*Optimal Portfolio Application with Double-Uniform Jump Model**

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Overview

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1. Introduction

1.1 Background:

- ***Merton's 1971 pioneering J.E.T. paper on the optimal portfolio and consumption problem*** for geometric diffusions used HARA (hyperbolic absolute risk-aversion) utility. However, there were errors, in particular with the bankruptcy boundary conditions and vanishing consumption, some errors were due to the HARA model. See also Merton's 1969 lifetime portfolio paper in R.E.&S.

- ***Merton's optimal portfolio errors*** are throughly discussed in the seminal collection of papers with coauthors in Sethi's bankruptcy book in 1997. See his introduction, the KLS(ethi)S M.O.R. 1986 paper and the J.E.T. 1988 paper with Taksar.

1.2 Market Jump Properties:

- ***Statistical evidence*** that jumps are significant in financial markets:
  - Stock and Option Prices in Ball and Torous ('85);
  - Capital Asset Pricing Model in Jarrow and Rosenfeld ('84);
  - Foreign Exchange and Stocks in Jorion ('89).
- Log-return market distributions usually ***skewed negative***, $\eta_3 \equiv M_3/(M_2)^{1.5} < 0$, if data time interval sufficiently long, compared to the skew-less normal distribution.
- Log-return market distributions usually ***leptokurtic***, $\eta_4 \equiv M_4/(M_2)^2 > 3$, i.e., more peaked than normal.
- Log-return market distribution have ***fatter or heavier tails*** than the normal distribution's exponentially small tails.
- ***Time-dependence*** of rate coefficients is important, i.e., non-constant coefficients are important; and stochastic volatility.

1.3 Jump-Diffusion Models:

- ***Merton (J.F.E., 1976) in his pioneering jump-diffusion option pricing model*** used IID log-normally distributed jump-amplitudes with a compound Poisson process. Other authors have also used the normal jump-applitude model.
- ***Kou*** (Mgt.Sci. 2002, and 2004 with Wang) used the IID ***log-double-exponential (Laplace)*** for option pricing.
- ***Hanson and Westman*** (2001-2004) have a number of optimal portfolio papers using various log-return jump-amplitude distributions such as ***discrete, normal and uniform distributions***.
- ***Jump-diffusions give skewness and excess-leptokurtosis*** to market distributions.

1.5 Jump Considerations:

- ***Extreme jumps*** in the market are ***relatively rare (statistical outliers)*** among the large number of daily fluctuations.
- Aït-Sahalia (J.F.E., 2004) shows ***difficulty in separating the jumps from the diffusion*** by the usual maximum likelihood methods.
- ***NYSE have had circuit breakers installed*** since 1988 to suppress extreme market changes, like in the 1987 crash.
- ***Uniform jump-amplitudes*** have the ***fattest of tails and finite range***, consistent with circuit breakers and parsimony.
- ***Bankruptcy conditions*** also need to be considered for the ***jump-integrals of the jump-diffusion PIDE*** as we shall see for the optimal portfolio problem; unlike the option pricing problem.

2.0 Log-Return Double-Uniform Amplitude Density:

- ***Linear Stochastic Differential Equation (SDE)***:

$$dS(t) = S(t)(\mu_d(t)dt + \sigma_d(t)dG(t)) + \sum_{k=1}^{dP(t)} S(T_k^-)J(T_k^- Q_k), \quad (1)$$

where $S(0) = S_0 > 0$ and

  - $\mu_d(t) =$ ***expected rate of return*** in absence of asset jumps, i.e., diffusive drift;
  - $\sigma_d(t) =$ ***diffusive volatility*** (standard deviation);
  - $G(t) =$ ***Brownian motion or diffusion process***, normally distributed such that $\mathrm{E}[dG(t)] = 0$ and $\mathrm{Var}[dG(t)] = dt$;
  - $P(t) =$ ***Poisson jump counting process***, Poisson distributed such that $\mathrm{E}[dP(t)] = \lambda(t)dt = \mathrm{Var}[dP(t)]$;

2.0 Continued: Stock Price Dynamics:

- $J(t, Q)$ = ***Poisson jump-amplitude*** with underlying ***random mark variable Q***, selected for log-return so that $Q = \ln(J(t, Q) + 1)$, such that $J(t, Q) > -1$;
- T_k^- is the ***pre-jump time*** and Q_k is an independent and identically distributed ***(IID) mark*** realization at the kth jump;
- The ***processes $G(t)$ and $P(t)$ along with Q_k are independent***, except that Q_k is conditioned on a jump-event at T_k.

2.1 Double-Uniform Probability Jump-Amplitude Q Mark Distribution:

$$\Phi_Q(q;t) = p_1(t)\frac{q-a(t)}{|a|(t)} I_{\{a(t)\le q<0\}} + \left(p_1(t) + p_2(t)\frac{q}{b(t)}\right) I_{\{0\le q<b(t)\}}$$
$$+ I_{\{b\le q<\infty\}}, \quad p_1(t)+p_2(t)=1, \quad a(t)<0<b(t),$$

- ***Mark Mean:*** $\mu_j(t) \equiv \mathrm{E}_Q[Q] = (p_1(t)a(t)+p_2(t)b(t))/2;$
- ***Mark Variance:***
  $\sigma_j^2(t) \equiv \mathrm{Var}_Q[Q] = (p_1(t)a^2(t)+p_2(t)b^2(t))/3 - \mu_j^2(t);$
- ***Mark Higher Central Moments:***

$$\begin{aligned} M_3^{(\mathrm{duq})}(t) &\equiv \mathrm{E}_Q\big[(Q-\mu_j(t))^3\big] \\ &= (p_1(t)a^3(t)+p_2(t)b^3(t))/4 - \mu_j(t)(3\sigma_j^2(t)+\mu_j^2(t)) \end{aligned}$$

$$\begin{aligned} M_4^{(\mathrm{duq})}(t) &\equiv \mathrm{E}_Q\big[(Q-\mu_j(t))^4\big] = (p_1(t)a^4(t)+p_2(t)b^4(t))/5 \\ &\quad -4\mu_j(t)M_3^{(\mathrm{duq})}(t) - 6\mu_j^2(t)\sigma_j^2(t) - \mu_j^4(t). \end{aligned}$$

- ***More motivation:*** Double-uniform distribution unlinks the different behaviors in crashes and rallies.

