Applied Stochastic Processes and Control for Jump-Diffusions: Modeling, Analysis and Computation

Floyd B. Hanson
University of Illinois
Chicago, Illinois, USA

Chapter 7 Kolmogorov Equations

January 5, 2007

Chapter 7

Kolmogorov Forward and Backward Equation and Their Applications

The theory of probability as mathematical discipline can and should be developed from axioms in exactly the same way as Geometry and Algebra.
—Andrey Nikolaevich Kolmogorov (1903-1987), *Wikipedia*, March 2006.

Here, the Kolmogorov forward (Fokker-Planck) and backward equations are treated, including their inter-relationship and their use in finding transition distributions, densities, moments and optimal state trajectories. There is a close relationship between the PDE representations in the Kolmogorov equations and the SDE representation. Unlike the SDE which is a symbolic representation that requires specification of the stochastic ne integration rule to be well posed, the Kolmogorov equations are deterministic. They can be derived from an SDE using expectations and a chain rule such as Itô's chain rule. Some investigators prefer to solve problems with the Kolmogorov PDEs rather than directly from the underlying SDEs.

7.1 Dynkin's Formula and the Backward Operator

Prior to deriving the Kolmogorov PDEs, a useful formula due to Dynkin is derived. Dynkin's formula relates the expectation of a function of a jump-diffusion process and a functional of the backward jump-diffusion operator. There are many variants of Dynkin's formula [77], but here a derivation of Schuss [244] for pure-diffusions is modified for jump-diffusions in the time-inhomogeneous case and in one-dimension to start.

Theorem 7.1. ***Dynkin's Formula for Jump-Diffusions on*** $[t_0, t]$ ***in One Space Dimension:***

Let $X(t)$ be a jump-diffusion process satisfying the SDE,

$$dX(t) \overset{\text{sym}}{=} f(X(t),t)dt + g(X(t),t)dW(t) + h(X(t),t,Q)dP(t;Q,X(t),t)\,, \tag{7.1}$$

with smooth (continuously differentiable) coefficients $\{f,g,h\}$ with bounded spatial gradients. The diffusion process is the Wiener process $W(t)$ and the jump process is the Poisson process $P(t;Q,X(t),t)$ such that $\mathrm{E}[dP(t;Q,X(t),t)|X(t)=x]=\lambda(t;x,t)dt$ and Q is the jump amplitude mark random variable with density $\phi_Q(q;X(t),t)$. Let $v(x,t)$ be twice continuously differentiable in x and once in t, while bounded at infinity. Then the conditional expectation of the composite process $v(X(t),t)$ satisfies Dynkin's formula in integral form,

$$\begin{aligned} u(x_0,t_0) &= \overline{v}(x_0,t_0;t) \equiv \mathrm{E}[v(X(t),t)|X(t_0)=x_0] \\ &= v(x_0,t_0)+\mathrm{E}\left[\int_{t_0}^{t}\left(\frac{\partial v}{\partial t}(X(s),s)+\mathcal{B}_x[v](X(s),s)\right)ds\middle|X(t_0)=x_0\right], \end{aligned} \tag{7.2}$$

where the dependence on the parameter t is suppressed in $u(x_0,t_0)$. The jump-diffusion ***backward operator*** *$\mathcal{B}_{x_0}$ with respect to the state x_0 for time t dependent coefficients, in backward coordinates, is*

$$\begin{aligned} \mathcal{B}_{x_0}[v](x_0,t_0) \equiv{}& f(x_0,t_0)\frac{\partial v}{\partial x_0}(x_0,t_0) + \frac{1}{2}g^2(x_0,t_0)\frac{\partial^2 v}{\partial x_0^2}(x_0,t_0) \\ &+\widehat{\lambda}(x_0,t_0)\int_{\mathcal{Q}}\Delta_h[v](x_0,t_0,q)\phi_Q(q;x_0,t_0)dq, \end{aligned} \tag{7.3}$$

where $\widehat{\lambda}(x_0,t_0)\equiv\lambda(t;x_0,t_0)$ suppresses the forward time t and the Poisson h-jump is

$$\Delta_h[v](x_0,t_0,q) \equiv v(x_0+h(x_0,t_0,q),t) - v(x_0,t_0)\,. \tag{7.4}$$

Note that the subscript x_0 on the backward operator $\mathcal{B}_{x_0}$ only denotes that the operator operates with respect to the backward state variable x_0 for jump-diffusions and only denotes partial differentiation in the pure-diffusion ($h(x_0,t_0,q)\equiv 0$) case.

In the time-homogeneous case, $f(x,t)=f(x)$, $g(x,t)=g(x)$ and $h(x,t,q)=h(x,q)$, so $v(x,t)=v(x)$ and

$$\begin{aligned} u(x_0) &\equiv \mathrm{E}[v(X(t))|X(t_0)=x_0] \\ &= v(x_0) + \mathrm{E}\left[\int_{t_0}^{t}\mathcal{B}_x[v](X(s))ds\middle| X(t_0)=x_0\right]\,, \end{aligned} \tag{7.5}$$

dropping the t dependence of the backward operator here.

Proof. Dynkin's formula follows from Itô's chain rule for jump-diffusions here. Thus,

$$\begin{aligned} dv(X(t),t) \overset{\text{dt}}{=}{}& \left(\frac{\partial v}{\partial t}+f\frac{\partial v}{\partial x}+\frac{1}{2}g^2\frac{\partial^2 v}{\partial x^2}\right)(X(t),t)dt + \left(g\frac{\partial v}{\partial x}\right)(X(t),t)dW(t) \\ &+\int_{\mathcal{Q}}\Delta_h[v](X(t),t,q)\mathcal{P}(\mathbf{dt},\mathbf{dq};X(t),t)\,, \end{aligned} \tag{7.6}$$

where common arguments have been condensed. Upon integrating in t,

$$v(X(t),t) = v(x_0,t_0) + \int_{t_0}^{t}\left(\left(\frac{\partial v}{\partial t} + f\frac{\partial v}{\partial x} + \frac{1}{2}g^2\frac{\partial^2 v}{\partial x^2}\right)(X(s),s)ds + \left(g\frac{\partial v}{\partial x}\right)(X(s),s)dW(s) + \int_{\mathcal{Q}} \Delta_h[v](X(s),s,q)\mathcal{P}(\mathbf{ds},\mathbf{dq};X(s),s)\right). \tag{7.7}$$

Next taking expectations while using the facts that follow from the independent increment property of Markov processes,

$$\mathrm{E}\left[\int_{t_0}^{t} G(X(s),s)dW(s)\right] = 0$$

after (2.43) and with the zero mean jump process

$$\mathrm{E}\left[\int_{t_0}^{t} H(X(s),s)\widehat{\mathcal{P}}(\mathbf{ds},\mathbf{dq};X(s),s)\right] = 0,$$

generalized from (3.27) with $d\widehat{P}(s)$, where here the mean-zero Poisson random measure is

$$\widehat{\mathcal{P}}(\mathbf{dt},\mathbf{dq};X(t),t) \equiv \mathcal{P}(\mathbf{dt},\mathbf{dq};X(t),t) - \lambda(t;X(t),t)\phi_Q(q;X(t),t)dqdt, \tag{7.8}$$

then using the definition of the backward operator $\mathcal{B}_x[v]$,

$$E[v(X(t),t)|X(t_0)\!=\!x_0] = v(x_0,t_0) + \mathrm{E}\left[\int_{t_0}^{t}\left(\frac{\partial v}{\partial t} + \mathcal{B}_x[v]\right)(X(s),s)ds\,|X(t_0)\!=\!x_0\right]. \tag{7.9}$$

In the time-homogeneous case, without time-dependent coefficients, we need only use the x-dependent test function $v = v(x)$ and the Dynkin formula reduces to (7.5). □

Example 7.2. ***Application of Dynkin's Formula to Final Value Problems:*** *Consider the final value problem for the backward problem with PDE*

$$\begin{aligned}\frac{\partial v}{\partial t_0}(x_0,t_0) + \mathcal{B}_{x_0}[v](x_0,t_0) &= \alpha(x_0,t_0) \quad x_0 \in \Omega, \quad t_0 < t_f\,,\\ v(x_0,t_f) &= \gamma(x_0,t_f) \quad x_0 \in \Omega\,,\end{aligned} \tag{7.10}$$

where the general functions $\alpha(x,t)$ and $\gamma(x,t)$ are given, while $\mathcal{B}_{x_0}[v](x_0,t_0)$ is the jump-diffusion backward operator defined in (7.3). From Dynkin's formula (7.2) with $t = t_f$,

$$\mathrm{E}[\gamma(X(t_f),t_f)|X(t_0)=x_0] = v(x_0,t_0) + \mathrm{E}\left[\int_{t_0}^{t_f}\alpha(X(s),s)\bigg|\,X(t_0)=x_0\right],$$

where the jump-diffusion process is given by the SDE (7.1). By simple rearrangement, the formal solution to the final value problem is given by

$$v(x_0,t_0) = \mathrm{E}\left[\gamma(X(t_f),t_f) - \int_{t_0}^{t_f} \alpha(X(s),s)\,\middle|\, X(t_0)=x_0\right], \tag{7.11}$$

in a more useful form, suitable for stochastic simulations using the given problem functions and the SDE.

The final problem (7.10) can be called the ***Dynkin's equation*** *corresponding to* ***Dynkin's formula*** *(7.2).*

7.2 Backward Kolmogorov Equations

Many exit and stopping time problems rely on the backward Kolmogorov equations, since they represent perturbations of the initial condition when the final condition for exit or stopping is known. Another very useful application is a PDE governing the behavior of the transition density as a function of the initial state. First the general backward equation in the sense of Kolmogorov is derived using an infinitesimal form of Dynkin's equation.

