Optimal Harvesting with Coupled Population and Price Dynamics

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Overview

1. Noninflationary, Deterministic Model.
2. Inflationary, Stochastic Control Model.
4. Numerical Results.
5. Conclusions.
Outline of Abstract

• Optimal Control of Stochastic Resource in Continuous Time.

• Model Effects of Large Random Price Fluctuations.

• Influence of Continuous growth and Jump Stochastic Noise.

• Computational Stochastic Dynamic Programming.

• Pronounced Effect of Inflationary Prices on Optimal Return.
Part 1. Noninflationary, Deterministic Model: 

Introduction.

1.1. Ordinary Differential Equation (ODE):

- Nonlinear (Logistic) Dynamics:
  \[ \frac{dX(s)}{ds} = [r_1X(s)(1 - X(s)/K) - H(s)] \quad 0 < t < s < T. \]

- Initial Conditions: \( X(0) = x_0 \); \( 0 < t < T \)

- State Variable (Resource Size): \( X(t) = [X_i(t)]_{1 \times 1} \)

- BiLinear Control-State Dynamics Assumption (for Resource Harvesting): \( H(t) = qU(t)X(t) \)

- \( q \) = Efficiency (Catchability) Coefficient;
• Control Variable (Harvesting Effort):

\[ U(t) = \left[ U_i\{X(t), t}\right]_{1 \times 1} , \quad U_{\min} \leq U(t) \leq U_{\max} < \infty ; \]

• Growth Parameters:

\( r_1 = \) Resource Intrinsic Growth Rate;
\( K = \) Environment Carrying (Saturation) Capacity.
1.2. Quadratic Performance Index:

\[ V(X, U, t) = \int_t^T e^{-\delta(s-t)} [pqU(s)X(s) - c(U(s))] \, ds , \]

where

- \( V(x, u, t) \) = Current Value of Future Resources (i.e., \( \exp(\delta t) \) times Present Value);
- \( T = \) Time Horizon \( (T \geq t) \);
- \( \delta = \) Nominal Discount Rate (NOT adjusted for inflation);
- \( p = \) Price of Resource per Unit Harvest Rate;
- \( c(u) = c_1 u + c_2 u^2 = \) Quadratic Costs (Assume Increasing, Convex Quadratic Costs: \( c_1 > 0 \) and \( c_2 > 0 \));
- Instantaneous Net Return: \( R(x, u) = pqux - c(u) \).
1.3. Deterministic Dynamic Programming:

- Optimization Goal = Maximize Total Return:

\[ v^*(x, t) = V(x, u^*, t) = \max_u [V(x, u, t)] ; \]

- PDE of Deterministic Dynamic Programming:

\[ v_t^*(x, t) + r_1 x(1 - x/K)v_x^*(x, t) - \delta v^*(x, t) + S^*(x, t) = 0; \]

- Control Switching Term:

\[ S^*(x, t) = \max_u \left[ \left( p - v_x^*(x, t) \right) q x - c_1 u - c_2 u^2 \right] ; \]

- Regular (Unconstrained) Control:

\[ u_R(x, t) = \frac{(p - v_x^*(x, t)) q x - c_1}{2 \cdot c_2} , \quad c_2 > 0; \]
• Optimal (Constrained) Control:

\[ u^*(x, t) = \begin{cases} 
  U_{\text{max}}, & U_{\text{max}} \leq u_R(x, t) \\
  u_R(x, t), & U_{\text{min}} \leq u_R(x, t) \leq U_{\text{max}} \\
  U_{\text{min}}, & u_R(x, t) \leq U_{\text{min}} 
\end{cases} \]

• Final Boundary Condition: \( v^*(x, T) = 0; \)

• Extinction Natural Boundary Condition:

\[ v^*(0, t) = -\frac{(c_1 + c_2 U_{\text{min}}) U_{\text{min}}}{\delta} \left(1 - e^{-\delta(T-t)}\right), \quad \delta > 0. \]
Part 2. Inflationary, Stochastic Control Model