2.2 *Log-Return* $\ln(S(t))$ *S∆E:*

- According to a discrete form of ***Itô's stochastic chain rule*** for jump-diffusions

$$\begin{aligned}\Delta \ln(S(t)) \equiv\ & \ln(S(t+\Delta t)) - \ln(S(t)) \\ \simeq\ & (\mu_{ld}(t) + \lambda(t)\mu_j(t))\Delta t + \sigma_d(t)\Delta G(t) \\ & +\mu_j(t)(\Delta P(t) - \lambda(t)\Delta t) + \textstyle\sum_{k=1}^{\Delta P(t)}(Q_k - \mu_j(t)),\end{aligned}$$

separated into convenient zero-mean stochastic terms, where $\mu_{ld}(t) \equiv \mu_d(t) - \sigma_d^2(t)/2$ and $0 < \Delta t \ll 1$.

- Some Moments on $\boldsymbol{\Delta \ln(S(t))}$:

$$M_1^{(\mathrm{dujd})}(t) \equiv\ \mathrm{E}[\Delta \ln(S(t))] = (\mu_{ld}(t) + \lambda(t)\mu_j(t))\Delta t,$$

$$M_2^{(\mathrm{dujd})}(t) \equiv\ \mathrm{Var}[\Delta \ln(S(t))] = \left(\sigma_d^2(t) + \lambda(t)\left(\mu_j^2(t) + \sigma_j^2(t)\right)\right)\Delta t,$$

$$\begin{aligned}M_3^{(\mathrm{dujd})}(t)\ &\equiv \mathrm{E}\left[\left(\Delta[\ln(S(t))] - M_1^{(\mathrm{dujd})}(t)\right)^3\right] \\ &= (p_1(t)a^3(t) + p_2(t)b^3(t))\lambda(t)\Delta t/4,\end{aligned}$$

2.3 Log-Return Double-Uniform Probability Density

Theorem 2.3 Let

$$\Delta\ln(S(t)) = \mathcal{G}(t) + \sum_{k=1}^{\Delta P(t)} Q_k$$

where $\mathcal{G}(t) \equiv \mu_{ld}(t)\Delta t + \sigma_d \Delta G(t)$ is the Gaussian term.
Then the probability density of $\Delta\ln(S(t))$ is

$$\begin{aligned} \phi^{(\mathrm{dujd})}_{\Delta\ln(S(t))}(x) &\simeq \sum_{k=0}^{\infty} p_k(\lambda(t)\Delta t)\phi^{(\mathrm{dujd})}_{\mathcal{G}(t)+\sum_{i=1}^{k} Q_i}(x) \\ &\equiv \sum_{k=0}^{\infty} p_k(\lambda(t)\Delta t)\phi^{(\mathrm{dujd})}_k(x), \end{aligned}$$

for sufficiently small Δt and $-\infty < x < +\infty$, where $p_k(\lambda(t)\Delta t)$ is the Poisson distribution with parameter $\lambda(t)\Delta t$ and the multiple-convolution, Poisson distribution coefficients are

$$\phi^{(\mathrm{dujd})}_k(x) = \left(\phi_{\mathcal{G}(t)} \prod_{i=1}^{k} (*\phi_{Q_i})\right)(x).$$

2.3 Some theorem special details:

In the case of the corresponding ***normalized second order approximation***,

$$\phi^{(\mathrm{dujd},2)}_{\Delta \ln(S(t))}(x) = \sum_{k=0}^{2} p_k(\lambda(t)\Delta t)\phi^{(\mathrm{dujd})}_k(x) \Big/ \sum_{k=0}^{2} p_k(\lambda(t)\Delta t)\ ,$$

where the density coefficients are given by

$$\phi^{(\mathrm{dujd})}_0(x) = \phi^{(n)}\left(x;\mu,\sigma^2\right)\ ,$$

for $k = 0$, where $\phi^{(n)}\left(x;\mu,\sigma^2\right)$ is the normal distribution with mean μ and variance σ^2, while here $\left(\mu,\sigma^2\right) = \left(\mu_{ld}(t),\sigma^2_d(t)\right)\Delta t$, for $k = 1$,

$$\begin{aligned}\phi^{(\mathrm{dujd})}_1(x) = \quad & +\tfrac{p_1(t)}{|a|(t)}\Phi^{(n)}\left(a(t),0;x-\mu,\sigma^2\right)\\ & +\tfrac{p_2(t)}{b(t)}\Phi^{(n)}\left(0,b(t);x-\mu,\sigma^2\right)\ ,\end{aligned}$$

where $\Phi^{(n)}\left(a,b;\mu,\sigma^2\right)$ is the normal distribution on (a,b) with density $\phi^{(n)}\left(x;\mu,\sigma^2\right)$, and for $k = 2$, see the Zhu and Hanson in the 2006 Sethi volume for $\phi^{(\mathrm{dujd})}_2(x)$ since the formula and proof are too long to present here.

