \tag{7}$$
$$\lim_{s\to S^*} (\partial Y/\partial s)(s,v,G(t)) = \left(-1 - \left(\partial P^{(E)}/\partial S\right)(S^*,v,t)\right)/\,G.$$

- By constant-volatility jump-diffusion (CVJD) ad hoc approach (Bates, 1996) reformulated, we assume that the dependence on the volatility variable v is weak and replace v by the constant time averaged quasi-deterministic approximation of $V(t)$:

$$\overline{\overline{V}} \equiv \frac{1}{T}\int_0^T \overline{V}(t)dt = \theta_v + (V(0) - \theta_v)\left(1 - e^{-k_v T}\right)\Big/\,(k_v T).$$

The PIDE (6) becomes the linear constant coefficient OIDE, with argument suppressed parameters G and $\overline{\overline{V}}$,

$$\begin{aligned} 0 \;=\; & +\left(r-\lambda\bar{J}\right)s\widehat{Y}'(s)-\frac{r}{G}\widehat{Y}(s)+\frac{1}{2}\overline{\overline{V}}s^2\widehat{Y}''(s) \\ & +\lambda\int_{-\infty}^{\infty}\left(\widehat{Y}(se^q)-\widehat{Y}(s)\right)\phi_Q(q)dq. \end{aligned} \tag{8}$$

- Solution to the linear OIDE (8) has the power form:

$$\widehat{Y}(s) = c_1 s^{A_1} + c_2 s^{A_2},$$

where $c_1 = 0$ because the positive root A_1 is excluded by the vanishing boundary condition in (7).

- The last two boundary conditions in (7) give the equations satisfied by $S^*(t)$ and c_2. Then $S^* = S^*(t)$ can be calculated by fixed point iteration method with the expression:

$$S^* = \frac{A_2 \left(K - P^{(E)}\left(S^*, \overline{\overline{V}}, t; K, T\right)\right)}{A_2 - 1 - (\partial P^{(E)}/\partial s)\left(S^*, \overline{\overline{V}}, t; K, T\right)}$$

and

$$c_2 = \left(K - S^* - P^{(E)}\left(S^*, \overline{\overline{V}}, t; K, T\right)\right) \Big/ \left(G \cdot (S^*)^{A_2}\right).$$

5. Finite Differences for American Put Options Linear Complementarity Problem:

- Free boundary problem is transferred to partial integro-differential complementarity problem (PIDCP) formulated as follows

$$\begin{gathered} P^{(A)}(s,v,t;K,T) - F(s) \geq 0, \qquad \partial P^{(A)}/\partial\tau - \mathcal{A}P^{(A)} \geq 0, \\ \left(\partial P^{(A)}/\partial\tau - \mathcal{A}P^{(A)}\right)\left(P^{(A)} - F\right) = 0, \end{gathered} \tag{9}$$

  where $F(s) \equiv \max[K-s,0]$ and $\tau \equiv T-t$ is the time-to-go.

- Crank-Nicolson scheme with discrete state operator $\mathcal{A} \simeq L$,

$$P^{(A)}(S_i,V_j,T-\tau_k;K,T) \equiv U(S_i,V_j,\tau_k) \simeq U_{i,j}^{(k)}, \;\; U^{(k)} = \left[U_{i,j}^{(k)}\right],$$

$$\partial P^{(A)}/\partial\tau \simeq \frac{U^{(k+1)} - U^{(k)}}{\Delta\tau} \quad \& \quad \mathcal{A}P^{(A)} \simeq \frac{1}{2}L\left(U^{(k+1)} + U^{(k)}\right).$$

- Standard Linear Algebraic Definitions: Let $\widehat{\mathbf{U}}^{(k)} = \left[\widehat{U}_i^{(k)}\right]$, the single subscripted version of $U^{(k)} = \left[U_{i,j}^{(k)}\right]$, with corresponding $\widehat{\mathbf{F}}$, $\widehat{L}$, $\widehat{M}$ and $\widehat{\mathbf{b}}^{(k)}$, so

$$\widehat{M} \equiv I - \frac{\Delta\tau}{2}\widehat{L} \quad \& \quad \widehat{\mathbf{b}}^{(k)} \equiv \left(I + \frac{\Delta\tau}{2}\widehat{L}\right)\widehat{\mathbf{U}}^{(k)}.$$

