*A Monte-Carlo Option-Pricing Algorithm for Log-Uniform Jump-Diffusion Model**

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Finance and Time Series (WeB11) December 14, 2005

in Proceedings of 44th IEEE Conference on Decision and Control

and European Control Conference 2005, Seville SPAIN,

pp. 5221-5226, 12 December 2005.

* **This material is based upon work supported by the National Science Foundation under Grant No. 0207081 in Computational Mathematics. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the author(s) and do not necessarily reflect the views of the National Science Foundation.**

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1. Introduction

1.1 Background:

- ***Black-Scholes-Merton*** (2 seminal papers in Spring ’73 leading to ’97 Nobel Prize in Economics for Scholes and Merton, Black dying in ’95) option pricing formula is based upon a purely geometric (linear) diffusion process and its associated log-normal distribution.

- ***Statistical evidence*** that jumps are significant in financial markets:
  - Stock and Option Prices in Ball and Torous (’85);
  - Capital Asset Pricing Model in Jarrow and Rosenfeld (’84);
  - Foreign Exchange and Stocks in Jorion (’89).

1.2 Market Jump Properties:

- Log-return market distributions usually ***skewed negative*** if data time interval sufficiently long compared to the skew-less normal distribution.

- Log-return market distributions usually ***leptokurtic***, i.e., more peaked than the normal distribution.

- Log-return market distribution have ***fatter or heavier tails*** than the normal distribution's exponentially small tails.

- ***Stochastic dependence of volatility*** (standard deviation) is important.

- ***Time-dependence*** of rate coefficients is important, i.e., non-constant coefficients are important.

1.3 Merton's Jump-Diffusion Option Pricing Model:

- In ***Merton's ('76) pioneering jump-diffusion option pricing model***, he used log-normally distributed jump-amplitudes in a compound Poisson process.
- Merton argued that the ***portfolio volatility could not be hedged*** as in the Black-Scholes' pure diffusion case, but that the risk-neutral property could preserve the no-arbitrage strategy by ensuring that the discounted, expected return would be at the market rate, all without relying on measure theory.
- Merton's solution is the expected value of an ***infinite set of Black-Scholes' call option pricing formulas*** each one the initial stock price shifted by a jump factor depending on the number of jumps which have a Poisson distribution.

1.4 Risky Asset Stock Price, $S(t)$, Dynamics at time t:

- ***Linear, Constant Rate Stochastic Differential Equation (SDE)***:

$$dS(t) = S(t)\left(\mu dt + \sigma dW(t) + \sum_{k=1}^{dN(t)} S(T_k^-)J(Q_k)\right),$$

where $S(0) = S_0 > 0$ and

- μ = ***expected rate of return*** in absence of asset jumps, i.e., diffusive drift;
- σ = ***diffusive volatility*** (standard deviation);
- $W(t)$ = ***Wiener*** (diffusion or Brownian motion) ***process***, normally distributed such that $\mathrm{E}[W(t)] = 0$ and $\mathrm{Var}[W(t)] = t$;
- $J(Q)$ = ***Poisson jump-amplitude*** with underlying ***random mark variable*** Q, selected for log-return so that $Q = \ln(J(Q) + 1)$, such that $J(Q) > -1$;
- $N(t)$ = ***Poisson jump counting process***, Poisson distributed such that $\mathrm{E}[N(t)] = \lambda t = \mathrm{Var}[N(t)]$;

1.4 Continued: Stock Price Dynamics:

- T_k^- is the ***pre-jump time*** and Q_k is an independent and identically distributed ***(IID) mark*** realization at the kth jump;
- The processes $W(t)$ and $P(t)$ along with Q_k are ***independent***, except that Q_k depends on a jump-event at T_k.

- ***Uniform Probability Jump-Amplitude Q Density***:

$$\phi_Q(q) = \frac{1}{b-a} \left\{ \begin{array}{ll} 1, & a \le q \le b \\ 0, & \text{else} \end{array} \right\},$$

  - ***Mark Mean*** $\mu_j \equiv \mathrm{E}_Q[Q] = 0.5(b+a)$;
  - ***Mark Variance*** $\sigma_j^2 \equiv \mathrm{Var}_Q[Q] = (b-a)^2/12$;
  - ***Jump-Amplitude Mean***:

$$\bar{J} \equiv \mathrm{E}[J(Q)] \equiv \mathrm{E}[\exp(Q)-1] = \frac{(\exp(b)-\exp(a))}{(b-a)} - 1.$$

1.5 Uniform Distribution Motivation:

- ***Extreme jumps*** in the market are ***relatively rare*** among the large number of daily fluctuations and as statistical outliers they are very difficult, some say impossible, to include in statistical analysis of financial market data. With little information on the jump component, we focus here on the ***uniform jump-amplitude*** with the ***fattest of tails and finite range***, that is consistent with the ***NYSE circuit breakers*** (since 1988) on extreme market changes.