2.4 Jump-Diffusion Parameter Estimation:

- ***S&P 500 Index Data:*** from 1988 to 2003 with $n^{(\mathrm{sp})} = 4036$ daily closings, so $\Delta[\ln(SP_i)] \equiv \ln(SP_{i+1}) - \ln(SP_i)$ for $i = 1\!:\!n^{(\mathrm{sp})}-1$ discrete log-returns.
- ***Basic Statistics:*** $M_1^{(\mathrm{sp})} \simeq 3.640 \times 10^{-4}$, $M_2^{(\mathrm{sp})} \simeq 1.075 \times 10^{-4}$,
  $\eta_3^{(\mathrm{sp})} \equiv M_3^{(\mathrm{sp})}/(M_2^{(\mathrm{sp})})^{1.5} \simeq -0.1952 < 0$,
  $\eta_4^{(\mathrm{sp})} \equiv M_4^{(\mathrm{sp})}/(M_2^{(\mathrm{sp})})^{2} \simeq 6.974 > 3$.
- ***Yearly Partitioning:*** $\Delta[\ln(SP^{(\mathrm{spy})}_{j_y,k})]$ for $k = 1\!:\!n^{(\mathrm{sp})}_{y,j_y}$ data points per year for $j_y = 1\!:\!16$ years.
- ***Six-Dimensional Parameter Space:*** Given $\Delta T_{j_y} \simeq 1/252$ years/day,

$$\mathbf{y}_{j_y} = \left(\mu_{ld,j_y}, \sigma^2_{d,j_y}, a_{j_y}, b_{j_y}, p_{1,j_y}, \lambda_{j_y}\right).$$

- ***Maximum Likelihood Objective:***

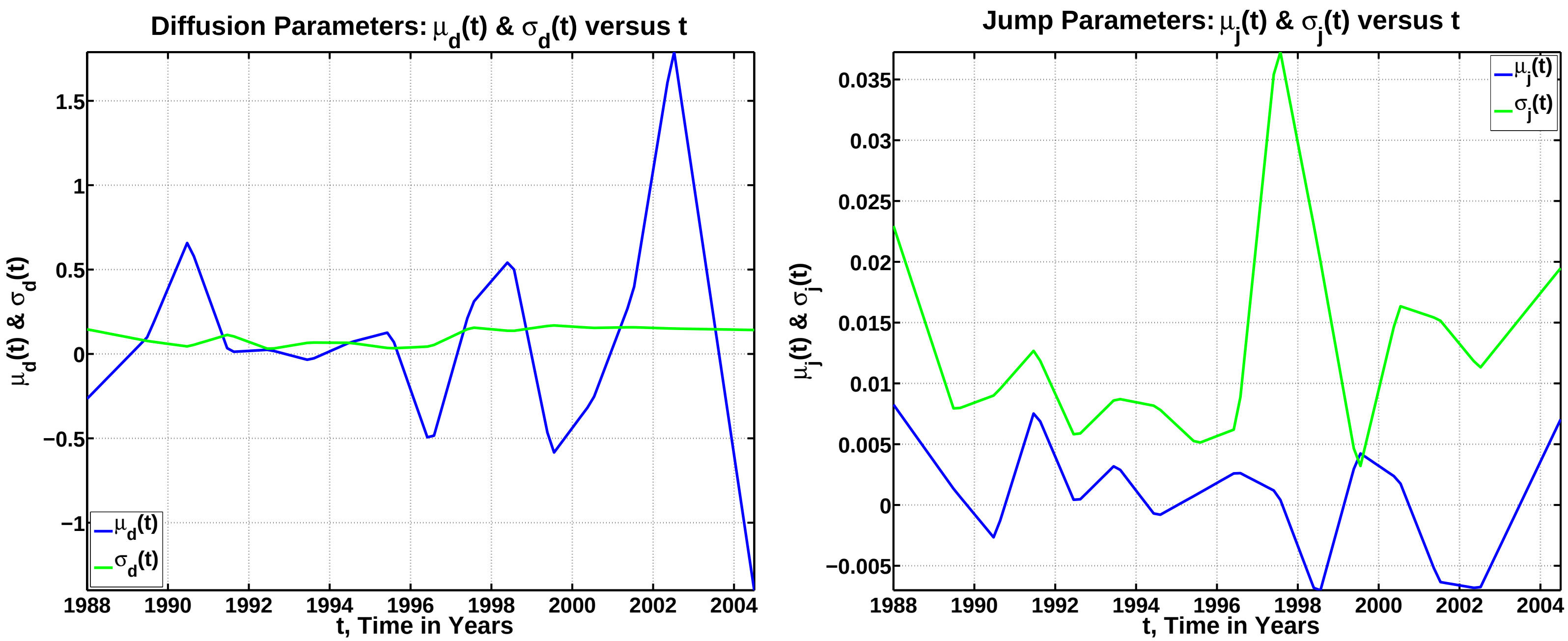
$$f(\mathbf{y}_{j_y}) = -\sum_{k=1}^{n^{(\mathrm{sp})}_{y,j_y}} \log\left(\phi^{(\mathrm{dujd},2)}_{\Delta\ln(S(t))}(x_k;\mathbf{y}_{j_y})\right).$$

2.5 Computational Procedures:

- ***Optimization Techniques: Nelder-Mead down-hill simplex method*** using the `fminsearch` function implementation of MATLAB™, needing only one new function evaluation for each successive step to test for best new search direction.
- ***Constraint Techniques: Barrier techniques*** used to enforce $\sigma^2_{d,j_y} > 0$, $a_{j_y} < 0$, $b_{j_y} > 0$, $p_{1,j_y} \in [0,1)$ and $\lambda_{j_y} > 0$.
- ***Some Average Values:***

$$(\mu_d, \sigma_d, \mu_j, \sigma_j) \simeq \left(0.17, 0.10, 3.1 \times 10^{-4}, 8.6 \times 10^{-3}\right).$$