- Discretized LCP (Cottle et al., 1992; Wilmott et al., 1995, 1998):

$$\begin{gathered} \widehat{\mathbf{U}}^{(k+1)} - \widehat{\mathbf{F}} \geq \mathbf{0}, \qquad \widehat{M}\widehat{\mathbf{U}}^{(k+1)} - \widehat{\mathbf{b}}^{(k)} \geq \mathbf{0}, \\ \left(\widehat{\mathbf{U}}^{(k+1)} - \widehat{\mathbf{F}}\right)^{\top}\left(\widehat{M}\widehat{\mathbf{U}}^{(k+1)} - \widehat{\mathbf{b}}^{(k)}\right) = 0, \end{gathered} \tag{10}$$

- Projective Successive OverRelaxation (PSOR = projected SOR on max) algorithm with acceleration parameter ω for LCP (10) by iterating $\widetilde{U}_i^{(n+1)}$ for $\widehat{U}_i^{(k+1)}$ until changes are sufficiently small:

$$\widetilde{U}_i^{(n+1)} = \max\left(\widehat{F}_i\ ,\ \widetilde{U}_i^{(n)} + \omega\widehat{M}_{i,i}^{-1}\left(\widehat{b}_i^{(k)} - \sum_{j<i}\widehat{M}_{i,j}\widetilde{U}_j^{(n+1)} - \sum_{j\geq i}\widehat{M}_{i,j}\widetilde{U}_j^{(n)}\right)\right).$$

- Full Boundary Conditions for $U(s,v,\tau)$:

$$U(0,v,\tau)=F(0) \;\text{ for }\; v\ge 0 \;\text{ and }\; \tau\in[0,T],$$
$$U(s,v,\tau)\to 0 \;\text{ as }\; s\to\infty \;\text{ for }\; v\ge 0 \;\text{ and }\; \tau\in[0,T],$$
$$U(s,0,\tau)=F(s) \;\text{ for }\; s\ge 0 \;\text{ and }\; \tau\in[0,T],$$
$$\partial U(s,v,\tau)/\partial v=0 \;\text{ as }\; v\to\infty \;\text{ for }\; s\ge 0 \;\text{ and }\; \tau\in[0,T].$$

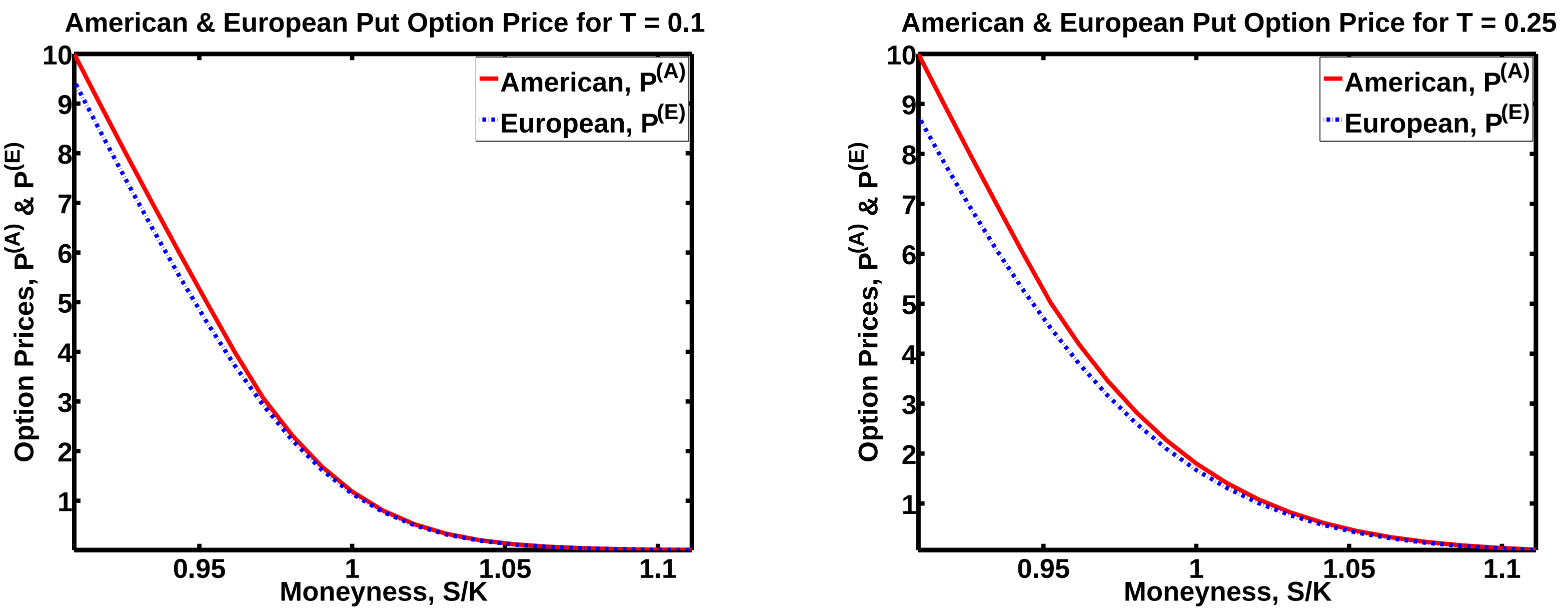
- Initial Condition for $U(s,v,\tau)$:
  $U(s,v,0)=F(s) \;\text{ for }\; s\ge 0 \;\text{ and }\; v\ge 0.$