1.6 *Log-Return* $\ln(S(t))$ *SDE and Solution:*

- According to ***Itô's stochastic chain rule*** for jump-diffusions

$$d\ln(S(t)) = (\mu - \sigma^2/2)dt + \sigma dW(t) + \sum_{k=1}^{dN(t)} Q_k \,.$$

- Easily integrated in continuous and jump components:

$$S(t) = S_0 \exp\left((\mu - \sigma^2/2)t + \sigma W(t) + \sum_{k=1}^{N(t)} Q_k\right). \qquad (1)$$

2. *Risk-Neutral Constant-Coefficient Problems*

First a critical theorem:

Theorem 2.1: Expected Jump-Diffusion Stock Price

$$\mathrm{E}[S(t)] = S_0 \exp((\mu + \lambda \bar{J})t),$$

where $S(t)$ is given in (1) and $\bar{J}$.

Proof: Using the mutual independence of the diffusion, Poisson counting and IID mark processes with separated and iterated expectations,

$$\begin{aligned}
\mathrm{E}[S(t)] &= S_0 e^{(\mu-\sigma^2/2)t}\mathrm{E}\left[e^{\sigma W(t)} e^{\sum_{i=1}^{N(t)} Q_i}\right] \\
&= S_0 e^{(\mu-\sigma^2/2)t}\mathrm{E}_{W(t)}\left[e^{\sigma W(t)}\right]\mathrm{E}_{N(t)}\left[\mathrm{E}_{Q|N}\left[\prod_{i=0}^{N(t)} e^{Q_i}\,\middle|\, N(t)\right]\right] \\
&= S_0 e^{(\mu-\sigma^2/2)t} e^{\sigma^2 t/2} \sum_{k=0}^{\infty} p_k(\lambda t) \prod_{i=0}^{k} \mathrm{E}_Q\left[e^{Q_i}\right] = S_0 e^{(\mu+\lambda\bar{J})t},
\end{aligned}$$

where the Poisson distribution $p_k(\lambda t) \equiv e^{-\lambda t}(\lambda t)^k/k!$ has been used.

□

2.2 Risk-Neutral Assumptions:

After Merton's discontinuous paper (1976),

- Jumps are due to Extreme Changes in Firm's Specifics, i.e., Non-Systematic Risks, e.g., bankruptcy, adverse legal rulings, unfavorable publicity, important discoveries, etc.
- Portfolio-Market Return Correlation *beta* (i.e., $\mathrm{Cov}[R_S, R_M]/\mathrm{Var}[R_M]$, where return $R_X = \Delta X/X$ for $X = S$ or M) is Zero and can be constructed by *delta* (i.e., $\partial V/\partial S$) Hedging.
- Thus, Jump-Diffusion Model is Arbitrage-Free.
- $\therefore$ Risk-Neutral World
  $\Longrightarrow \mathrm{E}[S(t)] = S_0 \exp rt \Longrightarrow \mu + \lambda \bar{J} = r \Longrightarrow \mu = \mu_{\mathrm{rn}} \equiv r - \lambda \bar{J}$.
  - Similarly, for time-dependent coefficients,

$$\mu(t) = \mu_{\mathrm{rn}}(t) \equiv r - \lambda \mathrm{E}[J(t, Q)].$$

2.3 Risk-Neutral Jump-Diffusion SDE:

Under Risk-Neutral Measure $\mathcal{M}_{\mathrm{rn}}$, in principle,

$$dS(t)/S(t) = \left(r - \lambda \bar{J}\right) dt + \sigma dW(t) + \sum_{k=1}^{dN(t)} J(Q_k)$$

$$= r dt + \sigma dW(t) + \sum_{k=1}^{dN(t)} \left(J(Q_k) - \bar{J}\right) + \bar{J}\left(dN(t) - \lambda dt\right),$$

where jump terms are separated into the zero-mean forms of the compound Poisson process for convenience.