Theorem 7.3. ***General Backward Kolmogorov Equation for Jump-Diffusions on*** $[t_0, t]$ ***in One Space Dimension:***
Let the jump-diffusion process $X(t)$ at time t with $X(t_0) = x_0$ at initial or backward time t_0 satisfy (7.1) along with associated conditions and let the test function $v(X(t))$ also satisfy relevant conditions. Let

$$u(x_0,t_0) = \overline{v}(x_0,t_0;t) \equiv \mathrm{E}[v(X(t))|X(t_0)=x_0] = \mathrm{E}_{(t_0,t]}[v(X(t))|X(t_0)=x_0], \tag{7.12}$$

suppressing the ***forward time*** *t in favor of the* ***backward time*** *t_0. Then $u(x_0,t_0)$ satisfies the following backward PDE with backward arguments,*

$$0 = \frac{\partial u}{\partial t_0}(x_0,t_0) + \mathcal{B}_{x_0}[u](x_0,t_0)\,, \tag{7.13}$$

where the backward operator with respect to x_0 operating on u is

$$\begin{aligned}\mathcal{B}_{x_0}[u](x_0,t_0) = {} & f(x_0,t_0)\frac{\partial u}{\partial x_0}(x_0,t_0) + \frac{1}{2}g^2(x_0,t_0)\frac{\partial^2 u}{\partial x_0^2}(x_0,t_0) \\ & + \widehat{\lambda}(x_0,t_0)\int_{\mathcal{Q}} \Delta_h[u](x_0,t_0,q)\phi_Q(q;x_0,t_0))dq\,,\end{aligned} \tag{7.14}$$

the h-jump of u is

$$\Delta_h[u](x_0,t_0,q) \equiv u(x_0+h(x_0,t_0,q),t_0) - u(x_0,t_0)\,, \tag{7.15}$$

with final condition

$$\lim_{t_0 \uparrow t} u(x_0,t_0) = v(x_0). \tag{7.16}$$

Proof. This formal proof is a modified version of the one for pure diffusions in Schuss [244] modified to include Poisson jump processes. First, the objective is to calculate the backward time partial derivative

$$u(x_0,t_0)-u(x_0,t_0-dt)\overset{\mathrm{dt}}{=}\frac{\partial u}{\partial t_0}dt\equiv\left.\frac{\partial u}{\partial t_0}\right|_{x_0\ \mathrm{fixed}}dt\,,$$

so consider the infinitesimal backward difference in the spirit of Dynkin's formula, noting that the initial time t_0 is perturbed one step backward in time to $t_0 - dt$ with fixed x_0. On the other hand, using the representation (7.12), splitting the expectation at t_0 using the new random variable $X(t_0)$ and expanding by the stochastic chain rule,

$$\begin{aligned}
u(x_0,t_0)-u(x_0,t_0-dt) &= u(x_0,t_0)-E[v(X(t))|X(t_0-dt)=x_0]\\
&= u(x_0,t_0)-\mathrm{E}[\mathrm{E}[v(X(t))|X(t_0)]|X(t_0-dt)=x_0]\\
&= u(x_0,t_0)-\mathrm{E}[u(X(t_0),t_0)|X(t_0-dt)=x_0]\\
&= \mathrm{E}[u(x_0,t_0)-u(X(t_0),t_0)|X(t_0-dt)=x_0]\\
&\overset{\mathrm{dt}}{=} \mathrm{E}[\mathcal{B}_{x_0}[u](x_0,t_0)dt+g(x_0,t_0)dW(t_0)\\
&\quad +\int_{\mathcal{Q}}\Delta_h[u](X(s),s,q)\widehat{\mathcal{P}}(\mathbf{ds},\mathbf{dq};X(s),s)|X(t_0-dt)=x_0]\\
&= \mathrm{E}[\mathcal{B}_{x_0}[u](x_0,t_0)dt|X(t_0-dt)=x_0]\\
&= \mathcal{B}_{x_0}[u](x_0,t_0)dt\\
&= \left[f(x_0,t_0)\frac{\partial u}{\partial x_0}(x_0,t_0)+\frac{1}{2}g^2(x_0,t_0)\frac{\partial^2 u}{\partial x_0^2}(x_0,t_0)\right.\\
&\quad \left.+\,\widehat{\lambda}(x_0,t_0)\int_{\mathcal{Q}}\Delta_h[u](x_0,t_0,q)\phi_Q(q;x_0,t_0)dq\right]dt\,,
\end{aligned}$$

where the stochastic chain rule (5.41) was used, marked by the dt-precision step, along with expectations over the zero-mean jump-diffusion differentials. Just equating the two about results for $u(x_0,t_0)-u(x_0,t_0-dt)$ and eliminating the dt factor yields the backward Kolmogorov equation (7.13) result. The final condition (7.16) simply follows from the definition of $u(x_0,t_0)$ in (7.12) and taking the indicated limit from the backward time t_0 to the forward time t for fixed x_0,

$$\lim_{t_0\uparrow t}u(x_0,t_0)=\lim_{t_0\uparrow t}\mathrm{E}[v(X(t))|X(t_0)=x_0]=\mathrm{E}[v(X(t))|X(t)=x_0]=v(x_0).$$

□

Transition Probability Distribution $\Phi_{X(t)}(x,t;x_0,t_0)$:

One of the most important applications of the backward Kolmogorov equation is for the transition probability whose distribution is given by

$$\Phi_{X(t)}(x,t;x_0,t_0)\equiv\mathrm{Prob}[X(t)\le x|X(t_0)=x_0] \tag{7.17}$$

with density

$$\phi_{X(t)}(x,t;x_0,t_0) = \frac{\partial \Phi_{X(t)}}{\partial x}(x,t;x_0,t_0) \tag{7.18}$$

or alternatively by

$$\begin{aligned}\phi_{X(t)}(x,t;x_0,t_0)dx &\stackrel{\text{dx}}{=} \text{Prob}[x < X(t) \le x+dx | X(t_0)=x_0] \\ &= \text{Prob}[X(t) \le x+dx | X(t_0)=x_0] \\ &\quad -\text{Prob}[X(t) \le x | X(t_0)=x_0]\,,\end{aligned} \tag{7.19}$$

in dx-precision, provided the density exists, including the case of generalized functions (see Section B.12) as assumed in this book. In terms of the transition density, the conditional expectation can be rewritten such that

$$\begin{aligned}u(x_0,t_0) = \overline{v}(x_0,t_0;t) &= \text{E}_{(t_0,t]}[v(X(t))|X(t_0)=x_0] \\ &= \int_{-\infty}^{+\infty} v(x)\phi_{X(t)}(x,t;x_0,t_0)dx\,.\end{aligned} \tag{7.20}$$

Thus, if we let

$$v(x) \stackrel{\text{gen}}{=} \delta(x-\xi),$$

then

$$u(x_0,t_0) = \overline{v}(x_0,t_0;t) = \phi_{X(t)}(\xi,t;x_0,t_0)$$

by definition of the Dirac delta function, and so the transition density satisfies the general backward Kolmogorov equation (7.13) in the backward or initial arguments (x_0,t_0).

Corollary 7.4. ***Backward Kolmogorov Equation for Jump-Diffusion Transition Density:***
Let $\widehat{\phi}(x_0,t_0) \equiv \phi_{X(t)}(x,t;x_0,t_0)$, suppressing the parametric dependence on the forward coordinates (x,t), where the process satisfies the jump-diffusion SDE (7.1) under the specified conditions. Then

$$0 = \frac{\partial \widehat{\phi}}{\partial t_0}(x_0,t_0) + \mathcal{B}_{x_0}[\widehat{\phi}](x_0,t_0) \tag{7.21}$$

$$\begin{aligned}&= \frac{\partial \widehat{\phi}}{\partial t_0}(x_0,t_0) + f(x_0,t_0)\frac{\partial \widehat{\phi}}{\partial x_0}(x_0,t_0) + \frac{1}{2}g^2(x_0,t_0)\frac{\partial^2 \widehat{\phi}}{\partial x_0^2}(x_0,t_0) \\ &\quad + \widehat{\lambda}(x_0,t_0)\int_{\mathcal{Q}} \Delta_h\left[\widehat{\phi}\right](x_0,t_0,q)\phi_Q(q;x_0,t_0)dq\,,\end{aligned} \tag{7.22}$$

subject to the final condition,

$$\lim_{t_0 \uparrow t} \widehat{\phi}(x_0,t_0) = \delta(x_0 - x)\,. \tag{7.23}$$

The final condition (7.23) follows from the alternate, differential definition (7.19) of the transition probability density.

Often the transition density backward equation (7.21) is referred to as the **backward Kolmogorov equation**. It is useful for problems in which the final state is known, such as an exit time problem or a stopping time problem where a state boundary is reached, in the case of finite state domains. For some stochastic researchers, the backward equation is considered more basic than the forward equation, since in the backward equation some final goal may be reached as in stochastic dynamic programming, or some significant event may occur, such as the extinction time for a species. The evolution of the moments or expectations of powers of the state are governed by transition probability density.

7.3 Forward Kolmogorov Equations

In contrast to the backward time problems of the previous section, the forward equation will be needed to find the evolution of the transition density forward in time given an initial state. The basic idea is that the **forward operator** $\mathcal{F}_x$ and the **backward operator** are (formal) **adjoint** operators, i.e., under suitable conditions on the transition density

$$\phi(x,t) = \phi_{X(t)}(x,t;x_0,t_0),$$

with truncated arguments to focus on forward variables, and a well-behaved test function $v(x)$, well-behaved particularly at infinity. Then the operators are related through an inner product equality,

$$(\mathcal{B}_x[v], \phi) = (\mathcal{F}_x[\phi], v)\,, \tag{7.24}$$

which is derived in Theorem 7.5 below. The conditional expectations in Dynkin's formula can be considered an inner product over a continuous state space with the transition density such that

$$(v, \phi) = \mathrm{E}[v(X(t))|X(t_0) = x_0] = \int_{-\infty}^{+\infty} v(x)\phi(x,t)dx\,,$$

emphasizing forward variables (x, t).

