Optimal Harvesting with Coupled Population and Price Dynamics

Floyd B. Hanson *
Laboratory for Advanced Computing
University of Illinois at Chicago

and

Dennis Ryan
Division of Science and Mathematics
McKendree College

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Overview

1. **Noninflationary, Deterministic Model.**

2. **Inflationary, Stochastic Control Model.**

3. **Numerical Approximations.**

4. **Numerical Results.**

5. **Conclusions.**

Outline of Abstract

- **Optimal Control of Stochastic Resource in Continuous Time.**
- **Model Effects of Large Random Price Fluctuations.**
- **Influence of Continuous growth and Jump Stochastic Noise.**
- **Computational Stochastic Dynamic Programming.**
- **Pronounced Effect of Inflationary Prices on Optimal Return.**

Part 1. Noninflationary, Deterministic Model: Introduction.

1.1. Ordinary Differential Equation (ODE):

- **Nonlinear (Logistic) Dynamics:**

$$\boldsymbol{d\mathbf{X}(s) \;=\; [r_1\mathbf{X}(s)(1-\mathbf{X}(s)/K)-\mathbf{H}(s)]\,ds,}$$
$$\boldsymbol{0 < t < s < T.}$$

- **Initial Conditions:** $\mathbf{X}(0) = \mathbf{x}_0$; $\quad 0 < t < T$
- **State Variable (Resource Size):** $\mathbf{X}(t) = [X_i(t)]_{1\times 1}$;
- **BiLinear Control-State Dynamics Assumption (for Resource Harvesting):** $\mathbf{H}(t) = q\mathbf{U}(t)\mathbf{X}(t)$;
- q = Efficiency (Catchability) Coefficient;

- **Control Variable (Harvesting Effort):**

  $$\mathbf{U}(t) \;=\; [U_i\{\mathbf{X}(t), t\}]_{1\times 1}\;, \qquad U_{\min} \le \mathbf{U}(t) \le U_{\max} < \infty\;;$$

- **Growth Parameters:**

  r_1 = Resource Intrinsic Growth Rate;

  K = Environment Carrying (Saturation) Capacity.

1.2. Quadratic Performance Index:

$$\boldsymbol{V(\mathrm{X}, \mathrm{U}, t) = \int_t^T e^{-\delta(s-t)} \left[pq\mathrm{U}(s)\mathrm{X}(s) - c(\mathrm{U}(s))\right] ds} \ ,$$

where

- $V(\mathbf{x}, \mathbf{u}, t) =$ **Current Value of Future Resources** (i.e., $\exp(\delta t)$ times Present Value);
- $T =$ **Time Horizon** $(T \geq t)$;
- $\delta =$ **Nominal Discount Rate** (NOT adjusted for inflation);
- $p =$ **Price of Resource per Unit Harvest Rate**;
- $c(\mathbf{u}) = c_1\mathbf{u} + c_2\mathbf{u}^2 =$ **Quadratic Costs** (Assume Increasing, Convex Quadratic Costs: $c_1 > 0$ and $c_2 > 0$);
- **Instantaneous Net Return:** $\mathbf{R}(\mathbf{x}, \mathbf{u}) = pq\mathbf{ux} - c(\mathbf{u})$.

1.3. Deterministic Dynamic Programming:

- **Optimization Goal = Maximize Total Return:**

$$\mathrm{v}^*(\mathrm{x},t) = V(\mathrm{x},\mathrm{u}^*,t) = \max_{u}\left[V(\mathrm{x},\mathrm{u},t)\right] \;;$$

- **PDE of Deterministic Dynamic Programming:**

$$\mathrm{v}_t^*(\mathrm{x},t) + r_1\mathrm{x}(1-\mathrm{x}/K)\mathrm{v}_x^*(\mathrm{x},t) - \delta\mathrm{v}^*(\mathrm{x},t) + S^*(\mathrm{x},t) = 0;$$