3. Risk-Neutral Option Price Solutions

3.1 Risk-Neutral European Call Option:

Under Risk-Neutral Valuation with Measure $\mathcal{M}_{\rm rn}$ with drift $\mu_{\rm rn}$, the Payoff for European Call Option using Stock Price $S(t)$ having exercise price K at exercise time T is

$$\begin{aligned}
\mathcal{C}(S_0,T) &\equiv e^{-rT}\mathrm{E}_{\mathcal{M}}[\max(S(T)-K,0)] \\
&= \frac{e^{-rT}}{\sqrt{2\pi}}\sum_{k=0}^{\infty} p_k(\lambda T)\int_{ka}^{kb}\int_{Z_0(s_k)}^{\infty}\left(S_0 e^{DJ(z,s_k)} - K\right) \\
&\quad \cdot e^{-z^2/2}\phi_{\widetilde{\mathcal{S}}_k}(s_k)\,dz\,ds_k \\
&= \frac{1}{\sqrt{2\pi}}\sum_{k=0}^{\infty} p_k(\lambda T)\mathrm{E}_{\widetilde{\mathcal{S}}_k}\Bigg[\int_{Z_0(\widetilde{\mathcal{S}}_k)}^{\infty}\left(S_0 e^{DJ(z,\widetilde{\mathcal{S}}_k))-rT} - Ke^{-rT}\right) \\
&\quad \cdot e^{-z^2/2}dz\Bigg],
\end{aligned}$$

3.1 Continued: Risk-Neutral Options:

where the scaled jump-diffusion exponent is

$$DJ(z,s) \equiv (r - \lambda \bar{J} - \sigma^2/2)T + \sigma\sqrt{T}z + s,$$

the ***At-The-Money*** for standard normal integration variable is

$$Z_0(s) \equiv (\ln(K/S_0) - (r - \lambda \bar{J} - \sigma^2/2)T - s)/(\sigma\sqrt{T})$$

and the partial sum of k uniformly distributed IID marks is

$$\widetilde{\mathcal{S}}_k = \sum_{i=1}^{k} Q_i$$

on $[a,b]$ and s_k is the corresponding realized variable, such that $\widetilde{\mathcal{S}}_0 = \sum_{i=1}^{0} Q_i \equiv 0$.

3.2 Black-Scholes Type Splitting of Integral:

Let

$$A(s) \equiv \frac{1}{\sqrt{2\pi}} \int_{Z_0(s)}^{\infty} S_0 e^{-(\lambda\bar{J}+\sigma^2/2)T+\sigma\sqrt{T}z+s} e^{-z^2/2} dz$$
$$= S_0 e^{s-\lambda\bar{J}T} \Phi\left(d_1\left(S_0 e^{s-\lambda\bar{J}T}\right)\right),$$

$$B(s) \equiv \frac{1}{\sqrt{2\pi}} \int_{Z_0(s)}^{\infty} K e^{-rT} e^{-z^2/2} dz = K e^{-rT} \Phi\left(d_2\left(S_0 e^{s-\lambda\bar{J}T}\right)\right),$$

$$\Phi(y) \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} e^{-z^2/2} dz$$

is the standardized normal distribution and

$$d_1(x) \equiv (\ln(x/K) + (r+\sigma^2/2)T)/(\sigma\sqrt{T}) \quad \& \quad d_2(x) \equiv d_1(x) - \sigma\sqrt{T}$$

are the usual ***Black-Scholes normal distribution argument functions***.

3.2 Continued: Splitting:

$$
\begin{aligned}
\therefore\ \mathcal{C}(S_0,T) &= \sum_{k=0}^{\infty} p_k(\lambda T)\mathrm{E}_{\widetilde{\mathcal{S}}_k}\left[A(\widetilde{\mathcal{S}}_k)-B(\widetilde{\mathcal{S}}_k)\right] \\
&= \sum_{k=0}^{\infty} p_k(\lambda T)\mathrm{E}_{\widetilde{\mathcal{S}}_k}\Big[S_0 e^{\widetilde{\mathcal{S}}_k-\lambda\bar{J}T}\Phi\Big(d_1\Big(S_0 e^{\widetilde{\mathcal{S}}_k-\lambda\bar{J}T}\Big)\Big) \\
&\qquad -Ke^{-rT}\Phi\Big(d_2\Big(S_0 e^{\widetilde{\mathcal{S}}_k-\lambda\bar{J}T}\Big)\Big)\Big]\,. \\
&= \sum_{k=0}^{\infty} p_k(\lambda T)\mathrm{E}_{\widetilde{\mathcal{S}}_k}\Big[\mathcal{C}^{(BS)}\Big(S_0 e^{\widetilde{\mathcal{S}}_k-\lambda\bar{J}T},T;K,\sigma^2,r\Big)\Big],
\end{aligned}
\tag{2}
$$