Theorem 7.5. ***Forward Kolmogorov Equation or Fokker-Planck Equation for the Transition Density*** $\phi(\mathbf{x},\mathbf{t};\mathbf{x_0},\mathbf{t_0})$***:***
Let $\phi(x,t;x_0,t_0)$ be the transition probability density for the jump-diffusion process $X(t)$ that is symbolically represented by the SDE (7.1) along with the coefficient conditions specified in Dynkin's Formula Theorem 7.1. Let $v(x)$ be a bounded and twice differentiable but otherwise arbitrary test function such that the integrated ***conjunct*** *vanishes, i.e.,*

$$\left[\left((f\phi)(x,t) - \frac{1}{2}\frac{\partial(g^2\phi)}{\partial x}(x,t)\right)v(x) + \frac{1}{2}(g^2\phi)(x,t)v'(x)\right]_{-\infty}^{+\infty} = 0\,, \tag{7.25}$$

where $(f\phi)(x,t) \equiv f(x,t)\phi(x,t)$, $g^2(x,t) \equiv (g(x,t))^2$ and $v'(x) \equiv (dv/dx)(x)$. Then, in the weak sense, ϕ satisfies the forward Kolmogorov equation in forward *space-time variables* (x, t),

$$\frac{\partial\phi}{\partial t}(x,t) = \frac{1}{2}\frac{\partial^2(g^2\phi)}{\partial x^2}(x,t) - \frac{\partial(f\phi)}{\partial x}(x;t) - (\widehat{\lambda}\phi)(x,t) + \int_{\mathcal{Q}}(\widehat{\lambda}\phi)(x-\eta,t)|1-\eta_x|\phi_Q(q;x-\eta,t)dq\,, \quad (7.26)$$

where $\eta = \eta(x;t,q)$ *is related to the inverse jump amplitude such that*

$$x = \xi + h(\xi, t, q)$$

is the new state value corresponding to the old state value ξ*, such that*

$$\eta(x;t,q) = h(\xi,t,q),$$

assuming h *is monotonic in* ξ *so that* h *is invertible with respect to* ξ*, that the Jacobian*

$$(1-\eta_x) = \left(1 - \frac{\partial\eta}{\partial x}(x;t,q)\right)\,,$$

is non-vanishing, and that the inverse transformation from ξ *to* x *maps* $(-\infty,+\infty)$ *onto* $(-\infty,+\infty)$*.*

The transition probability density satisfies the delta function intial condition,

$$\phi(x,t_0^+) = \phi_{X(t_0^+)}(x,t_0^+;x_0,t_0) = \delta(x-x_0)\,. \quad (7.27)$$

Proof. The main idea of this proof is to perform several integrations by parts to move the partial differentiation from the backward operator on the arbitrary test function $v(x)$ to differentiation of the jump-diffusion transition probability $\phi(x,t) = \phi_{X(t)}(x,t;x_0,t_0)$, deriving the adjoint backward-forward operator relation (7.24) in principle. Differentiating Dynkin's formula (7.2) in forward time t for fixed initial conditions (x_0,t_0) and for some well-behaved test function $v(x)$,

$$\begin{aligned}\frac{\partial\overline{v}}{\partial t}(x_0,t_0;t) &= \mathrm{E}\left[\frac{\partial}{\partial t}\int_{t_0}^{t}\mathcal{B}_x[v](X(s))ds\,\middle|\, X(t_0)=x_0\right]\\ &= \mathrm{E}\left[\mathcal{B}_x[v](X(t))|\,X(t_0)=x_0\right]\end{aligned} \quad (7.28)$$

assuming that differentiation and expectation can be interchanged, where the backward operator $\mathcal{B}$ is given in (7.3). However, the conditional expectation of $\mathcal{B}$ on the RHS of (7.28) can be written in terms of the transition probability ϕ (7.20),

$$\mathrm{E}[\mathcal{B}_x[v](X(t))|X(t_0)=x_0] = \int_{-\infty}^{+\infty}\mathcal{B}_x[v](x)\phi(x,t)dx\,. \quad (7.29)$$

Combining (7.28) and (7.29) , substituting for $\mathcal{B}$ using (7.3), and using two integration by parts on the spatial derivatives to move the spatial derivatives from v to ϕ, then

$$
\begin{aligned}
\frac{\partial \overline{v}}{\partial t}(x_0,t_0;t) &= \int_{-\infty}^{+\infty} v(x)\frac{\partial \phi}{\partial t}(x,t)dx = \int_{-\infty}^{+\infty} \mathcal{B}_x[v](x)\phi(x,t)dx \\
&= \int_{-\infty}^{+\infty} \Bigg(f(x,t)v'(x) + \frac{1}{2}g^2(x,t)v''(x) \\
&\quad + \widehat{\lambda}(x,t)\int_{\mathcal{Q}} \Delta_h[v](x,t,q)\phi_Q(q;x,t)dq \Bigg)\phi(x,t)dx \\
&= \int_{-\infty}^{+\infty} \Bigg(-v(x)\frac{\partial (f\phi)}{\partial x}(x,t) - \frac{1}{2}\frac{\partial (g^2\phi)}{\partial x}(x,t)v'(x) \\
&\quad + (\widehat{\lambda}\phi)(x,t)\int_{\mathcal{Q}} \Delta_h[v](x,t,q)\phi_Q(q;x,t)dq \Bigg) dx \\
&\quad + \left[(f\phi)(x,t)v(x) + \frac{1}{2}(g^2\phi)(x,t)v'(x) \right]_{-\infty}^{+\infty} \\
&= \int_{-\infty}^{+\infty} \Bigg(v(x)\left(\frac{1}{2}\frac{\partial^2 (g^2\phi)}{\partial x^2}(x,t) - \frac{\partial (f\phi)}{\partial x}(x,t) \right) \\
&\quad + (\widehat{\lambda}\phi)(x,t)\int_{\mathcal{Q}} \Delta_h[v](x,t,q)\phi_Q(q;x,t)dq \Bigg) dx \\
&\quad + \left[\left(f\phi - \frac{1}{2}\frac{\partial (g^2\phi)}{\partial x} \right)(x,t)v(x) + \frac{1}{2}(g^2\phi)(x,t)v'(x) \right]_{-\infty}^{+\infty} .
\end{aligned}
$$

The last term is the integrated **conjunct** from two integrations by parts. By the hypothesis in (7.25), this conjunct is required to be zero, so that the forward and backward operators will be genuine adjoint operators. Otherwise, the forward and backward operators would be called *formal adjoints*.

So far only the adjoint diffusion part of the forward operator has been formed with respect to the test function v as an integration weight. There still remains more work to form the corresponding adjoint jump part and this is done inverting the jump amplitude function $h(x,t,q)$ with respect to x, assuming that $h(x,t,q)$ is monotonic x. Let the post-jump state value be $y = x + h(x,t,q)$ for each fixed (t,q) with inverse written as $x = y - \eta(y;t,q)$ relating the pre-jump state to the post-jump state. Technically, with fixed (t,q), if $y = (I+h)(x)$ where here I denotes the identity function so $I(x) = x$, then the inverse argument is $x = (I+h)^{-1}(y) = (I-\eta)(y)$ for convenience and $\eta \stackrel{\text{op}}{=} I - (I+h)^{-1}$. Thus, $dx = (1-\eta_y(y;t,q))dy$, where $(1-\eta_y(y;t,q))$ is the Jacobian of the inverse transformation. Further, it is assumed that the state domain $(-\infty,+\infty)$ is transformed back onto itself, modulo the sign of the Jacobian. Consequently, we have

$$\int_{-\infty}^{+\infty} v(x)\frac{\partial\phi}{\partial t}(x,t)dx = \int_{-\infty}^{+\infty} v(x)\left(\frac{1}{2}\frac{\partial^2(g^2\phi)}{\partial x^2}(x,t) - \frac{\partial(f\phi)}{\partial x}(x,t) - (\widehat{\lambda}\phi)(x,t)\right.$$
$$+\int_{\mathcal{Q}}(\widehat{\lambda}\phi)(x-\eta(x;t,q),t)|1-\eta_x(x;t,q,t)|$$
$$\left.\cdot\phi_Q(q;x-\eta(x;t,q),t)dq\right)dx\,,$$

upon replacing y as a dummy variable in the state integral back to x so a common factor of the test function $v(x)$ can be collected. Finally, since the test function is assumed to be arbitrary, then the coefficients of $v(x)$ must be equivalent on the left and right sides of the equation *in the weak sense*. The argument is that of the *Fundamental Lemma of the Calculus of Variations* [40, 15, 163]. This leads to the forward Kolmogorov equation for the transition density $\phi(x,t) = \phi_{X(t)}(x,t;x_0,t_0)$ given in the concluding equation (7.26) of Theorem 7.5,

$$\begin{aligned}\frac{\partial\phi}{\partial t}(x,t) &= \mathcal{F}_x[\phi](x,t)\\ &\equiv \frac{1}{2}\frac{\partial^2(g^2\phi)}{\partial x^2}(x,t) - \frac{\partial(f\phi)}{\partial x}(x;t) - (\widehat{\lambda}\phi)(x,t)\\ &+\int_{\mathcal{Q}}(\widehat{\lambda}\phi)(x-\eta(x;t,q),t)|1-\eta_x(x;t,q)|\phi_Q(q;x-\eta(x;t,q),t)dq.\end{aligned} \tag{7.30}$$

Note that the subscript x on the forward operator $\mathcal{F}_x$ only denotes that the operator operates with respect to the forward variable x for jump-diffusions and only denotes partial differentiation in the pure-diffusion ($h(x,t,q)\equiv 0$) case.

The initial condition (7.27), $\phi_{X(t_0^+)}(x,t_0^+;x_0,t_0) = \delta(x-x_0)$, is very obvious for the continuous pure diffusion process, but the jump-diffusion processes undergo jumps triggered by the Poisson process $P(t;Q,X(t),t)$ and so $X(t)$ can be discontinuous. However, a jump is very unlikely in a small time interval since by (1.42) modified by replacing $\lambda(t)$ by the composite time dependence $\lambda(t;X(t),t)$,

$$\mathrm{Prob}[dP(t;Q,X(t),t)=0] = p_0(\lambda(t;X(t),t)dt) = e^{-\lambda(t;X(t),t)dt} = 1+\mathrm{O}(dt)\sim 1\,,$$

as $dt\to 0^+$, so the initial state is certain with probability one by conditioning, i.e.,

$$\phi(x,t) = \phi_{X(t)}(x,t;x_0,t_0)\to\delta(x-x_0)\quad\text{as}\quad t\to t_0^+\,.$$

□

Remarks 7.6.

- *Another applied approach to derive the forward equation for pure diffusions is to use the diffusion approximation as given by Feller [84], but this requires strong assumptions about truncating a Taylor expansion just for diffusion processes alone. This approach does not apply to jump-diffusions, since the jump difference term $D_h[\phi]$ would require an infinite expansion.*

- *For the jump amplitude, a good illustration could be the affine model that is the sum of a state-independent term plus a term purely linear in the state, i.e.,* $h(x,t,q) = \nu_0(t,q) + \nu_1(t,q)x$ *for suitable time-mark coefficients, so the inverse of* $y = x + h(x,t,q)$ *is* $x = (y - \nu_0(t,q))/(1+\nu_1(t,q)) = y - \eta(y;t,q)$ *and* $\eta(y;t,q) = (\nu_0(t,q) + \nu_1(t,q)y)/(1+\nu_1(t,q))$. *For comparison, different cases of this model are tabulated in Table 7.1.*

Table 7.1. *Some Simple jump amplitude models and inverses.*

State Dependence	Direct $h(x,t,q)$	Forward Arg. x=$y - \eta(y;t,q)$	Inverse $\eta(y;t,q)$
constant	$\nu_0(t,q)$	$y - \nu_0(t,q)$	$\nu_0(t,q)$
pure linear	$\nu_1(t,q)x$	$\dfrac{y}{1+\nu_1(t,q)}$	$\dfrac{\nu_1(t,q)y}{1+\nu_1(t,q)}$
affine	$\nu_0(t,q)+\nu_1(t,q)x$	$\dfrac{y-\nu_0(t,q)}{1+\nu_1(t,q)}$	$\dfrac{\nu_0(t,q)+\nu_1(t,q)y}{1+\nu_1(t,q)}$

A mistake is sometimes made by incorrectly generalizing the inverse of the linear jump case $x + \nu_1(t,q)x = y$, *so that* $(1-\nu_1(t,q))y$ *is incorrectly used for the forward argument (x) in the linear case instead of the correct argument, which is* $x = y/(1+\nu_1(t,q))$.