- Discretization of the PIDE: The first-order and second-order spatial derivatives and the cross-derivative term are all approximated with the standard second-order accurate finite differences, using a nine-point computational molecule. Linear interpolation is applied to the jump integral term and quadratic extrapolation of the solution is used for the critical stock price $S^*(t)$ calculation.

6. Implementation and Methods Comparison:

- The Heuristic Quadratic Approximation and LCP/PSOR approaches for American put option pricing are implemented and compared. All computations are done on a 2.40GHz Celeron$^{(R)}$ CPU. For the quadratic approximation analytic formula, one American put option price and critical stock price can be computed in about 7 seconds. The finite difference method can give a series of option prices for different stock prices and maturity for a specific strike price by one implementation. A single implementation, with $51 \times 101 \times 51$ grids and acceleration parameter $\omega = 1.35$, takes 17 seconds.

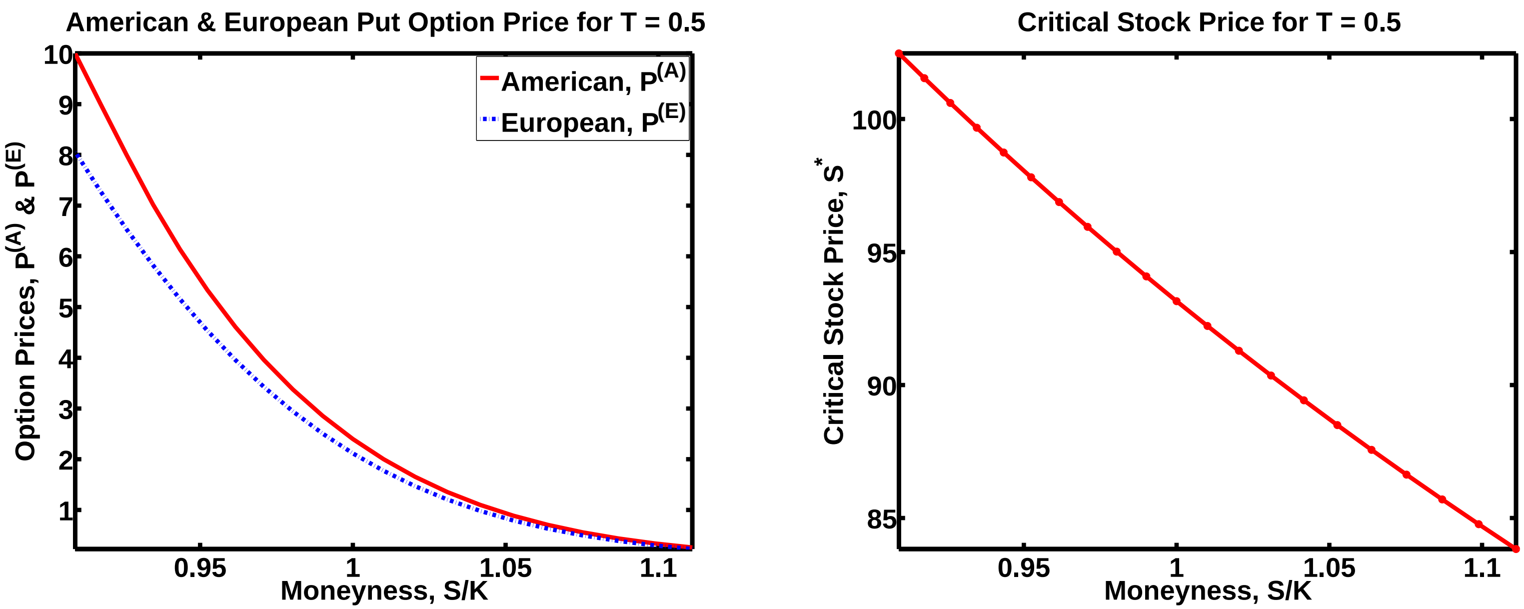
- The American put option prices are implemented for Parameters: $r = 0.05$, $S_0 = \$100$; the stochastic volatility part: $V = 0.01$, $k_v = 10$, $\theta_v = 0.012$, $\sigma_v = 0.1$, $\rho = -0.7$; and the uniform jump part: $a = -0.10$, $b = 0.02$ and $\lambda = 0.5$.