where

$$
\mathcal{C}^{(BS)}(x,T;K,\sigma^2,r)\equiv x\Phi(d_1(x))-Ke^{-rT}\Phi(d_2(x))
$$

is the ***Black-Scholes formula (1973)***, but with the stock price argument shifted by a jump factor $\exp(\widetilde{\mathcal{S}}_k-\lambda\bar{J}T)$.

3.3 Put-Call Parity Generally Valid:

Put-call parity is founded on ***basic maximum function properties*** (Merton (1973), Hull (2000) and Higham (2004)), so is independent of the particular process,

$$\mathcal{C}(S_0, T) + Ke^{-rT} = \mathcal{P}(S_0, T) + S_0$$

or solving for European put option price $\mathcal{P}(S_0, T)$,

$$\mathcal{P}(S_0, T) = \mathcal{C}(S_0, T) + Ke^{-rT} - S_0, \tag{3}$$

assuming no dividends or transaction fees and that European call option price $\mathcal{C}(S_0, T)$ is known.

4. *Monte Carlo Simulations*

For Monte Carlo simulations (e.g., see Glasserman (2004)), the European call option price formula (2) can be equivalently written compactly as

$$\mathcal{C}(S_0,T) = \mathrm{E}_{\widetilde{\mathcal{S}}(T)}\Big[\mathcal{C}^{(BS)}\Big(S_0 e^{\widetilde{\mathcal{S}}(T)-\lambda\bar{J}T}, T\Big)\Big], \tag{4}$$

directly in terms of the compound Poisson process $\widetilde{\mathcal{S}}(T)=\sum_{i=1}^{N(T)} Q_i$ with uniformly distributed IID random variables Q_i on $[a,b]$.

Remark: If the zero mean process $\widehat{\mathcal{S}}(T)\equiv\widetilde{\mathcal{S}}(T)-\lambda T\bar{J}$, where $\exp(\lambda T\bar{J}) = \mathrm{E}[\exp(\widetilde{\mathcal{S}}(T))]$, then $\exp(\widehat{\mathcal{S}}(T))$ is an exponential compound Poisson process with ***exponential martingale*** property on $[0,T]$ that $\mathrm{E}[\exp(\widehat{\mathcal{S}}(T))]=\exp(\widehat{\mathcal{S}}(0))=1$.

4.1 Compound Poisson Simulation Samples:

Let $\boldsymbol{N_i}$ be a ***IID Poisson variate sample point*** taken from the distribution of $N(T)$ for $i=1:n$ sample points. Given a jump in N_i, let $\boldsymbol{U_{i,j}}$ be ***IID uniformly generated on [0, 1] sample points*** for $j=1:N_i$, then $\widehat{\mathcal{S}}_i = \sum_{j=1}^{N_i}(a+(b-a)U_{i,j}) = aN_i + (b-a)\sum_{j=1}^{N_i} U_{i,j}$ will be IID compound Poisson random variables on [a, b] having the same distribution as $\widehat{\mathcal{S}}(T)$.

An ***elementary Monte Carlo (EMC) estimate*** for $\mathcal{C}(S_0, T)$ is

$$\widehat{\mathcal{C}}_n = \frac{1}{n}\sum_{i=1}^{n} \mathcal{C}^{(BS)}\left(S_0 e^{\widehat{\mathcal{S}}_i - \lambda \bar{J} T}, T\right) \equiv \frac{1}{n}\sum_{i=1}^{n} \widehat{\mathcal{C}}_i^{(BS)} ,$$

such that the $\widehat{\mathcal{C}}_i^{(BS)}$ are IID random variables based on $\widehat{\mathcal{S}}_i$.