- *The difference in the jump argument between the backward and forward equation is that in the backward case the post-jump or forward value* $y = x + h(x,t,q)$ *is used, while in the forward case the pre-jump or backward value* $x = y - h(x,t,q) = y - \eta(y;t,q)$ *is used.*

7.4 Multi-dimensional Backward and Forward Equations

For many applications, there can be multiple state variables and multiple sources of random disturbances. In biological problems there can be several interacting species each suffering from species specific and common random changes, that can be detrimental or beneficial in effect and range in magnitude from small to large fluctuations. Such effects may be due to the weather, diseases, natural disasters or inter-species predation. In finance, there are the usual background fluctuations in market values, and then there is the occasional market crash or buying frenzy. In manufacturing systems, there may be a large number of machines which randomly fail with the time to repair being randomly distributed due to the many causes of failure.

Consider again the multi-dimensional SDE from Chapter 5 for the n_x-dimensional state process $\mathbf{X}(t) = [X_i(t)]_{n_x \times 1}$,

$$d\mathbf{X}(t) \overset{\text{sym}}{=} \mathbf{f}(\mathbf{X}(t),t)dt + g(\mathbf{X}(t),t)d\mathbf{W}(t) + h(\mathbf{X}(t),t,\mathbf{Q})d\mathbf{P}(t;Q,\mathbf{X}(t),t)\,, \quad (7.31)$$

where

$$\mathbf{W}(t) = [W_i(t)]_{n_w \times 1}$$

is an n_w-dimensional vector diffusion process and

$$\mathbf{P}(t; Q, \mathbf{X}(t), t) = [P_i(t; Q_i, \mathbf{X}(t), t)]_{n_p \times 1}$$

is an n_p-dimensional vector state-dependent Poisson jump process. The state-dependent coefficient functions are dimensionally specified by

$$\begin{aligned} \mathbf{f} &= [f_i(\mathbf{X}(t), t)]_{n_x \times 1} \,, \\ g(\mathbf{X}(t), t) &= [g_{i,j}(\mathbf{X}(t), t)]_{n_x \times n_w} \,, \\ h(\mathbf{X}(t), t, \mathbf{Q}) &= [h_{i,j}(\mathbf{X}(t), t, Q_j)]_{n_x \times n_p} \end{aligned}$$

and have dimensions that are commensurate in multiplication. The mark vector, $\mathbf{Q} = [Q_i)]_{n_p \times 1}$, in the last coefficient function is assumed to have components corresponding to all Poisson vector process components. The coefficient $h(\mathbf{X}(t), t, \mathbf{Q})$ of $d\mathbf{P}(t; Q, \mathbf{X}(t), t)$ is merely the mark $\mathbf{Q}$ dependent symbolic form of the jump amplitude operator-coefficient $h(\mathbf{X}(t), t, \mathbf{q})$, using similar notation, in the corresponding Poisson random mark integral (5.83), i.e.,

$$h(\mathbf{X}(t), t, \mathbf{Q}) d\mathbf{P}(t; Q, \mathbf{X}(t), t) \overset{\text{sym}}{=} \int_{\mathcal{Q}} h(\mathbf{X}(t), t, \mathbf{q}) \boldsymbol{\mathcal{P}}(\mathbf{dt}, \mathbf{dq}; \mathbf{X}(t), t).$$

Dynkin's formula remains unchanged, except for converting the state variable $X(t)$ to a vector $\mathbf{X}(t)$ and making the corresponding change in the backward operator $\mathcal{B}_{\mathbf{x}}[v]$ using the multi-dimensional stochastic chain rule (5.98),

$$\begin{aligned} \overline{v}(\mathbf{x}_0, t_0; t) &\equiv \mathrm{E}[v(\mathbf{X}(t)) | \mathbf{X}(t_0) = \mathbf{x}_0] \\ &= v(\mathbf{x}_0) + \mathrm{E}\left[\int_{t_0}^{t} \mathcal{B}_{\mathbf{x}}[v](\mathbf{X}(s); \mathbf{X}(s), s) ds \,\middle|\, \mathbf{X}(t_0) = \mathbf{x}_0 \right], \end{aligned} \tag{7.32}$$

where the backward operator is given below. The multi-dimensional backward and forward Kolmogorov equations are summarized in the following theorem, with the justification left as an exercise for the reader.

Theorem 7.7. ***Kolmogorov Equations for Jump-Diffusions in Multi-dimensions on*** $[t_0, t]$***:***
Let

$$u(\mathbf{x}_0, t_0) = \overline{v}(\mathbf{x}_0, t_0; t) = \mathrm{E}[v(\mathbf{X}(t)) | \mathbf{X}(t_0) = \mathbf{x}_0].$$

Then $u(\mathbf{x}_0, t_0)$ *satisfies the following multi-dimensional backward Kolmogorov PDE with backward arguments,*

$$0 = \frac{\partial u}{\partial t_0}(\mathbf{x}_0, t_0) + \mathcal{B}_{\mathbf{x}_0}[u](\mathbf{x}_0, t_0; \mathbf{x}_0, t_0)\,, \tag{7.33}$$

where the backward Kolmogorov operator is defined as

$$\begin{aligned}\mathcal{B}_{\mathbf{x}_0}[u](\mathbf{x}_0,t_0;\mathbf{x}_0,t_0) \equiv{}& \mathbf{f}^\top(\mathbf{x}_0,t_0)\nabla_{\mathbf{x}_0}[u](\mathbf{x}_0,t_0)\\ &+\frac{1}{2}\left(gR'g^\top\right):\nabla_{\mathbf{x}_0}\left[\nabla_{\mathbf{x}_0}^\top[u]\right](\mathbf{x}_0,t_0)\\ &+\sum_{j=1}^{n_p}\widehat{\lambda}_j(\mathbf{x}_0,t_0)\int_{\mathcal{Q}}\Delta_j[u](\mathbf{x}_0,t_0,q_j)\phi_{Q_j}(q_j;\mathbf{x}_0,t_0)dq_j,\end{aligned} \tag{7.34}$$

where R' is a correlation matrix defined in (5.95), $A\!:\!B$ is the double dot product (5.99),

$$\Delta_j[u](\mathbf{x}_0,t_0,q_j) \equiv u(\mathbf{x}_0+\widehat{\boldsymbol{h}}_j(\mathbf{x}_0,t_0,q_j),t_0)-u(\mathbf{x}_0,t_0)$$

is the jump of u corresponding to the jump amplitude

$$\widehat{\boldsymbol{h}}_j(\mathbf{x},t,q_j)\equiv[h_{i,j}(\mathbf{x},t,q_j)]_{n_x\times 1}$$

of the jth Poisson process P_j at the jth mark for $j=1:n_p$ and with final condition

$$u(\mathbf{x}_0,t^-)=\overline{v}(\mathbf{x}_0,t^-;t)=v(\mathbf{x}_0)\,.$$

Similarly, the forward Kolmogorov PDE in the multi-dimensional transition density $\phi(\mathbf{x},t;\mathbf{x}_0,t_0)$ as the adjoint of the backward equation is

$$\frac{\partial\phi}{\partial t}(\mathbf{x},t)=\mathcal{F}_{\mathbf{x}}[\phi](\mathbf{x},t)\,, \tag{7.35}$$

where the forward Kolmogorov operator is defined as

$$\begin{aligned}\mathcal{F}_{\mathbf{x}}[\phi](\mathbf{x},t)\equiv{}&\frac{1}{2}\nabla_{\mathbf{x}}\left[\nabla_{\mathbf{x}}^\top:\left[gR'g^\top\phi\right]\right](\mathbf{x},t)\\ &-\nabla_{\mathbf{x}}^\top[\mathbf{f}\phi](\mathbf{x};t)-\sum_{j=1}^{n_p}(\widehat{\lambda}_j\phi)(\mathbf{x},t)\\ &+\sum_{j=1}^{n_p}\int_{\mathcal{Q}}(\widehat{\lambda}_j\phi)(\mathbf{x}-\boldsymbol{\eta}_j(\mathbf{x};t,q_j),t)\left|1-\frac{\partial(\boldsymbol{\eta}_j(\mathbf{x};t,q_j))}{\partial(\mathbf{x})}\right|\\ &\quad\cdot\phi_{Q_j}(q_j;\mathbf{x}-\boldsymbol{\eta}_j(\mathbf{x};t,q_j),t)dq_j\,,\end{aligned} \tag{7.36}$$

where the backward to forward transformation and its Jacobian are

$$\mathbf{x}-\mathbf{x}_0=\boldsymbol{\eta}_{j'}(\mathbf{x},t,q_{j'})=\widehat{\boldsymbol{h}}_{j'}(\mathbf{x}_0,t,q_{j'})\,;$$

$$\frac{\partial(\boldsymbol{\eta}_{j'}(\mathbf{x};t,q_{j'}))}{\partial(\mathbf{x})}=\mathrm{Det}\left[\left[\frac{\partial\eta_{j',i}(\mathbf{x};t,q_{j'})}{\partial x_j}\right]_{n_x\times n_x}\right]=\mathrm{Det}\left[\left(\nabla_{\mathbf{x}}\left[\boldsymbol{\eta}_{j'}^\top\right]\right)^\top\right]$$

for $j'=1:n_p$.