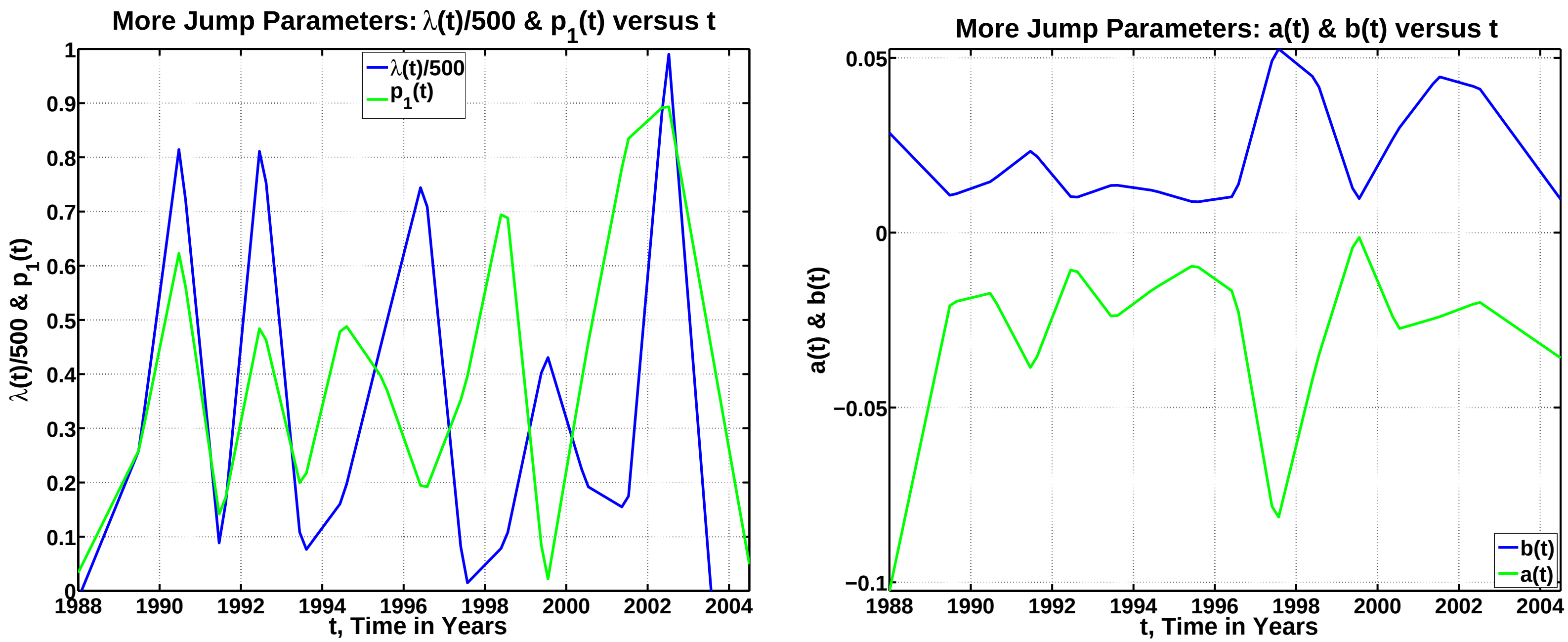
2.6 Results for $(\mu_d(t), \sigma_d(t))$ and $(\mu_j(t), \sigma_j(t))$:



(a) Diffusion parameters: $\mu_d(t)$ & $\sigma_d(t)$. (b) Jump parameters: $\mu_j(t)$ & $\sigma_j(t)$.

Figure 1: Jump-diffusion mean and variance parameters, $(\mu_d(t), \sigma_d(t))$ and $(\mu_j(t), \sigma_j(t))$ on $t \in [1988, 2004.5]$, represented as piecewise linear interpolation of yearly averages assigned to the mid-year.

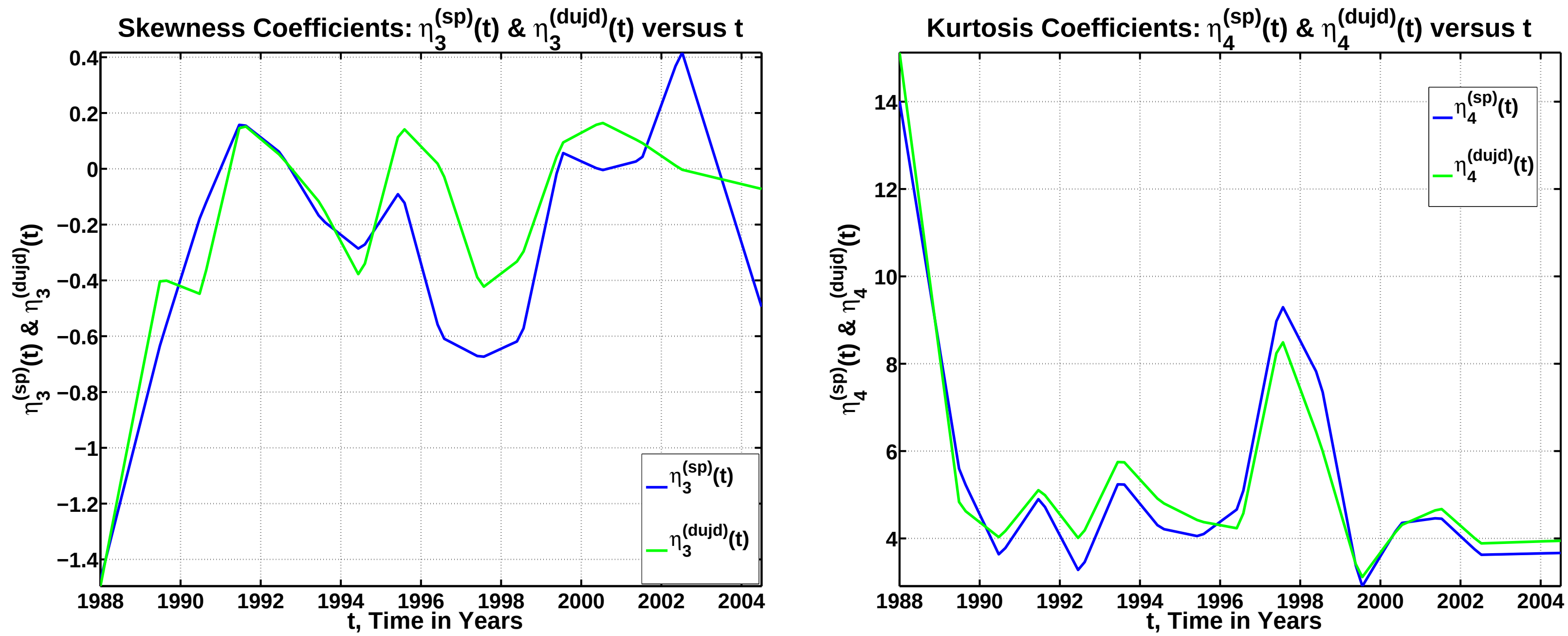
2.7 Results for $(\boldsymbol{\lambda(t)/500}, \boldsymbol{p_1(t)})$ and $(\boldsymbol{a(t)}, \boldsymbol{b(t)})$:



(a) Jump parameters: $\lambda(t)/500$ & $p_1(t)$. (b) Jump parameters: $a(t)$ & $b(t)$.