4.2 Variance Reduction Techniques: Antithetic Variates:

Variance reduction techniques can reduce the size of $\sigma_{\widehat{\mathcal{C}}_n}$ at reasonable computational cost. If $U_{i,j}$ is uniformly distributed on $[0,1]$, then the $\boldsymbol{\widehat{Q}_{i,j} = a + (b-a)U_{i,j}}$ are uniformly distributed (***thetic***) random variables on $[a,b]$, so are the ***antithetic*** counterparts $\boldsymbol{\widehat{Q}^{(a)}_{i,j} = a + (b-a)(1-U_{i,j})}$ and $\widehat{\mathcal{S}}^{(a)}_i \equiv (b+a)N_i - \widehat{\mathcal{S}}_i$ are IID random variables for $i = 1\!:\!n$ having the same compound Poisson distribution as $\widehat{\mathcal{S}}(T)$. Let the ***thetic-antithetic averaged***, Black-Scholes risk-neutral, discounted payoff be

$$\boldsymbol{X_i = 0.5\left(\widehat{\mathcal{C}}^{(BS)}_i + \widehat{\mathcal{C}}^{(aBS)}_i\right),} \tag{5}$$

where the antithetic $\widehat{\mathcal{C}}^{(aBS)}_i \equiv \mathcal{C}^{(BS)}(S_0 e^{\widehat{\mathcal{S}}^{(a)}_i - \lambda \bar{J} T}, T)$, for $i = 1\!:\!n$, with thetic-antithetic averaged jump factor

$$\boldsymbol{Y_i = 0.5\left(\exp\left(\widehat{\mathcal{S}}_i\right) + \exp\left(\widehat{\mathcal{S}}^{(a)}_i\right)\right),} \tag{6}$$

So the antithetic and thetic variates can be use together to double the sample size without significant computational cost (Boyle (1977)).

4.3 Variance Reduction Techniques: Control Variates:

Since $\mathrm{E}[\exp(\widehat{\mathcal{S}}_i(T))] = \exp(\lambda T \bar{J})$ from Theorem 2.1 and $\exp(\widehat{\mathcal{S}}_i(T))$ has positive correlation with $\widehat{\mathcal{C}}_i^{(BS)}$, so the ***control variates*** technique can also be used to reduce the variance of Monte Carlo estimation, working faster the higher the correlation. The antithetic variate and control variate variance reduction techniques can be combined, the control adjusted payoff is

$$\boldsymbol{Z_i(\alpha) = X_i - \alpha \cdot (Y_i - \exp(\lambda T \bar{J}))\,,} \tag{7}$$

where $\boldsymbol{(Y_i - \exp(\lambda T \bar{J}))}$ is the ***control deviation*** and $\boldsymbol{\alpha}$ is an ***adjustable control parameter***. The sample mean of $\boldsymbol{Z_i(\alpha)}$ produces the ***Monte Carlo estimator*** for $\boldsymbol{\mathcal{C}(S_0, T)}$, since $\overline{Z}_n(\alpha) \equiv \sum_{i=1}^{n} Z_i(\alpha)/n = \overline{X}_n - \alpha(\overline{Y}_n - \exp(\lambda T \bar{J}))$, is an unbiased estimation with $\mathrm{E}[\overline{Z}_n(\alpha)] = \mathcal{C}(S_0, T)$ using IID mean properties $\mathrm{E}[\overline{X}_n] = \mathrm{E}[X_i] = \mathcal{C}(S_0, T)$ and $\mathrm{E}[\overline{Y}_n] = \mathrm{E}[Y_i] = \exp(\lambda T \bar{J})$ from the proof of Thm. 2.1.

4.3 Continued: Combined Antithetic and Control Variates:

The variance of $\overline{Z}_n(\alpha)$ is $\boldsymbol{\sigma^2_{\overline{Z}_n(\alpha)} \equiv \mathrm{Var}\left[\overline{Z}_n(\alpha)\right] = \mathrm{Var}[Z_i(\alpha)]/n}$ **,** following from IID property of the $Z_i(\alpha)$. However,

$$\mathrm{Var}[Z_i(\alpha)] = \mathrm{Var}[X_i] - 2\alpha \mathrm{Cov}[X_i, Y_i] + \alpha^2 \mathrm{Var}[Y_i].$$