2.1. Stochastic Dynamics Equation (SDE (1)):

- Nonlinear Dynamics with Gaussian (G) and Poisson (Z) Noise:

\[
\begin{align*}
\frac{dX(s)}{ds} &= [r_1X(s)(1 - X(s)/K) - H(s)] \, ds \\
&+ \sigma_1 X(s) \, dW_1(s) + X(s) \sum_{j=1}^{n} a_j \, dZ_j(s, f_j),
\end{align*}
\]

\[X(t) = x,\]

- Initial Conditions: \(X(0) = x_0\), \(t_0 < t < s < T\);

- Gaussian (Wiener) Noise (Zero Mean and Normalized):

\[E[dW_1(t)] = 0, \quad Var[dW_1(t)] = dt, \quad \sigma_1 \leq 0;\]
• Poisson (Jump) Noise:

\[ E[dZ_j(t, f_j)] = f_j dt , \quad \text{Var}[dZ_j(t, f_j)] = f_j dt , \]

1 \leq j \leq n, where \( f_j = \) Jump Rate and \( a_j = \) Jump Amplitude Coefficient \((-1 < a_j);\)

• Independent (Uncorrelated) Processes Assumption:

\[ \text{Cov}[dW_1(t), dZ_j(t, f_j)] = 0 , \]

\[ \text{Cov}[dZ_j(t, f_j), dZ_{j'}(t, f_{j'})] = \delta_{j,j'} f_j dt ; \]
2.2. Inflationary Factor Model:

- Nonlinear Supply–Demand Model Relation:
  \[ P(t) = \left( \frac{p_0}{H(t)} + p_1 \right) Y(t), \]
  * \( P(t) \cdot H(t) = \) Gross Return on Harvest;
  * \( p_0 = \) Supply–Demand Price Coefficient;
  * \( p_1 = \) Constant Price per Unit Harvest;
  * \( Y(t) = \) Fluctuating Inflationary Factor;

- Linear Fluctuating Inflationary Factor SDE (2):
  \[ dY(s) = r_2 Y(s) \, ds + \sigma_2 Y(s) \, dW_2(s) + Y(s) \sum_{j=1}^{m} b_j \, dQ_j(s; g_j), \]
  * \( Y(t) = y; \)
  * \( r_2 = \) Annual Rate of Inflation without Fluctuations;
  * \( g_j = j^{th} \) component of Inflationary Jump Rate;
  * \( b_j = j^{th} \) component of Jump Amplitude Coefficient;
• Inflationary Gaussian (Wiener) Noise:

\[ E[dW_2(t)] = 0 , \quad Var[dW_2(t)] = dt \quad \sigma_2 \leq 0 ; \]

• Inflationary Poisson (Jump) Noise:

\[ E[dQ_j(t, g_j)] = g_j dt , \quad Var[dQ_j(t, g_j)] = g_j dt , \quad 1 \leq j \leq m ; \]

• Independent (Uncorrelated) Processes Assumption:

\[ Cov[dW_2(t), dQ_j(t, g_j)] = 0 , \]

\[ Cov[dQ_j(t, g_j), dQ_{j'}(t, g_{j'})] = \delta_{j,j'} g_j dt ; \]
Figure 1: Pacific halibut prices in US dollars per kilogram for each year from 1935 to 1985 (Raw Data: IPHC 1984 and 1985 Annual Reports).
Figure 2: U.S.-Canadian catch in millions of kilograms for each year from 1935 to 1985 (Raw Data: IPHC 1984 and 1985 Annual Reports).
Figure 3: Pacific halibut price in US dollars per kilogram versus catch in millions of kilograms for years from 1935 to 1985. Linear regression for price times catch as a function of catch from 1980 to 1985 displayed as smooth hyperbolic price curve. (Raw Data: IPHC 1984 and 1985 Annual Reports).
2.3. Mean Quadratic Performance Index:

\[
\bar{V}(x, y, u, t) = E \left[ \int_t^T e^{-\delta(s-t)} [(p_0 + p_1 q U(s)X(s))]Y(s) \right.