- **Control Switching Term:**

$$S^*(\mathrm{x},t) = \max_{u}\left[\left(p - \mathrm{v}_x^*(\mathrm{x},t)\right)q\mathrm{u}\mathrm{x} - c_1\mathrm{u} - c_2\mathrm{u}^2\right];$$

- **Regular (Unconstrained) Control:**

$$\mathrm{u}_R(\mathrm{x},t) = \frac{(p - \mathrm{v}_x^*(\mathrm{x},t))q\mathrm{x} - c_1}{2\cdot c_2}\,, \qquad c_2 > 0;$$

- **Optimal (Constrained) Control:**

$$\mathbf{u}^*(\mathbf{x},t) = \left\{ \begin{array}{ll} U_{\max}, & U_{\max} \le \mathbf{u}_R(\mathbf{x},t) \\ \mathbf{u}_R(\mathbf{x},t), & U_{\min} \le \mathbf{u}_R(\mathbf{x},t) \le U_{\max} \\ U_{\min}, & \mathbf{u}_R(\mathbf{x},t) \le U_{\min} \end{array} \right\};$$

- **Final Boundary Condition:** $\mathbf{v}^*(\mathbf{x},T) = 0;$

- **Extinction Natural Boundary Condition:**

$$\mathbf{v}^*(0,t) = -\frac{(c_1 + c_2 U_{\min})U_{\min}}{\delta}\left(1 - e^{-\delta(T-t)}\right), \qquad \delta > 0.$$

Part 2. Inflationary, Stochastic Control Model

2.1. Stochastic Dynamics Equation (SDE (1)):

- **Nonlinear Dynamics with Gaussian (G) and Poisson (Z) Noise:**

$$\begin{aligned} d\mathbf{X}(s) &= [r_1\mathbf{X}(s)(1-\mathbf{X}(s)/K) - \mathbf{H}(s)]\,ds \\ &+ \sigma_1\mathbf{X}(s)\,dW_1(s) + \mathbf{X}(s)\sum_{j=1}^{n} a_j\,dZ_j(s,f_j)\ , \\ \mathbf{X}(t) &= x\ , \end{aligned}$$

- **Initial Conditions:** $\mathbf{X}(0) = \mathrm{x}_0\ ,\quad t_0 < t < s < T\ \ ;$
- **Gaussian (Wiener) Noise (Zero Mean and Normalized):**

$$E[dW_1(t)] = 0\ , \qquad Var[dW_1(t)] = dt \qquad \sigma_1 \leq 0\ ;$$

- **Poisson (Jump) Noise:**

$$E[dZ_j(t,f_j)] = f_j dt\ , \qquad Var[dZ_j(t,f_j)] = f_j dt\ ,$$

$1 \le j \le n$, where f_j = Jump Rate and a_j = Jump Amplitude Coefficient $(-1 < a_j)$;

- **Independent (Uncorrelated) Processes Assumption:**

$$\mathrm{Cov}[dW_1(t),\ dZ_j(t,f_j)] \ = \ 0\ ,$$
$$\mathrm{Cov}[dZ_j(t,f_j),\ dZ_{j'}(t,f_{j'})] \ = \ \delta_{j,j'} f_j dt\ ;$$

2.2. Inflationary Factor Model:

- **Nonlinear Supply–Demand Model Relation:**

$$\mathbf{P}(t) = \left(\frac{p_0}{\mathbf{H}}(t) + p_1\right)\mathbf{Y}(t),$$

  * $\mathbf{P}(t) \cdot \mathbf{H}(t) =$ Gross Return on Harvest;
  * $p_0 =$ Supply–Demand Price Coefficient;
  * $p_1 =$ Constant Price per Unit Harvest;
  * $\mathbf{Y}(t) =$ Fluctuating Inflationary Factor;

- **Linear Fluctuating Inflationary Factor SDE (2):**

$$d\mathbf{Y}(s) \quad = \quad r_2\mathbf{Y}(s)\,ds + \sigma_2\mathbf{Y}(s)\,dW_2(s) + \mathbf{Y}(s)\sum_{j=1}^{m} b_j\,dQ_j(s; g_j)\ ,$$