So, the optimal parameter α^* to minimize $\mathrm{Var}[Z_i(\alpha)]$ is

$$\boldsymbol{\alpha^* = \mathrm{Cov}[X_i, Y_i]/\mathrm{Var}[Y_i].} \tag{8}$$

Using this optimal parameter α^*,

$$\mathrm{Var}[Z_i^*] \equiv \mathrm{Var}[Z_i(\alpha^*)] = \left(1 - \rho^2_{X_i,Y_i}\right) \mathrm{Var}[X_i],$$

where ρ_{X_i,Y_i} is the correlation coefficient between X_i and Y_i. We also know that

$$\mathrm{Var}[X_i] = \frac{1}{2}\left(1 + \rho_{\widehat{\mathcal{C}}_i^{(BS)}, \widehat{\mathcal{C}}_i^{(aBS)}}\right) \mathrm{Var}\left[\widehat{\mathcal{C}}_i^{(BS)}\right]$$

because $\mathrm{Var}\left[\widehat{\mathcal{C}}_i^{(aBS)}\right] = \mathrm{Var}\left[\widehat{\mathcal{C}}_i^{(BS)}\right]$.

4.4 Estimation of Optimal Parameter α^*:

In general, the parameter α^* is not known exactly, so estimation is needed along with the following results.

Lemma 4.1:

$$\mathrm{Var}\left[e^{\widehat{\mathcal{S}}_i}+e^{\widehat{\mathcal{S}}_i^{(a)}}\right]=2\left(e^{\lambda T\hat{J}}-2e^{2\lambda T\bar{J}}+e^{\lambda T(e^{a+b}-1)}\right),$$

where $\hat{J} = (\exp(2b) - \exp(2a))/(2(b-a))-1$ and $\bar{J} = (\exp(b) - \exp(a))/(b-a)-1$.

Proof: Follows from properties of the antithetic pair $\left(\widehat{\mathcal{S}}_i,\widehat{\mathcal{S}}_i^{(a)}\right)$.

Lemma 4.2: An unbiased estimator for α^* is

$$\widehat{\alpha} = \frac{n}{n-1}\frac{\overline{XY}_n - \overline{X}_n\overline{Y}_n}{\sigma_Y^2}, \qquad (9)$$

where $\overline{X}_n = \sum_{i=1}^{n} X_i/n$ is the sample mean, simlarly for $\overline{XY}_n$ and $\overline{Y}_n$.

Proof: Basically, the condition for an unbiased estimate $E[\widehat{\alpha}] = \alpha^*$ can be shown to be true.

4.4 Continued: Estimation of α^*:

Since $\widehat{\alpha}$ depends on Y_i for $i = 1:n$, the estimate $\widehat{\alpha}$ of α^* introduces a bias into the estimate of control adjusted payoff

$$\widehat{Z}_n \equiv \frac{1}{n}\sum_{i=1}^{n} X_i - \widehat{\alpha}\left(\frac{1}{n}\sum_{i=1}^{n} Y_i - e^{\lambda T \bar{J}}\right). \qquad (10)$$

Theorem 4.3: The estimate $\widehat{Z}_n$ of $\mathcal{C}(S_0, T)$ has bias

$$\mathcal{B}_n \equiv \mathrm{E}[\widehat{Z}_n] - \mathcal{C}(S_0, T) = \mathrm{Cov}[X, (2\mu_Y - Y])Y]]/(n\sigma_Y^2) = O(1/n),$$

where $\mu_Y = \mathrm{E}[Y_i] = \exp(\lambda T \bar{J})$, $\sigma_Y^2 = \mathrm{Var}[Y_i]$, Y has same distribution as Y_i, for $i = 1{:}n$.

Proof: This follows from the IID property of $\{X_i Y_i\}$.

Remark: The corrected unbiased estimate to $\widehat{Z}_n$ is $\widehat{\mathcal{Z}}_n = \widehat{Z}_n - \widehat{\mathcal{B}}_n$, where

$$\widehat{\mathcal{B}}_n = \frac{1}{n-1}\,\frac{\overline{XY'_n} - \overline{X}_n \overline{Y'}_n}{\sigma_Y^2}\,, \qquad (11)$$

while $Y_i' = Y_i(2\mu_Y - Y_i)$, $\overline{XY'}_n$, $\overline{X}_n$ and $\overline{Y'}_n$ are sample means.