Figure 2: More jump parameters, $(\lambda(t)/500,\ p_1(t)\ )$ and $(a(t),\ b(t))$ on $t \in [1988, 2004.5]$, represented as piecewise linear interpolation of yearly averages assigned to the mid-year.

2.8 Results for Skewness and Kurtosis Coefficients:



(a) Skewness coefficients: $\eta_3^{(\mathrm{sp})}$ & $\eta_3^{(\mathrm{dujd})}$. (b) Kurtosis coefficients : $\eta_4^{(\mathrm{sp})}$ & $\eta_4^{(\mathrm{dujd})}$.

Figure 3: Comparison of skewness and kurtosis coefficients for both the S&P500 data and the estimated double-uniform jump diffusion values on $t \in [1988, 2004.5]$, represented as piecewise linear interpolation of yearly averages assigned to the mid-year.

3.0 Optimal Portfolio and Consumption Policies:

- ***Portfolio:*** Riskless asset or *bond* at price $B(t)$ and Risky asset or *stock* at price $S(t)$ (1), with instantaneous portfolio change fractions $U_0(t)$ and $U_1(t)$, respectively, such that $U_0(t) + U_1(t) = 1$.
- ***Exponential Bond Price Process:***
$$dB(t) = r(t)B(t)dt\ , \quad B(0) = B_0\ .$$
- ***Jump-diffusion Portfolio Wealth Process $W(t)$, Less Consumption $C(t)$:***
$$dW(t) = W(t)\Bigg(r(t)dt + U_1(t)\Bigg((\mu_d(t) - r(t))dt + \sigma_d(t)dG(t) + \sum_{k=1}^{dP(t)}\left(e^{Q_k} - 1\right)\Bigg)\Bigg) - C(t)dt\ , \tag{2}$$
subject to constraints $W(t) \geq 0$, $0 \leq C(t) \leq C_0^{(\max)}W(t)$ and $U_0^{(\min)} \leq U_1(t) \leq U_0^{(\max)}$, allowing shortselling ($U_0^{(\min)} < 0$) and borrowing ($U_0^{(\max)} > 1$) .

3.1 Portfolio Optimal Objective

- ***Portfolio Objective:***

$$
\begin{aligned}
v^*(t,w) = \max_{\{u,c\}} \Bigg[\mathbf{E} \Bigg[& e^{-\beta(t,t_f)} \mathcal{U}_f(W(t_f)) \\
& + \int_t^{t_f} e^{-\beta(t,s)} \mathcal{U}(C(s))\, ds \\
& \Bigg| W(t) = w, U_1(t) = u, C(t) = c \Bigg] \Bigg].
\end{aligned} \tag{3}
$$

- ***Cumulative Discount:*** $\beta(t,s) = \int_t^s \widehat{\beta}(\tau) d\tau$, where $\widehat{\beta}(t)$ is the instantaneous discount rate.
- ***Consumption and Final Wealth Utility Functions:*** $\mathcal{U}(c)$ and $\mathcal{U}_f(w)$ are bounded, strictly increasing and strictly concave.
- ***Variable Classes:*** State variable is w, while control variables are u and c.
- ***Final Condition:*** $v^*(t_f, w) = \mathcal{U}_f(w)$.

3.2 Absorbing Natural Boundary Condition:

Approaching bankruptcy $w \to 0^+$, so by consumption constraint $c \to 0^+$ and by the objective (3),

$$v^*\left(t, 0^+\right) = \mathcal{U}_f\left(0^+\right) e^{-\beta(t,t_f)} + \mathcal{U}\left(0^+\right) \int_t^{t_f} e^{-\beta(t,s)} ds. \tag{4}$$

This is the simple variant what Merton gave as a correction in his 1990 book for his 1971 optimal portfolio paper. However, KLAS(ethi)S 1986 and Sethi with Taksar 1988 pointed out that it was necessary to enforce the non-negativity of wealth and consumption. See also Sethi's 1997 bankruptcy book for a large collection of papers as well as excellent summaries by Markowitz and Sethi, including the 1986 and 1988 papers, for a much greater variety of optimal portfolio and consumption problems.

3.3 Portfolio Stochastic Dynamic Programming PIDE:

$$
\begin{aligned}
0 = \; & v_t^*(t,w) - \widehat{\beta}(t) v^*(t,w) + \mathcal{U}(c^*(t,w)) \\
& + \left[(r(t) + (\mu_d(t) - r(t))u^*(t,w))w - c^*(t,w)\right] v_w^*(t,w) \\
& + \tfrac{1}{2}\sigma_d^2(t)(u^*)^2(t,w) w^2 v_{ww}^*(t,w) \\
& + \lambda(t)\left(\frac{p_1(t)}{|a|(t)}\int_{a(t)}^{0} + \frac{p_2(t)}{b(t)}\int_0^{b(t)}\right) \\
& \cdot \left(v^*(t, \underline{(1+(e^q-1)u^*(t,w))w}) - v^*(t,w)\right) dq,
\end{aligned}
\tag{5}
$$

where $u^* = u^*(t,w) \in \left[U_0^{(\min)}, U_0^{(\max)}\right]$ and $c^* = c^*(t,w) \in \left[0, C_0^{(\max)} w\right]$ are the optimal controls if they exist, while $v_w^*(t,w)$ and $v_{ww}^*(t,w)$ are the continuous partial derivatives with respect to wealth w when $0 \leq t < t_f$. Note that $\underline{(1+(e^q-1)u^*(t,w))w}$ is a wealth argument.

3.4 Non-Negativity of Wealth and Jump Distribution:

Since $(1+(e^q-1)u^*(t,w))w$ is a wealth argument in (5), it must satisfy the wealth nonnegativity condition, so

$$\kappa(u,q) \equiv 1 + (e^q - 1)u \geq 0$$

on the support $[a(t), b(t)]$ of the jump-amplitude mark density $\phi_Q(q;t)$.