  * $\mathbf{Y}(t) = \mathbf{y}$;
  * $r_2 =$ Annual Rate of Inflation without Fluctuations;
  * $g_j =$ jth component of Inflationary Jump Rate;
  * $b_j =$ jth component of Jump Amplitude Coefficient;

- **Inflationary Gaussian (Wiener) Noise:**

$$E[dW_2(t)] = 0\ , \qquad Var[dW_2(t)] = dt \qquad \sigma_2 \le 0\ ;$$

- **Inflationary Poisson (Jump) Noise:**

$$E[dQ_j(t, g_j)] = g_j dt\ , \qquad Var[dQ_j(t, g_j)] = g_j dt\ , 1 \le j \le m\ ;$$

- **Independent (Uncorrelated) Processes Assumption:**

$$\begin{aligned} \mathrm{Cov}[dW_2(t),\ dQ_j(t, g_j)] &= 0\ , \\ \mathrm{Cov}[dQ_j(t, g_j),\ dQ_{j'}(t, g_{j'})] &= \delta_{j,j'} g_j dt\ ; \end{aligned}$$

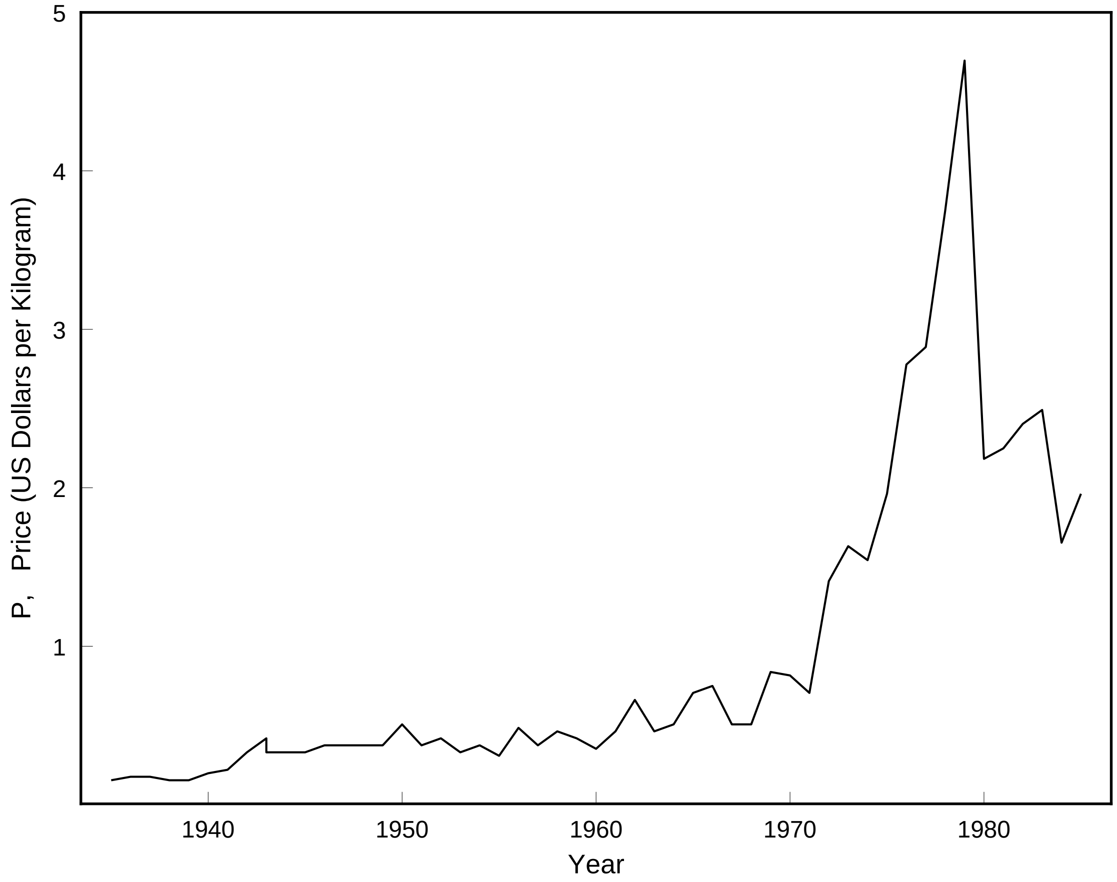

Figure 1: **Pacific halibut prices in USdollars per kilogram for each year from 1935 to 1985 (Raw Data: IPHC 1984 and 1985 Annual Reports).**