4.5 Monte Carlo Algorithm:

Finally, our Monte Carlo algorithm with antithetic and control variates (***ACV Monte Carlo***) variance reduction techniques :

for $i = 1\!:\!n$

  Randomly generate N_i by Inverse Transform Method;

  Randomly generate IID $U_{i,j}$, $j = 1\!:\!N_i$;

  Set $\widehat{\mathcal{S}}_i = aN_i + (b-a)\sum_{j=1}^{N_i} U_{i,j}$;

  Set $\widehat{\mathcal{S}}_i^{(a)} = (a+b)N_i - \widehat{\mathcal{S}}_i$;

  Set $\mathcal{C}_i^{(BS)} = \mathcal{C}^{(BS)}\left(S_0 \exp\left(\widehat{\mathcal{S}}_i - \lambda T\bar{J}\right), T\right)$;

  Set $\mathcal{C}_i^{(aBS)} = \mathcal{C}^{(BS)}\left(S_0 \exp\left(\widehat{\mathcal{S}}_i^{(a)} - \lambda T\bar{J}\right), T\right)$;

  Set $X_i = 0.5\left(\mathcal{C}_i^{(BS)} + \mathcal{C}_i^{(aBS)}\right)$;

  Set $Y_i = 0.5\left(\exp(\widehat{\mathcal{S}}_i) + \exp\left(\widehat{\mathcal{S}}_i^{(a)}\right)\right)$;

end %for i

Compute $\widehat{\alpha}$ according to (9);

Set $\widehat{Z}_n = \frac{1}{n}\sum_{i=1}^{n} X_i - \widehat{\alpha}(\frac{1}{n}\sum_{i=1}^{n} Y_i - e^{\lambda T\bar{J}})$;

Estimate bias $\widehat{\mathcal{B}}_n$ according to (11);

Get European call $\widehat{\mathcal{Z}}_n = \widehat{Z}_n - \widehat{\mathcal{B}}_n$;

Get European put $\widehat{\mathcal{P}}$ by (3).

5. *Monte Carlo Simulation Results*

Table 1: ***Elementary (no ACV) Monte Carlo Results*** (see Discussion)

σ	K/S_0	$\mathcal{C}$	$\mathcal{P}$	ϵ	t (sec.)	$\epsilon\sqrt{t}$
0.2	0.9	13.76	0.67	0.055	2.640	0.090
	1.0	5.26	3.28	0.035	2.578	0.056
	1.1	1.38	8.49	0.014	2.562	0.022
0.4	0.9	15.99	2.90	0.048	2.562	0.077
	1.0	8.45	6.47	0.033	2.578	0.053
	1.1	4.07	11.18	0.020	2.531	0.032
0.6	0.9	19.15	6.03	0.044	2.454	0.069
	1.0	11.79	9.81	0.033	2.500	0.052
	1.1	7.09	14.21	0.023	2.500	0.036

Option parameters: $K = 100$, $r = 0.1$, $T = 0.2$, $\lambda = 64$, $a = -0.028$, $b = 0.026$. Simulation count $n = 10,000$. Here, $\epsilon = \sigma_{\widehat{C}_n} = \sigma^{(BS)}/\sqrt{n}$.

Table 2: ***Improved ACV Monte Carlo Results*** (see Discussion)

σ	K/S_0	$\mathcal{C}$	$\mathcal{P}$	ϵ	t (sec.)	$\epsilon\sqrt{t}$
0.2	0.9	13.73	0.64	0.004	6.875	0.011
	1.0	5.23	3.25	0.008	6.828	0.021
	1.1	1.38	8.49	0.006	6.781	0.016
0.4	0.9	16.03	2.94	0.004	7.031	0.011
	1.0	8.42	6.44	0.004	6.922	0.011
	1.1	4.06	11.17	0.004	7.218	0.011
0.6	0.9	19.11	6.02	0.003	6.797	0.008
	1.0	11.81	9.83	0.003	6.859	0.008
	1.1	7.12	14.23	0.003	6.812	0.008

Option parameters: $K = 100$, $r = 0.1$, $T = 0.2$, $\lambda = 64$, $a = -0.028$, $b = 0.026$. Simulation count $n = 10,000$. Here, $\epsilon = \sigma_{\widehat{Z}_n} = \sigma_Z/\sqrt{n}$.