Lemma 1. ***Bounds on Optimal Stock Fraction due to Non-Negativity of Wealth Jump Argument:***
*If the support of $\phi_Q(q;t)$ is the **finite** interval $q \in [a(t), b(t)]$ with $a(t) < 0 < b(t)$, then $u^*(t,w)$ is restricted by (5) to*

$$\frac{-1}{J(t,b(t))} = \frac{-1}{e^{b(t)}-1} \leq u^*(t,w) \leq \frac{1}{1-e^{a(t)}} = \frac{-1}{J(t,a(t))}, \tag{6}$$

*but if the support of $\phi_Q(q)$ is fully **infinite**, i.e., $(-\infty, +\infty)$, then $u^*(t,w)$ is restricted by (5) to*

$$0 \leq u^*(t,w) \leq 1. \tag{7}$$

3.4 Remarks Continued: Non-Negativity of Wealth and Jump Distribution:

- ***Recall that u is the stock fraction, so that short-selling and borrowing will be overly restricted in the infinite support case (7) where $a(t) \to -\infty$ and $b(t) \to +\infty$, unlike the finite case (6) where $-\infty < a(t) < 0 < b(t) < +\infty$.***
- ***So, unlike option pricing, finite support of the mark density makes a big difference in the optimal portfolio and consumption problem!***
- ***Thus, it would not be practical to use either normally or double-exponentially distributed marks in the optimal portfolio and consumption problem with a bankruptcy condition.***
- If $[a_{\min}, b_{\max}] = [\min_t(a(t)), \max_t(b(t))]$, then the overall u^* range for the S&P500 data is

$$[u_{\min}, u_{\max}] = \left[\frac{-1}{(e^{b_{\max}} - 1)}, \frac{+1}{(1 - e^{a_{\min}})}\right] \simeq [-18, +12].$$

4.0 Unconstrained Optimal or Regular Control Policies:

In absence of control constraints and in presence of sufficient differentiability, the dual policy, implicit critical conditions are

- ***Regular Consumption $c^{(\mathbf{reg})}(t, w)$:***

$$\mathcal{U}'(c^{(\mathrm{reg})}(t,w)) = v_w^*(t,w). \tag{8}$$

- ***Regular Portfolio Fraction $u^{(\mathbf{reg})}(t, w)$:***

$$\begin{aligned} \sigma_d^2(t) w^2 v_{ww}^*(t,w) u^{(\mathrm{reg})}(t,w) &= -(\mu_d(t) - r(t)) w v_w^*(t,w) \\ -\lambda(t) w \left(\frac{p_1(t)}{|a|(t)} \int_{a(t)}^{0} + \frac{p_2(t)}{b(t)} \int_0^{b(t)} \right) (e^q - 1) \\ v_w^*(t, \kappa(u^{(\mathrm{reg})}(t,w), q) w) \, dq. \end{aligned} \tag{9}$$

4.1 Canonical Decomposition with CRRA Utilities:

- ***Constant Relative Risk-Aversion (CRRA $\subset$ HARA) Power Utilities:***

$$\mathcal{U}(x) = \mathcal{U}_f(x) = \frac{x^\gamma}{\gamma}, \;\; x \ge 0, \;\; 0 < \gamma < 1. \tag{10}$$

- $\Longleftarrow$ ***Relative Risk-Aversion (RRA):***

$$RRA(x) \equiv -\mathcal{U}''(x)/(\mathcal{U}'(x)/x) = (1-\gamma) > 0, \;\; \gamma < 1,$$

i.e., negative of ratio of marginal to average change in marginal utilility ($\mathcal{U}'(x) > 0$ & $\mathcal{U}''(x) < 0$) is a constant; the "risk-hating" singular utilities when $\gamma \le 0$ are excluded here.

- ***CRRA Canonical Separation of Variables:***

$$v^*(t,w) = \mathcal{U}(w)v_0(t), \quad v_0(t_f) = 1, \tag{11}$$

i.e., if valid, then wealth state dependence is known and only the time-dependent factor $v_0(t)$ need be determined.

4.2 Canonical Simplifications with CRRA Utilities:

- ***Regular Consumption Control is Linear in Wealth:***

$$c^{(\mathrm{reg})}(t,w) \equiv w \cdot c_0^{(\mathrm{reg})}(t) = w/v_0^{1/(1-\gamma)}(t), \tag{12}$$

with optimal consumption $c_0^*(t) = \max\left[\min\left[c_0^{(\mathrm{reg})}(t), C_0^{(\max)}\right], 0\right]$ per w.

- ***Regular Portfolio Fraction Control is Independent of Wealth:***

$$\begin{aligned} u^{(\mathrm{reg})}(t,w) \equiv\ & u_0^{(\mathrm{reg})}(t) \\ =\ & \frac{1}{(1-\gamma)\sigma_d^2(t)}\left[\mu_d(t) - r(t) + v\lambda(t)I_1\left(u_0^{(\mathrm{reg})}(t)\right)\right], \end{aligned} \tag{13}$$

in fixed point form and $u_0^*(t) = \max\left[\min\left[u_0^{(\mathrm{reg})}(t), U_0^{(\max)}\right], U_0^{(\max)}\right]$,
where $I_1(u) = \left(\frac{p_1(t)}{|a|(t)}\int_{a(t)}^{0} + \frac{p_2(t)}{b(t)}\int_0^{b(t)}\right)(e^q - 1)\kappa^{\gamma-1}(u,q)dq.$