\left. - c(U(s))] \, ds \mid X(t) = x, Y(t) = y, U(t) = u \right],
\]

- \( \bar{V}(x, y, u, t) \) = Expected Current Value of Future Resources (i.e., \( \exp(\delta t) \) times Present Value);
- \( \{x, y\} = 2\)-Dim State of Inflationary Stochastic Dynamics;
- \( T = \) Time Horizon \( (T \geq t) \);
- \( \delta = \) Nominal Discount Rate (NOT adjusted for inflation);
2.4. Stochastic Dynamic Programming:

- **Optimization Goal = Maximize Total Return:**
  \[ v^*(x, y, t) = \overline{V}(x, y, u^*, t) = \max_u [\overline{V}(x, y, u, t)] \]

- **PDE of Stochastic Dynamic Programming:**
  \[
  0 = v_t^*(x, y, t) + r_1 x (1 - x/K) v_x^*(x, y, t) - \delta v^*(x, y, t) \\
  + \frac{\sigma_1^2 x^2}{2} v_{xx}^* + \sum_j f_j [v^* ((1 + a_j)x, y, t) - v^*(x, y, t)] \\
  + r_2 y v_y^* + \frac{\sigma_2^2 y^2}{2} v_{yy}^* + \sum_j g_j [v^*(x, (1 + b_j)y, t) - v^*(x, y, t)] \\
  + S^*(x, y, t),
  \]
  by General Itô Chain Rule;

- **Control Switching Term:**
  \[
  S^*(x, y, t) = \max_u [p_0 y + (p_1 y - v_x^*(x, y, t)) q u x - c_1 u - c_2 u^2] ;
  \]
2.4.1. More Stochastic Dynamic Programming:

- Regular (Unconstrained) Control:
  \[ u_R(x, y, t) = \frac{(p_1y - v_x^*(x, y, t))qx - c_1}{2c_2}, \quad c_2 > 0; \]

- Optimal (Constrained) Control:
  \[
  u^*(x, y, t) = \begin{cases} 
  U_{\text{max}}, & U_{\text{max}} \leq u_R(x, y, t) \\
  u_R(x, y, t), & U_{\text{min}} \leq u_R(x, y, t) \leq U_{\text{max}} \\
  U_{\text{min}}, & u_R(x, y, t) \leq U_{\text{min}}
  \end{cases}
  \]

- Final Boundary Condition: \( v^*(x, y, T) = 0; \)

- Extinction Natural Boundary Condition*:
  \[
  v^*(0, 0, t) = -\frac{(c_1 + c_2U_{\text{min}})U_{\text{min}}}{\delta} \left(1 - e^{-\delta(T-t)}\right), \quad \delta > 0.
  \]

* see Kushner and Dupuis (1992) for proper handling of stochastic reflecting boundary conditions.
Part 3. Numerical Approximations

3.1 Basic Hybrid Numerical Procedures.

• Extrapolated, Predictor-Corrector for Nonlinear Iteration.

• Crank-Nicolson Implicit for 2nd Order in Time and State.

• Modifications for Poisson Functional Terms.

• Modifications for Optimization in Switching Term.
3.2. Numerical Discretizations:

- **State 1**: \( X_i \equiv (i - 1)\Delta x, \ i = 1, \cdots, N_x, \ \Delta x \equiv K/(N_x - 1); \)
- **State 2**: \( Y_j \equiv (j - 1)\Delta y, \ j = 1, \cdots, N_y, \ \Delta y \equiv e^{r_2 T}/(N_y - 1); \)
- **Time**: \( T_k \equiv T - (k - 1)\Delta t, \ k = 1, \cdots, N_t, \ \Delta t \equiv T/(N_t - 1); \)
- **Optimal Expected Value**: \( v^*(x_i, y_j, t_k) \longrightarrow V_{i,j,k}; \)
- **\( v^*_x(X_i, Y_j, T_k) \longrightarrow DVX_{i,j,k} \equiv 0.5(V_{i+1,j,k} - V_{i-1,j,k})/\Delta x; \)**
- **\( v^*_y(X_i, Y_j, T_k) \longrightarrow DVY_{i,j,k} \equiv 0.5(V_{i,j+1,k} - V_{i,j-1,k})/\Delta y; \)**
- **\( v^*_{xx}(X_i, Y_j, T_k) \longrightarrow DDVX_{i,j,k} \equiv \)**
  \[ (V_{i+1,j,k} - 2V_{i,j,k} + V_{i-1,j,k})/(\Delta x)^2; \]
- **\( v^*_{yy}(X_i, Y_j, T_k) \longrightarrow DDVY_{i,j,k} \equiv \)**
  \[ (V_{i,j+1,k} - 2V_{i,j,k} + V_{i,j-1,k})/(\Delta y)^2; \]
- **\( v^*_t(X_i, Y_j, T_{k+0.5}) \longrightarrow DVT_{i,j,k} \equiv -(V_{i,j,k+1} - V_{i,j,k})/\Delta t; \)**