(a) American and European put option prices for $T = 0.1$ years.

(b) American and European put option prices for $T = 0.25$ years.

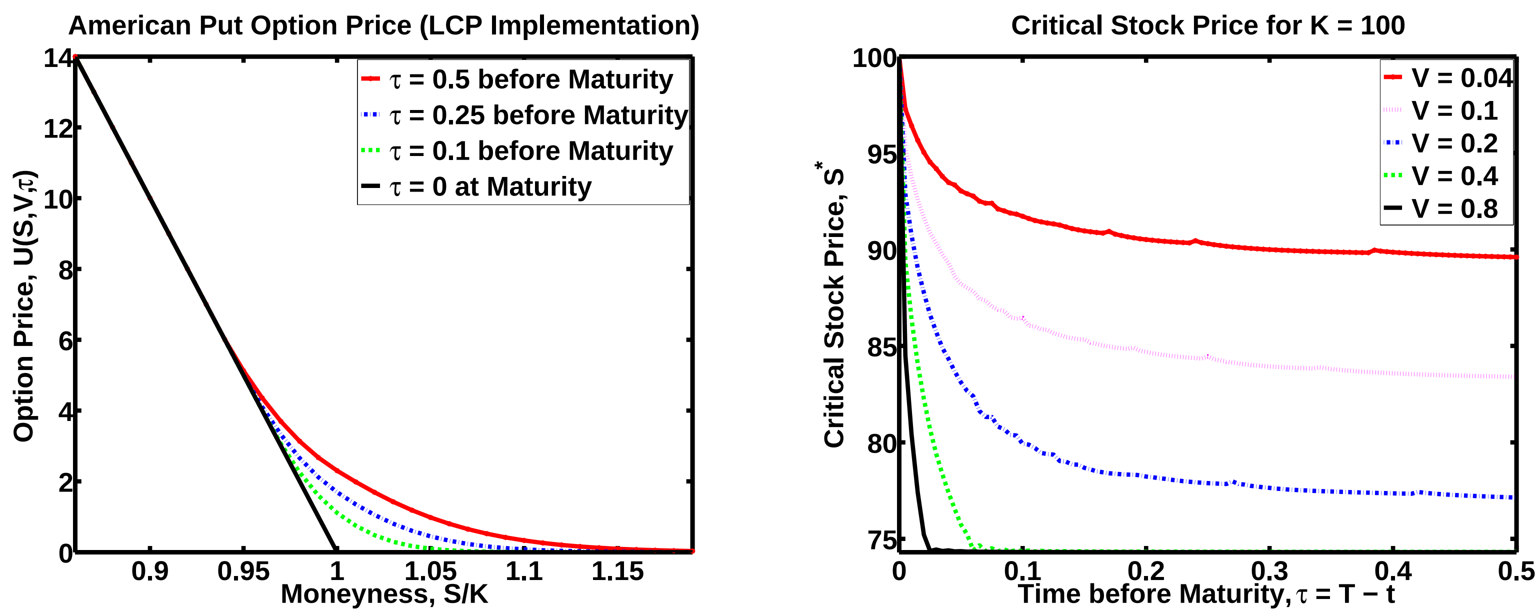
Figure 1: The heuristic quadratic approximation gives SVJD-Uniform American $P^{(A)} = P^{(A)}_{QA}$ compared to European $P^{(E)}$ put option prices for $T = 0.1$ and 0.25 years, with averaged approximation of $V(t)$.



(a) American and European put option prices for $T = 0.5$ years.

(b) Critical stock prices for $T = 0.5$.

Figure 2: The heuristic quadratic approximation gives SVJD-Uniform American $P^{(A)} = P^{(A)}_{QA}$ compared European $P^{(E)}$ put option prices and critical stock prices for $T = 0.5$ years, with averaged approximation of $V(t)$.



(a) American put option prices by LCP. (b) Critical stock prices for K = 100.