7.5 Chapman-Kolmogorov Equation for Markov Processes in Continuous Time

Alternate methods for deriving the Kolmogorov equations are based upon a fundamental functional equation of Chapman and Kolmogorov (see Bharucha-Reid [31] or other references at the end of this chapter). Let $\mathbf{X}(t)$ be a $n_x \times 1$ Markov process in continuous time, i.e., a jump-diffusion, on the state space Ω. The transition probability distribution function is given by

$$\Phi(\mathbf{x}, t; \mathbf{x}_0, t_0) = \text{Prob}[\mathbf{X}(t) < \mathbf{x} \mid \mathbf{X}(t_0) = \mathbf{x}_0]\,, \tag{7.37}$$

provided $t > t_0$, $\mathbf{X}(t) < \mathbf{x}$ means $X_i(t) < x_i$ for $i = 1 : n_x$, and assuming the probability density exists even if in the generalized sense,

$$\phi(\mathbf{x}, t; \mathbf{x}_0, t_0) = \left(\prod_{i=1}^{n_x} \frac{\partial \phi}{\partial x_i}\right)(\mathbf{x}, t; \mathbf{x}_0, t_0)\,. \tag{7.38}$$

Expressed as a Markov property for distributions, the *Chapman-Kolmogorov equation* for the transition between the start $(\mathbf{x}_0, t_0)$ and the current position $(\mathbf{x}, t)$ through all possible intermediate positions $(\mathbf{y}, s)$ is

$$\begin{aligned}\Phi(\mathbf{x}, t; \mathbf{x}_0, t_0) &= \int_\Omega \Phi(\mathbf{y}, s; \mathbf{x}_0, t_0)\Phi(\mathbf{x}, t; d\mathbf{y}, s) \\ &= \int_\Omega \Phi(\mathbf{y}, s; \mathbf{x}_0, t_0)\phi(\mathbf{x}, t; \mathbf{y}, s)d\mathbf{y}\,,\end{aligned} \tag{7.39}$$

where $t_0 < s < t$. Alternately, the *Chapman-Kolmogorov equation* solely in terms of transition probability densities is

$$\phi(\mathbf{x}, t; \mathbf{x}_0, t_0) = \int_\Omega \phi(\mathbf{y}, s; \mathbf{x}_0, t_0)\phi(\mathbf{x}, t; \mathbf{y}, s)d\mathbf{y}\,, \tag{7.40}$$

upon differentiating (7.39) according to (7.38), again with $t_0 < s < t$. See Bharucha-Reid [31] or other references at the end of this chapter for applications.

7.6 Jump-Diffusion Boundary Conditions

Many boundary value problems for stochastic diffusion processes are similar to their deterministic counterparts, but the stochastic justifications are different. When jump processes are included, then the situation is even more complicated. Since jump processes are discontinuous, jumps may over shoot the boundary making it more difficult to construct an auxiliary process that will implement the boundary with proper probability law.

7.6.1 Absorbing Boundary Condition

If the boundary is absorbing, i.e., the process that hits the boundary stays there [84, 98, 244, 162], it is quite easy to specify since the process can not reenter the interior and the transition probability for the process initially at $\mathbf{X}(0) = \mathbf{x}_0$ on the boundary $\Gamma = \partial\Omega$ can not reach $\mathbf{X}(t) = \mathbf{y}$ in the interior of the domain Ω. Thus, for pure-diffusions

$$\phi_{\mathbf{X}(t)}(\mathbf{x}, t; \mathbf{x}_0, t_0) = \text{Prob}[\mathbf{X}(t) = \mathbf{x} \in \Omega | \mathbf{X}(t_0) = \mathbf{x}_0 \in \Gamma, t > 0] = 0, \tag{7.41}$$

whereas for jump-diffusions

$$\phi_{\mathbf{X}(t)}(\mathbf{x}, t; \mathbf{x}_0, t_0) = \text{Prob}[\mathbf{X}(t) = \mathbf{x} \in \Omega | \mathbf{X}(0) = \mathbf{x}_0 \notin \text{Interior}[\Omega], t > 0] = 0, \tag{7.42}$$

since it is assumed that a jump over-shoot into the boundary or exterior of the region is absorbed. Kushner and Dupuis [179] have a more elaborate treatment of the absorbing boundary by stopping the process once it hits the boundary, assumed to be smooth and reachable in finite time (also called attainable or accessible). These are boundary conditions for the transition probability density backward equations, since they are specified on the backward variable x_0.

7.6.2 Reflecting Boundary Conditions

The reflecting boundary is much more complicated and the smoothness of the boundary, i.e., the boundary is continuously differentiable, is important for defining the reflection. Since a simple reflection at a boundary point, $\mathbf{x}_b$, will be in the plane of the nearby incoming trajectory at $\mathbf{x}_0$ and the normal vector $\mathbf{N}_b$ to the tangent plane of the boundary at $\mathbf{x}_b$. Let $\delta\mathbf{x} = \mathbf{x}_0 - \mathbf{x}_b$ be the distance vector to the point of contact and let $\mathbf{T}_b$ a tangent vector in the intersection of the tangent plane and the trajectory-normal plane. Using *stochastic reflection principle*, similar to the reflection principle used in PDEs, a stochastic reflection process is constructed such that $\delta\mathbf{x}_r = \mathbf{x}_r - \mathbf{x}_b$ is its current increment at the same time as $\delta\mathbf{x}$. The only difference is the opposite sign of its normal component, i.e., $\delta\mathbf{x}_r = -\delta_n\mathbf{N}_b + \delta_t\mathbf{T}_b$ if $\delta\mathbf{x}_0 = +\delta_n\mathbf{N}_b + \delta_t\mathbf{T}_b$, for sufficiently small and positive components d_n and δ_t. Since the reflected process at $\mathbf{x}_r$ by its construction must have the same probability as the original process at $\mathbf{x}_0$, then

$$\mathbf{N}_b^\top \nabla_{x_0}\left[\phi_{\mathbf{X}(t)}\right](\mathbf{x}, t\mathbf{x}_b, t_0) = \mathbf{N}_b^\top \nabla_{x_0}\left[\widehat{\phi}\right](\mathbf{x}_b, t_0) = 0\,, \tag{7.43}$$

upon expanding the difference between the two probability densities

$$\widehat{\phi}(\mathbf{x}_0, t_0') - \widehat{\phi}(\mathbf{x}_r, t_0') = \widehat{\phi}(\mathbf{x}_b + \delta_n\mathbf{N}_b + \delta_t\mathbf{T}_b, t_0') - \widehat{\phi}(\mathbf{x}_b - \delta_n\mathbf{N}_b + \delta_t\mathbf{T}_b, t_0') = 0,$$

in simplified backward notation at pre-hit time t_0' here, to order δ_n. The order δ_t cancels out.

See Kushner and Dupuis [179] about more reflecting boundary conditions and systematically constructing reflecting jump-diffusion processes. Also, see Karlin and Taylor [162] for a thorough discussion of other boundary conditions such as sticky and elastic, as well as an extensive boundary classification for pure diffusion problems.

7.7 Stopping Times: Expected Exit and First Passage Times

In many problems, an *exit time*, also called a *stopping time* or a *first passage time*, is of interest. For instance when a population falls to the zero level and thus ceases to exist, it is said to be extinct and the time of extinction is of interest. If it is a stochastic population, then the expected extinction time is of interest (Hanson and Tuckwell [119, 121]). For a neuron, stochastic fluctuations can be important and then the time to reach a threshold to fire a nerve pulse is of interest and in particular the expected firing time can be calculated (Stein [257], Tuckwell [269], Hanson and Tuckwell [120]). In cancer growth studies, the expected doubling time for the size of a tumor is often calculated (Hanson and Tier [117]). There are many other example of stopping times. First deterministic exit time problems are introduced as examples and as a basic reference.

Examples 7.8. ***Deterministic Exit Time Problems***

- ***Forward Exit Time Formulation:***
Let $X(t)$ be the state of the system at time t and be governed by the ODE

$$\frac{dX}{dt}(t) = f(X(t)), \quad X(0) = x_0 \in (a, b), \tag{7.44}$$

where $f(x)$ is strictly positive or strictly negative, $f(x)$ is continuous and $1/f(x)$ is integrable on $[a, b]$. Thus inverting 7.44, the forward running time is

$$dt = dT_F(x) = dx/f(x), \quad T_F(x_0) = 0,$$

so

$$T_F(x) = \int_{x_0}^{x} dy/f(y),$$

and the forward exit time is

$$T_F(b) \quad \text{if} \quad f(x) > 0 \quad \text{or} \quad T_F(a) \quad \text{if} \quad f(x) < 0.$$

- ***More Relevant Backward Exit Time Formulation:***
Since the stochastic exit time problem is more conveniently formulated as a backward time problem, let $x = c$ be the point of exit, so when $x_0 = c$ then we know the state $X(t)$ is already at the exit and the final condition is $T_B(c) \equiv 0$. Consequently, the backward exit time $T_B(x)$ problem is formulated with $T_B(x) = T_F(c) - T_F(x)$ or $T'_B(x) = -T'_F(x)$ as

$$dT_B(x) = -dx/f(x), \quad T_B(c) = 0$$

or in the more conventional backward form,

$$f(x)T'_B(x) = -1, \quad T_B(c) = 0, \tag{7.45}$$

so

$$T_B(x) = -\int_c^x dy/f(y)$$

or the backward exit time ending at $x = c$ *is*

$$T_B(x_0) = \int_{x_0}^c dy/f(y)$$

where $c = b$ *if* $f(x) > 0$ *or* $c = a$ *if* $f(x) < 0$.

7.7.1 Expected Stochastic Exit Time

First, the exit time is analytically defined, relevant for the piece-wise continuous jump-diffusion. For continuous, pure diffusion processes, it is sufficient to consider when the process hits a boundary. However, when the stochastic process also includes jumps, then it is possible that the process overshoots the boundary and ends up in the exterior of the domain. Here the domain will simply be an open interval in one state dimension.

Again let $X(t)$ be a jump-diffusion process satisfying the SDE,

$$dX(t) \overset{\text{sym}}{=} f(X(t),t)dt + g(X(t),t)dW(t) + h(X(t),t,Q)dP(t;Q,X(t),t), \quad (7.46)$$

with smooth (continuously differentiable) coefficients $\{f, g, h\}$ with bounded spatial gradients.

Definition 7.9. *In one state dimension, the exit time for the Markov process* $X(t)$ *in continuous time (7.46) from the open interval* (a, b) *is*

$$\tau_e(x_0, t_0) \equiv \inf_t \left[\, t \;|X(t) \notin (a, b); X(t_0) = x_0 \in (a, b)\right], \quad (7.47)$$

if it exists.

Before considering a more general formulation using probability theory, some applications of Dynkin's formula will be used to compute the expected extinction time and some higher moments.