4.3 CRRA Time-Dependent Component in Formal Bernoulli Equation:

$$0 = v_0'(t) + (1-\gamma)\left(g_1(t; u_0^*(t))v_0(t) + g_2(t)v_0^{\frac{\gamma}{\gamma-1}}(t)\right), \quad (14)$$

where

- ***Bernoulli Coefficients*** $g_1(t; u)$ ***and*** $g_2(t)$,
  $g_2(t) = g_2\left(t; c_0^*(t), c_0^{(\mathrm{reg})}(t)\right)$, introduce implicit nonlinear dependence on $u_0^*(t)$, $c_0^*(t)$ and $c_0^{(\mathrm{reg})}(t)$, so iterative approximations are required (Zhu and Hanson 2006).
- ***Formal (Implicit) Bernoulli Solution:***

$$v_0(t) = \left[e^{-\overline{g}_1(t;u_0^*(t))(t_f-t)}\left(1+\int_t^{t_f} g_2(\tau)e^{\overline{g}_1(t;u_0^*(t))(t_f-\tau)}d\tau\right)\right]^{1-\gamma},$$

where $\quad \overline{g}_1(t; u_0^*(t))(t_f - t) \equiv \int_t^{t_f} g_1(s; u_0^*(s))ds.$

5.1 Federal Funds Rates $r(t)$ and $\widehat{\beta}(t)$):

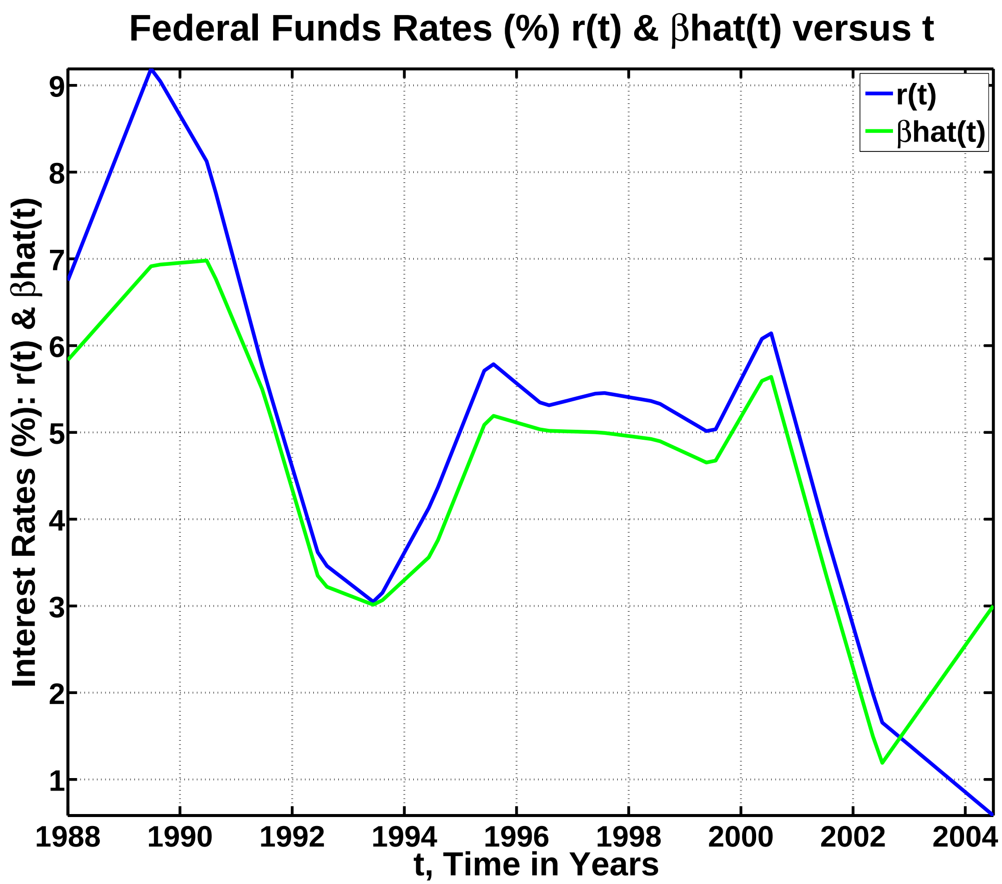
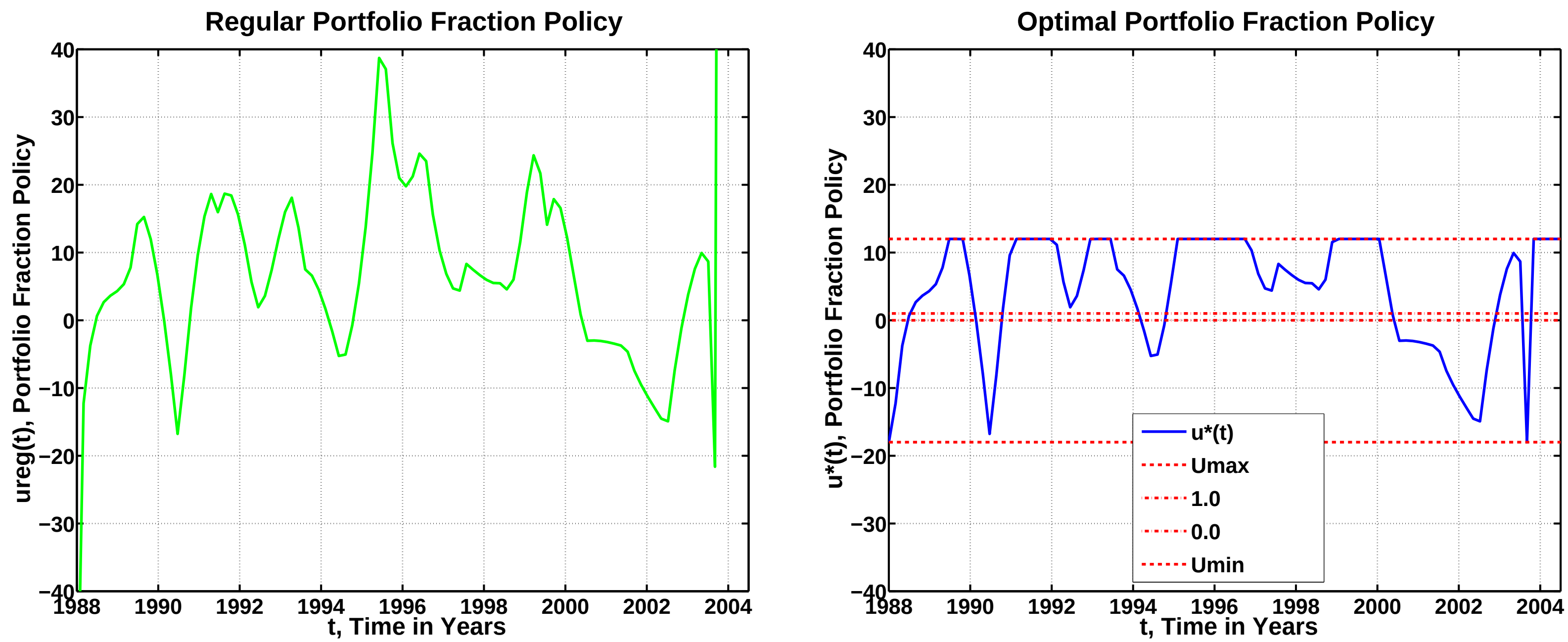