Figure 3: PSOR finite difference implementation of LCP gives SVJD-Uniform American put option prices $U(S,V,\tau) = P^{(A)}_{LCP}$ and critical stock prices $S^*(\tau;V)$ (using quadratic extrapolation approximations for smooth contact to the payoff function).

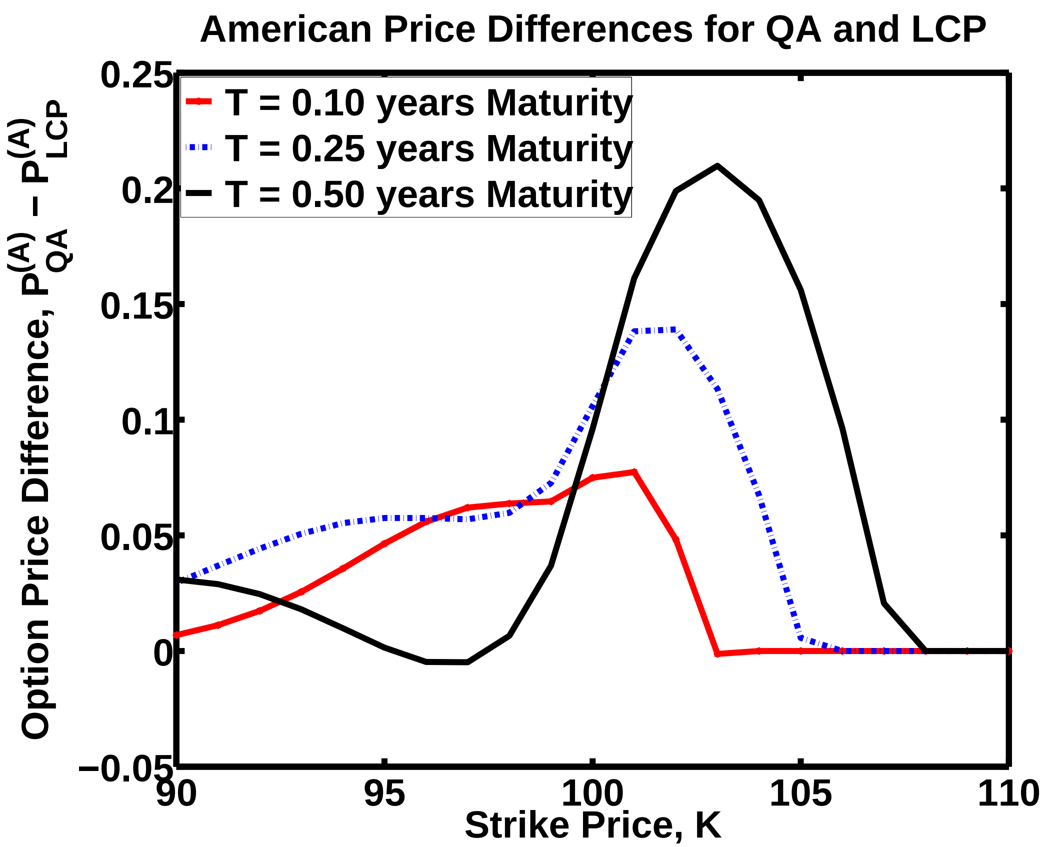


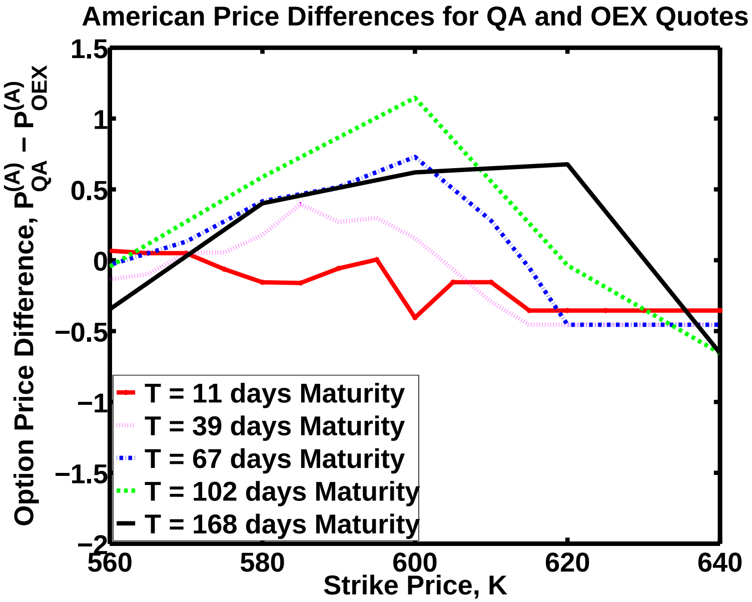
Figure 4: Comparison of American put option prices evaluated by quadratic approximation (QA) and LCP finite difference (FD) methods when $S = \$100$ and $V = 0.01$. Maximum price difference $P^{(A)}_{QA} - P^{(A)}_{LCP}$ is \$0.08, \$0.14, \$0.21 for $T = 0.1$, 0.25 and 0.5 years, respectively, so QA is probably good for practical purposes.