Examples 7.10. ***Expected Exit Time Applications of Dynkin's Formula:***

- ***Small modification of Dynkin's formula for exit times:***
  Consider the following boundary value problem of inhomogeneous backward Kolmogorov equation,

$$\frac{\partial v}{\partial t_0}(x_0, t_0) + \mathcal{B}_{x_0}[v](x_0, t_0) = \alpha(x_0, t_0), \quad x_0 \in (a, b), \quad (7.48)$$

$$v(x_0, t_0) = \beta(x_0, t_0), \quad x_0 \notin (a, b), \quad (7.49)$$

where $\mathcal{B}_{x_0}[v](x_0,t_0)$ *(7.14) is the jump-diffusion backward operator,* $\alpha(x_0,t_0)$ *is a given general state-independent homogeneous term and* $\beta(x_0,t_0)$ *is a given general exit boundary value. Both* $\alpha(x_0,t_0)$ *and* $\beta(x_0,t_0)$ *depend on the application. Sometimes (7.48) is called Dynkin's equation due to its relationship with Dynkin's formula.*

Prior to taking expectations, the integral form (7.9) of the stochastic chain rule was

$$\begin{aligned} v(X(t),t) = v(x_0,t_0) + \int_{t_0}^{t} \Bigg(&\left(\frac{\partial v}{\partial t} + f\frac{\partial v}{\partial x} + \frac{1}{2}g^2\frac{\partial^2 v}{\partial x^2}\right)(X(s),s)ds \\ &+ \left(g\frac{\partial v}{\partial x}\right)(X(s),s)dW(s) \qquad (7.50) \\ &+ \int_{\mathcal{Q}} \Delta_h[v](X(s),s,q)\mathcal{P}(\mathbf{ds},\mathbf{dq};X(s),s)\Bigg), \end{aligned}$$

but now make the random exit time substitution $t=\tau_e(x_0,t_0)$ *for the deterministic time variable which is simply abbreviated as* $t=\tau_e$ *and then take expectations getting an exit time version of Dynkin's formula,*

$$\begin{aligned} E\left[v(X(\tau_e),\tau_e)|X(t_0)=x_0\right] = v(x_0,t_0) \qquad (7.51) \\ +\mathrm{E}\left[\int_{t_0}^{\tau_e}\left(\frac{\partial v}{\partial t} + \mathcal{B}_x[v]\right)(X(s),s)ds\right]. \end{aligned}$$

Upon substituting Dynkin's equation (7.48) into Dynkin' Formula, it reduces to

$$E\left[\beta(X(\tau_e),\tau_e)|X(t_0)=x_0\right] = v(x_0,t_0) + \mathrm{E}\left[\int_{t_0}^{\tau_e}\alpha(X(s),s)ds\right]. \qquad (7.52)$$

- ***Ultimate Exit Time Distribution:***
  Let $\alpha(x_0,t_0)=0$*, while* $\beta(X(\tau_e),\tau_e)=1$ *since if* x_0 *starts at an exit, i.e.,* $x_0 \notin (a,b)$*, then exit is certain and the distribution function is 1. Hence, due to the jump-diffusion* $v(x_0,t_0)=1=\Phi_{\tau_e(x_0,t_0)}(+\infty)$ *on* (a,b) *under reasonable conditions for the existence of an exit.*

- ***Expected Exit Time:***
  Assuming that exit is certain, $\Phi_{\tau_e(x_0,t_0)}(+\infty)=1$*, let* $\alpha(x_0,t_0)=-1=-\Phi_{\tau_e(x_0,t_0)}(+\infty)$ *and* $\beta(X(\tau_e),\tau_e)=0$*, corresponding to* $x_0 \notin (a,b)$ *implying zero exit time, then*

  $$\mathrm{E}[\tau_e(x_0,t_0)] = t_0 + v^{(1)}(x_0,t_0)\,, \qquad (7.53)$$

  where $v^{(1)}(x_0,t_0)$ *is the solution to the problem (7.48-7.49) with* $\alpha(x_0,t_0)=0$ *and* $\beta(X(\tau_e),\tau_e)=0$.

- ***Second Moment of Exit Time:***
  Assuming that exit is certain, let $\alpha(x_0,t_0)=-2t_0$ *and* $\beta(X(\tau_e),\tau_e)=0$ *again, then*

  $$\mathrm{E}[\tau_e^2(x_0,t_0)] = t_0^2 + v^{(2)}(x_0,t_0)\,, \qquad (7.54)$$

where $v^{(2)}(x_0,t_0)$ is the solution to the problem (7.48-7.49) with $\alpha(x_0,t_0) = -2t_0$ and $\beta(X(\tau_e),\tau_e) = 0$. Hence, the variance of the exit time on (a,b) is

$$\begin{aligned}\mathrm{Var}[\tau_e(x_0,t_0)] &= \mathrm{E}[\tau_e^2(x_0,t_0)] - \mathrm{E}^2[\tau_e(x_0,t_0)] \\ &= v^{(2)}(x_0,t_0) - 2t_0 v^{(1)}(x_0,t_0) - (v^{(1)})^2(x_0,t_0)\end{aligned}$$

and the coefficient of variation (CV) of the exit time is

$$\begin{aligned}\mathrm{CV}[\tau_e(x_0,t_0)] &= \frac{\sqrt{\mathrm{Var}[\tau_e(x_0,t_0)]}}{\mathrm{E}[\tau_e(x_0,t_0)]} \\ &= \frac{\sqrt{v^{(2)}(x_0,t_0) - 2t_0 v^{(1)}(x_0,t_0) - (v^{(1)})^2(x_0,t_0)}}{v^{(1)}(x_0,t_0) + t_0}\,.\end{aligned}$$

- ***Higher Moments of Exit Time:***
  Assuming that exit is certain, let $\alpha(x_0,t_0) = -nt_0^{n-1}$ and again $\beta(X(\tau_e),\tau_e) = 0$, then

$$\mathrm{E}[\tau_e^n(x_0,t_0)] = t_0^n + v^{(n)}(x_0,t_0)\,, \tag{7.55}$$

  where $v^{(n)}(x_0,t_0)$ is the solution to the problem (7.48-7.49) with $\alpha(x_0,t_0) = -nt_0^{n-1}$ and $\beta(X(\tau_e),\tau_e) = 0$.

Often conditional exit time moments are of interest, but then the inhomogeneous term $\alpha(x_0,t_0)$ genuinely depends on the state x_0 which makes the (7.51) form of Dynkin's formula not too useful since then the $\alpha(X(s),s)$ in the integrand genuinely depends on the stochastic process $X(s)$ and the integral is no longer simple. Hence, for more conditional and more general problems a more general form is needed. This more general form is based upon a generalization of the time-homogeneous derivations in Schuss [244] and in the appendix of Hanson and Tier [117] to the time dependent coefficient case, obtaining a hybrid backward or Dynkin equation for the exit time density $\phi_{\tau_e(x_0,t_0)}(t)$.

Lemma 7.11. ***Exit Time Distribution and Density:***
Given the exit time $\tau_e(x_0,t_0)$ (7.47), then its probability distribution can be related to the distribution for $X(t)$ by

$$\Phi_{\tau_e(x_0,t_0)}(t) = 1 - \int_a^b \phi_{X(t)}(x,t;x_0,t_0)dx, \tag{7.56}$$

where $\phi_{X(t)}(x,t;x_0,t_0)$ is the transition probability density for the Markov process $X(t) = x$ in continuous time conditionally starting at $X(t_0) = x_0$, as given in (7.18). The density $\phi_{X(t)}(x,t;x_0,t_0)$ is assumed to exist.

Assuming the exit time distribution and the transition density are differentiable even in a generalized sense, the exit time probability density is

$$\phi_{\tau_e(x_0,t_0)}(t) = \frac{\partial \Phi_{\tau_e(x_0,t_0)}}{\partial t}(t)\,.$$

The $\phi_{X(t)}$ transition density is assumed to be twice differentiable in x_0 and once in t, leading to the Kolmogorov equation in the forward time but with the backward operator $\mathcal{B}_{x_0}$,

$$\begin{aligned}\frac{\partial\phi_{\tau_e(x_0,t_0)}}{\partial t}(t) &= \mathcal{B}_{x_0}\left[\phi_{\tau_e(x_0,t_0)}(t)\right] \\ &= f(x_0,t_0)\frac{\partial\phi_{\tau_e(x_0,t_0)}}{\partial x_0}(t) + \frac{1}{2}g^2(x_0,t_0)\frac{\partial^2\phi_{\tau_e(x_0,t_0)}}{\partial x_0^2}(t) \\ &\quad +\widehat{\lambda}(x_0,t_0)\int_{\mathcal{Q}}\Delta_h[\phi_{\tau_e(x_0,t_0)}(t)](x_0,t_0,q)\phi_Q(q;x_0,t_0)dq,\end{aligned} \tag{7.57}$$

where the jump function Δ_h is given in (7.4).

Proof. The Eq. (7.56) for the exit time distribution follows from the probability definitions

$$\begin{aligned}\Phi_{\tau_e(x_0,t_0)}(t) &= \mathrm{Prob}[\tau_e(x_0,t_0) < t] = \mathrm{Prob}[X(t) \notin (a,b)|X(t_0) = x_0] \\ &= 1 - \mathrm{Prob}[X(t) \in (a,b)|X(t_0) = x_0] \\ &= 1 - \int_a^b \phi_{X(t)}(x,t;x_0,t_0)dx,\end{aligned}$$

i.e., the fact that the exit time probability is the complement of the probability that the process $X(t)$ is in the interval (a,b) and thus yields the right-hand side of (7.56).