Figure 4: Federal funds rate (H.15-Historical Data) for interest $r(t)$ and discounting $\widehat{\beta}(t)$ on a daily bases, represented by piecewise linear interpolation with yearly averages assigned to the midpoint of each year for $t = 1988.5\!:\!2003.5$.

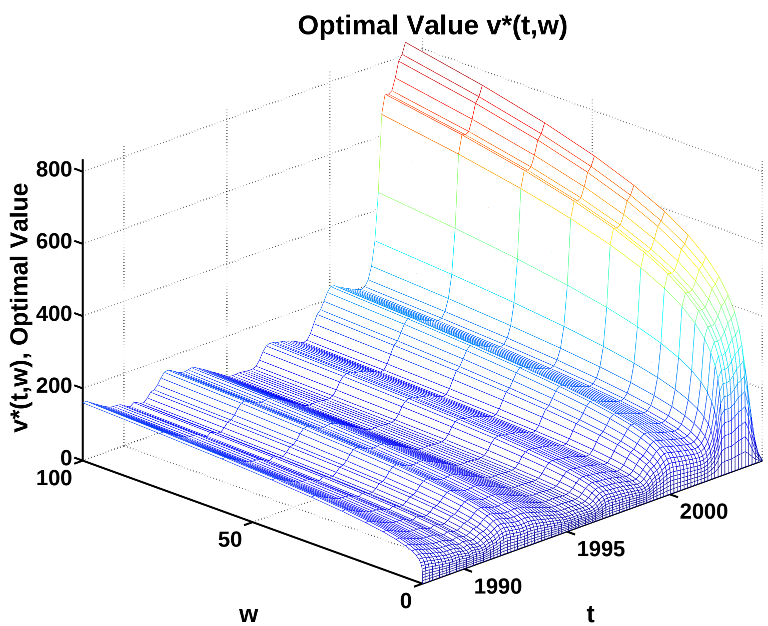
5.2 Results for Regular $u^{(\mathrm{reg})}(t)$ and Optimal $u^*(t)$ Portfolio Fraction Policies:



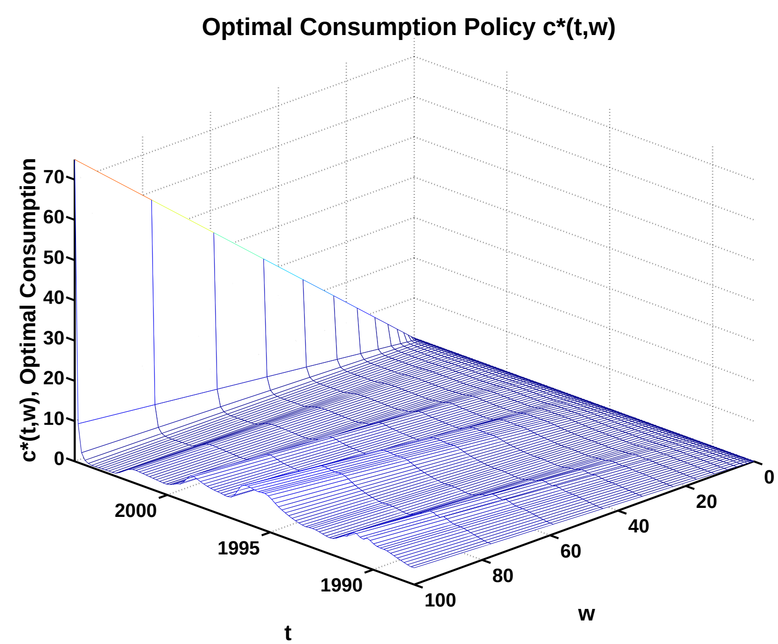
(a) Regular stock fraction policy $u^{(\mathrm{reg})}(t)$. (b) Optimal stock fraction policy, $u^*(t)$.

Figure 5: Regular and optimal portfolio stock fraction policies, $u^{(\mathrm{reg})}(t)$ and $u^*(t)$ on $t \in [1988, 2004.5]$, the latter subject to the control constraints set $[U^{(0)}_{\min}, U^{(0)}_{\max}] = [-18, 12]$.

5.3 Results for Optimal Valuel $v^*(t,w)$ and Optimal Consumption $c^*(t,w)$:



(a) Optimal portfolio value $v^*(t,w)$.



(b) Optimal consumption policy $c^*(t,w)$.

Figure 6: Optimal portfolio value $v^*(t,w)$ and optimal consumption policy $c^*(t,w)$ for $(t,w) \in [1988, 2004.5] \times [0, 100]$.

6. Conclusions

- Introduced ***log-double-uniform distribution of jump-amplitudes*** into jump-diffusion stock price models.
- Developed estimation of ***time-dependent jump-diffusion parameters*** for more realistic market models.
- Demonstrated significant effects on the variation of instantaneous stock fraction policy due to ***time-dependence of interest and discount rates***.
- Emphasized that double-uniform distribution is a ***reasonable assumption for rare, large jumps, crashes or buying-frenzies***.
- ***Showed jump-amplitude distributions with compact support are much less restricted on short-selling and borrowing in the optimal portfolio and consumption problem***.