7. Checking with Market Data:

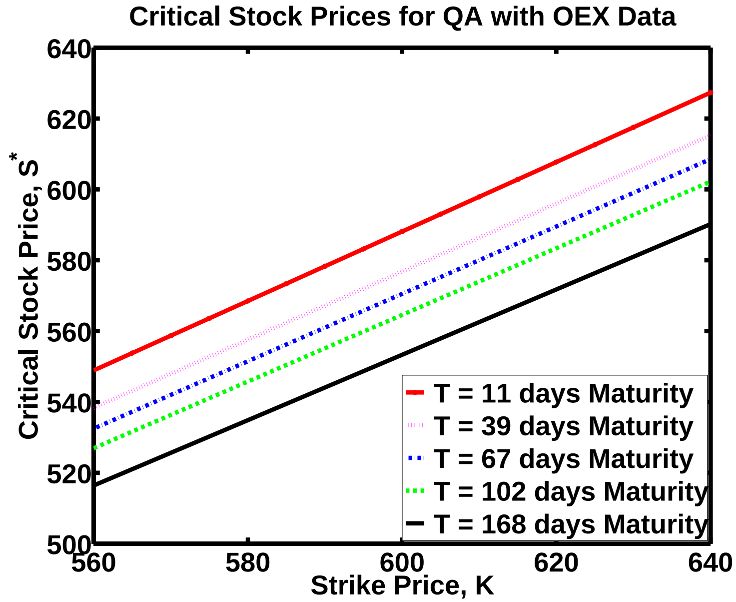
- Choose same time XEO (European options) and OEX (American options) quotes on April 10, 2006 from CBOE. They are based on same underlying S&P 100 Index.
- Use XEO put option quotes to estimate parameter values of the European put option pricing for the quadratic approximation.
- Calculate American put option prices by quadratic approximation formula with estimated parameter values and compare the results with OEX quotes. MSE = 0.137 is obtained, showing good fitting.

Table 1: SVJD-Uniform Parameters Estimated from XEO quotes on April 10, 2006

Parameters	k_v	θ_v	σ_v	ρ	a	b	λ	V	MSE
Values	10.62	0.0136	0.175	-0.547	-0.140	0.011	0.549	0.0083	0.195



(a) American put option price differences between QA and OEX Quotes.



(b) Critical stock prices using QA versus K with OEX quote data.

Figure 5: Comparison of American put option prices evaluated by quadratic approximation (QA) method and OEX quotes with critical stock price, when $S = \$100$ and $V = 0.01$. Maximum absolute price difference $P^{(A)}_{QA} - P^{(A)}_{OEX}$ is \$0.41, \$0.46, \$0.73, \$1.15, \$0.68 for $T = 11$, 39, 67, 102, 168 days, respectively.

8. Conclusions

- An **alternative stochastic-volatility jump-diffusion (SVJD)** model is proposed with square root mean reverting for stochastic-volatility **combined with log-uniform jump amplitudes**.

- The heuristic ***quadratic*** **approximation (QA)** and the **LCP finite difference** scheme for American put option pricing **are compared**, with QA being good for practical purposes.

- The QA results are also **calibrated against real market American option pricing data** OEX (with XEO for Euro. price base), yielding reasonable results considering the simpicity of QA.

Future Research Directions

- Validate the stochastic-volatility jump-diffusion models using high frequency time series underlying security market data to find actual behavior and decide the most accurate underlying dynamics.
- Explore application higher order numerical methods to the SVJD American option pricing problem (cf., Oosterliee (1993) nonlinear multigrid smoothing and review for the SVD American option pricing problem).
- Price other types of options based on stochastic-volatility jump-diffusion models, such as options with dividends, options with trading cost, exotic options, and others.
- Consider the optimal portfolio computations and approximate hedging using the stochastic-volatility jump-diffusion models and the estimated model parameters.