Under differentiability assumptions, the exit time density can be related to an integral of the forward operator $\mathcal{F}_x$ using the forward Kolomogorov

$$\begin{aligned}\phi_{\tau_e(x_0,t_0)}(t) = \frac{\partial\Phi_{\tau_e(x_0,t_0)}}{\partial t}(t) &= -\int_a^b \phi_{X(t),t}(x,t;x_0,t_0)dx \\ &= -\int_a^b \mathcal{F}_x[\phi](x,t;x_0,t_0)dx\,.\end{aligned}$$

Manipulating partial derivatives, first in forward form,

$$\phi_{X(t),t}(x,t;x_0,t_0)) = \phi_{X(t),t-t_0}(x,t;x_0,t_0) = -\phi_{X(t),t_0-t}(x,t;x_0,t_0)$$

and then in backward form,

$$\phi_{X(t),t_0}(x,t;x_0,t_0) = \phi_{X(t),t_0-t}(x,t;x_0,t_0)\,,$$

leads to

$$\phi_{\tau_e(x_0,t_0)}(t) = +\int_a^b \phi_{X(t),t_0}(x,t;x_0,t_0)dx = -\int_a^b \mathcal{B}_{x_0}[\phi](x,t;x_0,t_0)dx\,.$$

Again assuming sufficient differentiability along with the interchange of integral and differential operators,

$$\begin{aligned}\phi_{\tau_e(x_0,t_0),t}(t) &= -\int_a^b \mathcal{B}[\phi_{X(t),t}(x,t;x_0,t_0)]dx \\ &= -\int_a^b \mathcal{B}_{x_0}[\mathcal{F}[\phi_{X(t)}]](x,t;x_0,t_0)dx \\ &= -\mathcal{B}_{x_0}\left[\int_a^b \mathcal{F}[\phi_{X(t)}](x,t;x_0,t_0)dx\right] = +\mathcal{B}_{x_0}\left[\phi_{\tau_e(x_0,t_0)}(t)\right]\,.\end{aligned}$$

This is a hybrid Kolmogorov equation (7.57), since it is in forward time t on the left and the backward operator is on the far right. □

Examples 7.12. ***Conditionally Expected Exit Time Applications:***

- ***Ultimate Probability of Exit:***
  The ultimate probability of exit is

$$\Phi_e(x_0,t_0) \equiv \Phi_{\tau_e(x_0,t_0)}(+\infty) = \int_0^\infty \phi_{\tau_e(x_0,t_0)}(t)dt\,, \tag{7.58}$$

  assuming that the distribution is bounded for all t. Also under the same conditions,

$$\int_0^\infty \phi_{\tau_e(x_0,t_0),t}(t)dt = \phi_{\tau_e(x_0,t_0)}(t)\Big|_0^{+\infty} = 0$$

  and then from the exit time density equation (7.57), integration-operator interchange and (7.58) for $\Phi_e(x_0,t_0)$,

$$\int_0^\infty \mathcal{B}[\phi_{\tau_e(x_0,t_0)}(t)]dt = \mathcal{B}[\Phi_e(x_0,t_0)] = 0\,. \tag{7.59}$$

  For certain exit at both endpoints a and b, the obvious boundary conditions are $\Phi_e(a,t_0) = 1$ and $\Phi_e(b,t_0) = 1$ for continuous diffusion processes, but $[\Phi_e(x_0,t_0)]$ =1 for $x_0 \notin (a,b)$ for jump-diffusions. Presuming uniqueness, then the solution to the boundary value problem is $\Phi_e(x_0,t_0) = 1$.

- ***Conditional Exit on the Right of (a,b):*** *Now suppose the statistics of ultimate exit on one side of (a,b), say $x_0 \in [b,+\infty)$, i.e., on the right. The corresponding random exit time variable is*

$$\tau_e^{(b)}(x_0,t_0) = \inf_t[\ t\ |X(t) \ge b,\ X(s) \in (a,b),\ t_0 \le s < t,\ X(t_0) = x_0]\,,$$

  and the exit time distribution function is

$$\Phi_{\tau_e^{(b)}(x_0,t_0)}(t) \equiv \mathrm{Prob}[\tau_e^{(b)}(x_0,t_0) < t]$$

and the corresponding density is $\phi_{\tau_e^{(b)}(x_0,t_0)}(t)$. *Thus, the ultimate conditional distribution,*

$$\Phi_e^{(b)}(x_0,t_0) \equiv \int_0^{+\infty} \phi_{\tau_e^{(b)}(x_0,t_0)}(t)dt,$$

for counting only exits on the right, has boundary conditions $\Phi_e^{(b)}(x_0,t_0) = 1$ *if* $x_0 \in [b,+\infty)$, *but* $\Phi_e^{(b)}(x_0,t_0) = 0$ *if* $x_0 \in (-\infty,a]$. *(For counting only exits at the left,* $(-\infty,a]$, *then the boundary conditions are interchanged for* $\Phi_e^{(a)}(x_0,t_0)$.*) In general, the conditional distribution* $\Phi_e^{(b)}(x_0,t_0)$ *will not be one as in the certain ultimate probability in the prior item, so it is necessary to work in exit time moments rather than expected exit times. Let the conditional exit time first moment be*

$$M_e^{(b)}(x_0,t_0) \equiv \int_0^{+\infty} t\phi_{\tau_e^{(b)}(x_0,t_0)}(t)dt \tag{7.60}$$

and the expected conditional exit time is

$$T_e^{(b)}(x_0,t_0) \equiv M_e^{(b)}(x_0,t_0)/\Phi_e^{(b)}(x_0,t_0) \tag{7.61}$$

if $x_0 > a$. *Upon integration of both sides of (7.57), making the reasonable assumption*

$$t\phi_{\tau_e^{(b)}(x_0,t_0)}(t)\Big|_0^{+\infty} = 0$$

when apply integration by parts on the left, then the conditional moment equation, interchanging left and right sides, is

$$\mathcal{B}_{x_0}\left[M_e^{(b)}\right](x_0,t_0) = -\Phi_e^{(b)}(x_0,t_0) \tag{7.62}$$

with boundary condition $M_e^{(b)}(x_0,t_0) = 0$ *if* $x_0 \notin (a,b)$. *The conditions are zero on either side of* (a,b) *for different reasons, due to instant exit for* $x_0 \in [b,+\infty)$ *and due to excluded exit for* $x_0 \in (-\infty,a]$.

7.8 Diffusion Approximation Basis

Up until this point, stochastic diffusions have almost been taken as given. There are many derivations for physical diffusions in physics and engineering, such as the diffusion of a fluid concentration in a liquid or gas according to Fick's law for the flux or flow of concentration or the diffusion of heat in a conduction medium according to Fourier's law for the flux of heat. These types of physical diffusions lead to the same or similar diffusion equations as seen in this chapter when the jump terms are omitted. However, the stochastic diffusions are usually postulated on a different basis.

A fundamental property that distinguishes the pure diffusion process from the discontinuous jump process among Markov processes in continuous time is that the diffusion process is a continuous process. Let $\mathbf{X}(t) = [X_i(t)]_{n_x \times 1}$ be a continuous process, then it must satisfy the following continuity condition, given some $\delta > 0$,

$$\lim_{\Delta t \to 0} \frac{\text{Prob}[|\Delta \mathbf{X}(t)| > \delta \mid \mathbf{X}(t) = \mathbf{x}]}{\Delta t} = 0\,, \tag{7.63}$$

so jumps in the process are unlikely.

In addition, two basic moment properties are needed for the continuous process to have a diffusion limit and these are that the conditional mean increment process satisfy

$$\begin{aligned} \mathrm{E}[\Delta \mathbf{X}(t)|\mathbf{X}(t) = \mathbf{x}] &= \int_\Omega \phi_{X(t)}(\mathbf{y}, t + \Delta t; \mathbf{x}, t) d\mathbf{y}) \\ &= \boldsymbol{\mu}(\mathbf{x}, t)\Delta t + \mathrm{o}(\Delta t) \quad \text{as} \quad \Delta t \to 0\,, \end{aligned} \tag{7.64}$$

where $\boldsymbol{\mu}(\mathbf{x}, t) = [\mu_i(\mathbf{x}, t)]_{n_x \times 1}$, and that the conditional variance increment process satisfy

$$\mathrm{Cov}[\Delta \mathbf{X}(t), \Delta \mathbf{X}^\top(t)|\mathbf{X}(t) = \mathbf{x}] = \sigma(\mathbf{x}, t)\Delta t + \mathrm{o}(\Delta t) \quad \text{as} \quad \Delta t \to 0\,, \tag{7.65}$$

where $\sigma(\mathbf{x}, t) = [\sigma_{i.j}(\mathbf{x}, t)]_{n_x \times n_x} > 0$, i.e., positive definite, and $\phi_{X(t)}(\mathbf{x}, t; \mathbf{x}_0, x_0)d\mathbf{y})$ is the transition probability density for $\mathbf{X}(t)$. Alternatively, these two *infinitesimal moment conditions* can be written

$$\lim_{\Delta t \to 0} \frac{\mathrm{E}[\Delta \mathbf{X}(t)|\mathbf{X}(t) = \mathbf{x}]}{\Delta t} = \boldsymbol{\mu}(\mathbf{x}, t)$$

and

$$\lim_{\Delta t \to 0} \frac{\mathrm{Cov}[\Delta \mathbf{X}(t), \Delta \mathbf{X}^\top(t)|\mathbf{X}(t) = \mathbf{x}]}{\Delta t} = \sigma(\mathbf{x}, t)\,.$$

There are other technical conditions that are needed and the reader should consult references like Feller [84, Chapt. 10] or Karlin and Taylor [162, Chapt. 15] for the history and variations in these conditions. Another technical condition implies that higher order moments are negligible,

$$\lim_{\Delta t \to 0} \frac{\mathrm{E}[|\Delta \mathbf{X}(t)|^m \mid \mathbf{X}(t) = \mathbf{x}]}{\Delta t} = 0\,, \tag{7.66}$$

for $m \geq 3$.

Remarks 7.13.