with Error: \( O(\Delta x)^2 + O(\Delta y)^2 + O(\Delta t/2)^2; \)
3.2.1. More Numerical Discretizations:

- **X-Poisson Term:** \( v^*(1 + a_l)X_i, Y_j, T_k \rightarrow ZV_{i,j,k,l} \)
  by 2nd order accurate interpolation between nearest nodes;

- **Y-Poisson Term:** \( v^*(1 + b_l)Y_j, T_k \rightarrow QV_{i,j,k,l} \)
  by 2nd order accurate interpolation between nearest nodes;

- **Regular Control:**
  \[ u_R(X_i, Y_j, T_k) \rightarrow UR_{i,j,k} \equiv (p_1 Y_j - DVX_{i,j,k} \cdot q \cdot X_i - c_1)/(2c_2); \]

- **Optimal Control:**
  \[ u^*(X_i, Y_j, T_k) \rightarrow U_{i,j,k} \equiv \text{same as exact composite expression}; \]
3.3. Computational Stochastic Dynamic Programming:

For $k + 1 = 2$ to $N_t$ while $i = 1$ to $N_x$ & $j = 1$ to $N_y$:

- **Accelerating Extrapolating Start:**

  \[ \text{VE}_{i,j,k} \equiv 0.5(3V_{i,j,k}^{(c,*)} - V_{i,j,k-1}^{(c,*)}) \simeq V_{i,j,k+0.5}, \quad \text{if} \quad k \leq 2, \]

  which are used to get components $DVXE$, $DVYE$, $DDVXE$, $DDVYE$, $ZVE$, $QVE$, $URE$, $UE$ & $SE$, and where $V_{i,j,k}^{(c,*)}$ is the final correction from step $k$;

- **Extrapolated-Predictor Step:**

  \[
  V_{i,j,k+1}^{(p)} = V_{i,j,k}^{(c,*)} + \Delta t \left[ r_1 X_i (1 - X_i/K) DVXE_{i,j,k} 
  + \frac{1}{2} \sigma_1^2 X_i^2 DDVXE_{i,j,k} - \delta VE_{i,j,k} 
  + \Sigma_l f_l (ZVE_{i,j,k,l} - VE_{i,j,k}) 
  + r_2 Y_j DVYE_{i,j,k} + \frac{1}{2} \sigma_2^2 Y_j^2 DDVYE_{i,j,k} 
  + \Sigma_l g_l (QVE_{i,j,k,l} - VE_{i,j,k}) + SE_{i,j,k} \right],
  \]
• **Predictor Evaluation (Crank-Nicolson Midpoint):**

\[ VM^{(p)}_{i,j,k} \equiv 0.5(V_{i,j,k}^{(c,*)} + V_{i,j,k+1}^{(p)}) \simeq V_{i,j,k+0.5}, \]

which are used to get predicted components of \( DVXM, DVYM, DDVXM, DDVYM, ZVM, QVM, URM, UM & SM; \)
3.3.1 More Computational Dynamic Programming:

- \((L + 1)\)st Corrector Step:

\[
V_{i,j,k+1}^{(c,L+1)} = V_{i,j,k}^{(c,*)} + \Delta t \left[ r_1 x_i (1 - x_i / K) DVXM_{i,j,k}^{(c,L)} + \frac{1}{2} \sigma_1^2 x_i^2 DDVXM_{i,j,k}^{(c,L)} - \delta VM_{i,j,k}^{(c,L)} + \sum_l f_l \left( ZVM_{i,j,k,l}^{(c,L)} - VM_{i,j,k}^{(c,L)} \right) + r_2 y_j DVYM_{i,j,k}^{(c,L)} + \frac{1}{2} \sigma_2^2 y_j^2 DDVYM_{i,j,k}^{(c,L)} + \sum_l g_l \left( QVM_{i,j,k,l}^{(c,L)} - VM_{i,j,k}^{(c,L)} \right) + SM_{i,j,k}^{(c,L)} \right],
\]

for \(L + 1 = 1\) to \(L^*\), where \(VM_{i,j,k}^{(c,0)} = VM_{i,j,k}^{(p)}\);
Corrector Evaluation:

\[ VM_{i,j,k}^{(c,L)} = 0.5(V_{i,j,k}^{(c,*)} + V_{i,j,k+1}^{(c,L)}), \]

which are used to get corrected components of \( DVXM, DVYM, DDVXM, DDVYM, ZVM, QVM, URM, UM \& SM; \)
3.3.2 More Computational Dynamic Programming:

- **Corrector Relative Stopping Criterion:**
  \[ |V^{(c,L+1)}_{i,j,k+1} - V^{(c,L)}_{i,j,k+1}| < \varepsilon |V^{(c,L)}_{i,j,k+1}| \]

  for all \( \{i, j\} \) at fixed \( k + 1 \) and some relative tolerance \( \varepsilon > 0 \)
  with \( L + 1 = L^*_k \) and \( V^{(c,*)}_{i,j,k} = V^{(c,L^*_k)}_{i,j,k} \).

- **Mean Temporal-Spatial Mesh Corrector Convergence Condition:**
  \[ \Delta t < \frac{1}{2} \frac{1}{\sqrt{(2A/(\Delta \xi)^2)^2 + (B/\Delta \xi)^2}}, \]

  where for example \( B/\Delta \xi = 0.5(B_x/\Delta x + B_y/\Delta y) \) represents
  some mean reciprocal of state meshes weighted by respective linear comparison coefficients \( B_x \) and \( B_y \). This condition is a combined Parabolic-Hyperbolic (CFL) Mesh Ratio Condition.
Part 4. Numerical Results
Figure 4: Optimal current value, $V^*(K, y, t)$, in millions of US dollars versus scaled price factor, $y \cdot \exp(-r_2 \cdot T)$, with time parameter $t = 0.0, 2.0, 4.0, 6.0, 8.0, 10.0$ for each curve ordered from top to bottom, respectively, and with population size fixed at carrying capacity $x = K$. 
Figure 5: Optimal feedback effort, $q \cdot E^*/r_1(K, y, t)$, in dimensionless form versus scaled price inflation factor, $y \cdot \exp(-r_2 \cdot T)$, with time parameter covering $t = 0.0, 2.0, 4.0, 6.0, 8.0, 10.0$ for each curve closely spaced from bottom to top, respectively, and with population size fixed at carrying capacity $x = K$. 
Figure 6: Sensitivity of optimal current value, \( V^*(K, y, 0) \), to inflation price factor rate \( r_2 \), with curves parameterized by scaled inflation price factor, \( y \cdot \exp(-r_2 \cdot T) \), ranging from 1.0 at top to 0.2 at bottom in steps of 0.2, with time fixed at initial value \( t = 0.0 \), and with population size fixed carrying capacity \( x = K \).
Part 5. Conclusions

- Examined Effects of Random Price Fluctuations on Optimal Policy and Optimal Return.
- Successfully Applied Computational Stochastic Dynamic Programming.
- Random Price Jumps Strongly Affect Optimal Return.
- Random Price Jumps have Less Impact on Optimal Policy.
- Random Price Jumps needed as Serious Consideration as Hazardous Environments and other Environmental Effects.