5. Continued: Discussion of Table 1 and 2 Results:

- Table 1 and Table 2 show that the ***ACV Monte Carlo reduces the standard error*** ϵ by a factor ranging from 2 to about 14, but ***increases the computing time*** by 2 to 3 times. However, a better benchmark trade-off is $\epsilon\sqrt{t}$, (Boyle, Broadie and Glasserman (1997)).

- Results show that the ***European call $\mathcal{C}$*** option price is an ***increasing function of initial stock price***, S_0 and the ***European put $\mathcal{P}$*** is a ***decreasing function*** of S_0.

- Both the ***call $\mathcal{C}$ and put $\mathcal{P}$ option prices increase with volatility*** σ.

- The estimated model parameters used for following Table 3 are $\mu = 0.1626$, $\sigma = 0.1074$, $\lambda = 64.16$, $a = -0.028$, $b = 0.026$ from our double-unform distribution paper (Zhu and Hanson (2005)) to compute the Standard & Poor 500 index option prices.

Table 3: ***Comparison of Option Prices by ACV Monte Carlo*** (see Discussion)

$\frac{K}{S_0}$	$\mathcal{C}$	$\mathcal{P}$	ϵ	$\mathcal{C}^{(\mathcal{BS})}$	$\mathcal{P}^{(\mathcal{BS})}$	$\mathcal{C}^*$	$\mathcal{P}^*$
0.8	269.81	0.01	2.e-3	269.80	2.e-6	269.82	0.02
0.9	132.36	1.45	0.03	130.98	0.07	132.39	1.47
1.0	40.07	20.27	0.11	30.49	10.69	40.05	20.25
1.1	5.49	76.60	0.06	1.13	72.24	5.50	76.61
1.2	0.31	147.17	0.01	4.e-3	146.87	0.32	147.19

Option parameters: $K = 1000$, $r = 0.1$, $T = 0.2$. **S&P 500 estimated parameters:** $\sigma = 0.1074$, $\lambda = 64$, $a = -0.028$, $b = 0.026$. Simulation count $n = 10,000$. Here, $\epsilon = \sigma_{\widehat{Z}_n} = \sigma_Z/\sqrt{n}$. The $\mathcal{C}^*$ and $\mathcal{P}^*$ values are obtained by more simulations, say $n = 400,000$ sample points, as a good approximation of the true values.

5. *Continued*^{2}*: Discussion of Table 3 Results:*

- Computation of ***Black-Scholes call price*** $\mathcal{C}^{(BS)}(S_0, T; K, \sigma^2, r)$ and ***put price*** $\mathcal{P}^{(BS)}(S_0, T; K, \sigma^2, r) = \mathcal{C}^{(BS)}(S_0, T; K, \sigma^2, r) + K\exp(-rT) - S_0$ only gives a ***rough estimation*** of the true values.

- Table 3 shows that the estimated ***call*** $\mathcal{C}$ ***and put*** $\mathcal{P}$ ***values by ACV Monte Carlo are within the*** 95% ***confidence interval*** of the ***approximate true*** call $\mathcal{C}^*$ and put $\mathcal{P}^*$ values, i.e., $\mathcal{C} \in [\mathcal{C}^* - 1.96\epsilon, \mathcal{C}^* + 1.96\epsilon]$ or $\mathcal{P} \in [\mathcal{P}^* - 1.96\epsilon, \mathcal{P}^* + 1.96\epsilon]$ by the central limit theorem.

- ***Theorem 5.1 Jump-Diffusion European option prices are bigger than Black-Scholes option prices***, independent of the Q-mark distribution, i.e., $\mathcal{C}(S_0, T; K, \sigma^2, r) \geq \mathcal{C}^{(BS)}(S_0, T; K, \sigma^2, r)$, and $\mathcal{P}(S_0, T; K, \sigma^2, r) \geq \mathcal{P}^{(BS)}(S_0, T; K, \sigma^2, r)$.

6. Conclusions

- *Formulated a Risk-Neutral SDE appropriate for Compound Jump-Diffusions, with emphasis on log-uniformly distributed jump-amplitudes*.
- *Antithetic and Control Variates (ACV) Monte Carlo is a significant improvement of the elementary (no ACV) Monte Carlo lacking these variance reduction techniques*.
- *Jump-diffusion option prices are bigger than pure-diffusion Black-Scholes option prices*.
- *Framework of ACV Monte Carlo option pricing algorithm is quite general and can easily be applied to other jump-diffusion models*.