- *Note that since the focus is on diffusion, the* m*th central moment could be used here as in [84, 162], instead of the uncentered* m*th moment in (7.66), just as the 2nd moment could have been used in (7.65) instead of the covariance. For high moments, the central moment form may be easier to use since means of deviation are trivially zero.*

- *Karlin and Taylor [162] show that from the Chebyshev inequality (Chapter 1, Exercise 4),*

$$\frac{\mathrm{Prob}[|\Delta\mathbf{X}(t)| > \delta \mid \mathbf{X}(t) = \mathbf{x}]}{\Delta t} \le \frac{\mathrm{E}[|\Delta\mathbf{X}(t)|^m \mid \mathbf{X}(t) = \mathbf{x}]}{\delta^m \Delta t}, \tag{7.67}$$

that the high moment condition (7.66) for ***any*** $m \geq 3$ *can imply the continuity condition (7.63) for* $\delta > 0$. *Depending on the problem formulation, the high moment condition may be easier to demonstrate than estimating the tail of the probability distribution in the continuity condition.*

In terms of the general multi-dimensional jump-diffusion model (7.31), the corresponding infinitesimal parameters, in absence of the jump term ($h = 0$), are the infinitesimal vector mean

$$\boldsymbol{\mu}(\mathbf{x}, t) = \mathbf{f}(\mathbf{x}, t)$$

and the infinitesimal matrix covariance

$$\sigma(\mathbf{x}, t) = (gg^\top)(\mathbf{x}, t)\,.$$

These infinitesimal properties by themselves do not make a diffusion process, since adding jump processes to diffusion process invalidates the continuity condition (7.63). For instance, examining this continuity condition for the simplest case of a simple Poisson process $X(t) = P(t)$ but with a time-dependent jump rate $\lambda(t) > 0$, yields

$$\frac{\mathrm{Prob}[|\Delta P(t)| > \delta \mid P(t) = j]}{\Delta t} = \sum_{k=1}^{\infty} e^{-\Delta\Lambda(t)} \frac{(\Delta\Lambda)^k(t)}{k!\Delta t} = \frac{1 - e^{-\Delta\Lambda(t)}}{\Delta t}$$

assuming for continuity's sake that $0 < \delta < 1$ and where

$$\Delta\Lambda(t) = \int_t^{t+\Delta t} \lambda(s)ds \to \lambda(t)\Delta t \quad \text{as} \quad \Delta t \to 0^+\,.$$

Thus,

$$\lim_{\Delta t \to 0} \frac{\mathrm{Prob}[|\Delta P(t)| > \delta \mid P(t) = j]}{\Delta t} = \lambda(t) > 0$$

invalidating the continuity condition as expected, although the two basic infinitesimal moments can be calculated. In general, the higher moment criterion (7.66) will not be valid either, since for example,

$$\begin{aligned}\lim_{\Delta t \to 0} \frac{\mathrm{E}[|\Delta P(t)|^3 \mid \mathbf{X}(t) = \mathbf{x}]}{\Delta t} &= \lim_{\Delta t \to 0} \sum_{k=1}^{\infty} e^{-\Delta\Lambda(t)} \frac{(\Delta\Lambda)^k(t)}{k!\Delta t} k^3 \\ &= \lim_{\Delta t \to 0} \frac{\Delta\Lambda(t)(1 + 3\Delta\Lambda(t) + (\Delta\Lambda)^2(t))}{\Delta t} \\ &= \lambda(t) > 0\,,\end{aligned}$$

where incremental moment Table 1.2 has been used. It is easy to guess that the number of infinitesimal moments of the Poisson process will be infinite, extrapolating from Table 1.2, unlike the limit of two infinitesimal moments for diffusion processes. However, the table only can be used to confirm that cases $m = 3\!:\!5$ yield the infinitesimal expectation of $\lambda(t)$.

So far these conditions are merely general formulations of diffusion processes for which similar properties have been derived in the earlier chapters of this book. Where their power lies is when they are used to approximate other stochastic processes, such as in the stochastic tumor application using a diffusion approximation that can be solved for tumor doubling times in Subsection 11.2.1.

7.9 Exercises

1. *Derivation of the Forward Kolmogorov Equation in the Generalized Sense.* Let the jump-diffusion process $X(t)$ satisfy the SDE,

$$dX(t) = f(X(t),t)dt + g(X(t),t)dW(t) + h(X(t),t,Q)dP(t;Q,X(t),t)) \tag{7.68}$$

$X(t_0) = x_0$, where the coefficient functions (f, g, h) are sufficiently well-behaved, Q is the jump-amplitude random mark with density $\phi_Q(q; X(t), t)$ and $\mathrm{E}[dP(t;Q,X(t),t)|X(t)=x] = \lambda(t;Q,x,t)dt$.

   (a) Show (easy) that, in the generalize sense,

$$\phi(x,t) \stackrel{\text{gen}}{=} \mathrm{E}[\delta(X(t)-x)|X(t_0)=x_0]\,, \quad t_0 < t\,,$$

   where $\phi(x,t) = \phi_{X(t)}(x,t;x_0,t_0)$ is the transition probability density for the process $X(t)$ conditioned on the starting at $X(t_0) = x_0$ and $\delta(x)$ is the Dirac delta function.

   (b) Show that the Dirac delta function with composite argument satisfies

$$\int_{-\infty}^{+\infty} F(y)\delta(\gamma(y)-x)dy \stackrel{\text{gen}}{=} F\left(\gamma^{-1}(x)\right)\left|(\gamma^{-1})'(x)\right|\,,$$

   where $\gamma(y)$ is a monotonic function with non-vanishing derivative and inverse $y = \gamma^{-1}(z)$, such that $(\gamma^{-1})'(z) = 1/\gamma'(y)$ and $|\gamma^{-1}(\pm\infty)| = \infty$.

   (c) Apply the previous two results and other delta function properties from Section B.12 to derive the forward Kolmogorov equation (7.26) in the generalized sense.
   Hint: Regarding the proof of (7.26), the diffusion part is much easier given the delta function properties for the derivation, but the jump part is similar and is facilitated by the fact that $\gamma(y) = y + h(y;t,q)$ for fixed (t,q).

2. *Derivation of the **Feynman-Kac** (Dynkin with Integrating Factor) Formula for Jump-Diffusions.*
Consider the jump-diffusion process,

$$dX(t) = f(X(t),t)dt + g(X(t),t)dW(t) + h(X(t),t,Q)dP(t;Q,X(t),t),$$

$X(t_0) = x_0 \in \Omega$, $t_0 < t < t_f$ and related backward Feynman-Kac (pronounced Fineman-Katz) final value problem,

$$\frac{\partial v}{\partial t_0}(x_0, t_0) + \mathcal{B}[v](x_0, t_0) + \theta(x_0, t_0)v(x_0, t_0) = \alpha(x_0, t_0)\,, \tag{7.69}$$

$x_0 \in \Omega$, $0 \le t_0 < t_f$, with final condition

$$v(x_0, t_f) = \gamma(x_0, t_f)\,, \quad x_0 \in \Omega\,, \quad 0 \le t_0 < t_f\,,$$

where $\mathcal{B}[v](x_0, t_0)$ is the backward operator corresponding to the jump-diffusion process (7.3). The given coefficients, $\theta(x_0, t_0)$, $\alpha(x, t)$ and $\gamma(x, t)$ are bounded and continuous. The solution $v(x_0, t_0)$ is assumed to be twice continuously differentiable in x_0 while once in t.

(a) In preparation, apply the stochastic chain rule to the auxiliary function

$$w(X(t), t) = v(X(t), t)\exp(\Theta(t_0, t))$$

to use an integrating factor technique to remove the non-Dynkin linear source term $\theta(x_0, t_0)v(x_0, t_0)$ from (7.69) with integrating factor exponent process

$$\Theta(t_0, t) = \int_{t_0}^{t} \theta(X(s), s)ds.$$

Then show (best done using the usual time-increment form of the stochastic chain rule) that

$$\begin{aligned} dw(X(t), t) \stackrel{\text{dt}}{=} e^{\Theta(t_0,t)}\Bigg(&\left(\frac{\partial v}{\partial t} + \mathcal{B}[v] + \theta v\right)(X(t), t)dt \\ &+(gv\frac{\partial v}{\partial x})(X(t), t)dW(t) \\ &+ \int_{\mathcal{Q}} \delta_h[v](X(t), t, q)\widehat{\mathcal{P}}(\mathbf{dt}, \mathbf{dq}; X(t), t)\Bigg)\,, \end{aligned} \tag{7.70}$$

where $\delta_h[v]$ is defined in (7.4) and $\widehat{\mathcal{P}}$ is defined in (7.8).

(b) Next integrate the SDE (7.70) on $[t_0, t_f]$, solve for $v(x_0, t_0)$, then take expectations and finally apply the final value problem to obtain the Feynman-Kac formula corresponding to (7.69),

$$\begin{aligned} v(x_0, t_0) = \mathrm{E}\Bigg[&e^{+\Theta(t_0,t_f)}\gamma(X(t_f), t_f) \\ &- \int_{t_0}^{t_f} e^{+\Theta(t_0,s)}\alpha(X(s), s)ds\,\Bigg|\, X(t_0) = x_0\Bigg]\,. \end{aligned} \tag{7.71}$$

Hint: Follow the procedure in the derivation proof of Theorem 7.3 for this Feynman-Kac formula. See Schuss [244] or Yong and Zhou [288] for pure diffusion processes.

3. *Moments of Stochastic Dynamical Systems.* Consider first the linear stochastic dynamical system,

$$dX(t) = \mu_0 X(t)dt + \sigma_0 X(t)dW(t) + \nu_0 X(t)h(Q)dP(t;Q), \quad X(t_0) = x_0 \ ,$$

where $\{\mu_0, \sigma_0, \nu_0\}$ is a set of constant coefficients, x_0 is specified and $h(q)$ has finite moments with respect to a Poisson mark amplitude density $\phi_Z(z)$. Starting with a Dynkin's Formula (*or the Forward Kolmogorov Equation if you like deriving results the hard way*),

   (a) Show that the conditional first moment of the process

   $$\overline{X}(t) = \mathrm{E}[X(t)|X(t_0) = x_0]$$

   satisfies a first order ODE in $\overline{X}(t)$ only, (x_0, t_0) fixed, corresponding to the mean (quasi-deterministic) analog of the SDE. Solve the ODE in terms of the given initial conditions.

   (b) Derive the ODE for second moment

   $$\overline{X^2}(t) = \mathrm{E}[X^2(t)|X(t_0) = x_0]$$

   for the more general SDE

   $$dX(t) = f(X(t))dt + g(X(t))dW(t) + h(X(t), q)dP(t;Q) \ ,$$

   $X(t_0) = x_0$, in terms of expected coefficient values over both state and mark spaces.

   (c) Use the general second moment ODE of part (b) to derive the corresponding ODE for the state variance

   $$\mathrm{Var}[X(t)] = \overline{X^2}(t) - (\overline{X})^2(t)$$

   for the linear dynamical system in the part (a). Your result should show that the ODE is linear in $\mathrm{Var}[X](t)$ with an inhomogeneous term depending on the $\overline{X}(t)$ first moment solution and constants, so the ODE is closed in that it is independent of any higher moments beyond the second. Solve the ODE.

Suggested References for Further Reading

- Arnold, 1974 [13].
- Bharucha-Reid, 1960 [31].
- Feller, 1971 [84, II].
- Gihman and Skorohod, 1972 [94].
- Goel and Richter-Dyn, 1974 [98].

- Hanson and Tier, 1982 [117].
- Jazwinski, 1970 [154].
- Karlin and Taylor, 1981 [162, II].
- Kushner and Dupuis, 2001 [179].
- Ludwig, 1975 [188].
- Øksendal, 1998 [222].
- Schuss, 1980 [244].