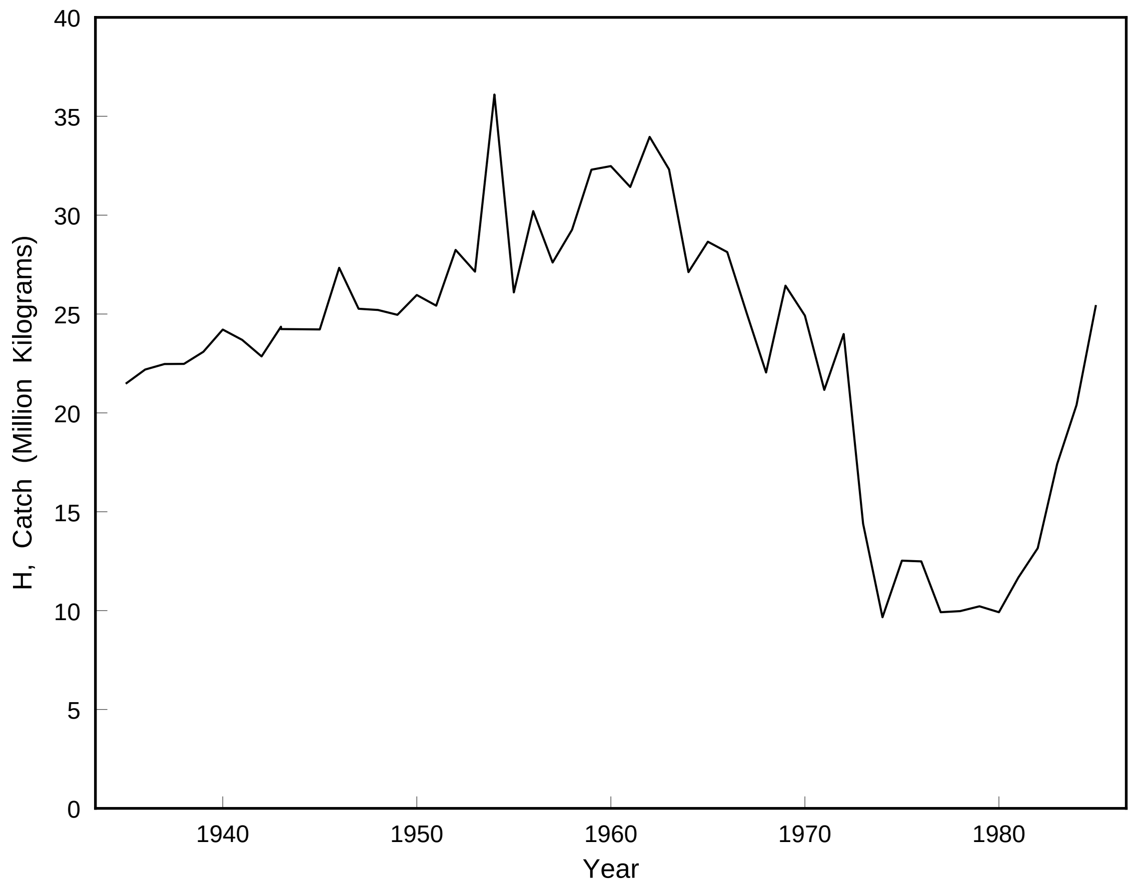

Figure 2: **U.S.-Canadian catch in millions of kilograms for each year from 1935 to 1985 (Raw Data: IPHC 1984 and 1985 Annual Reports).**

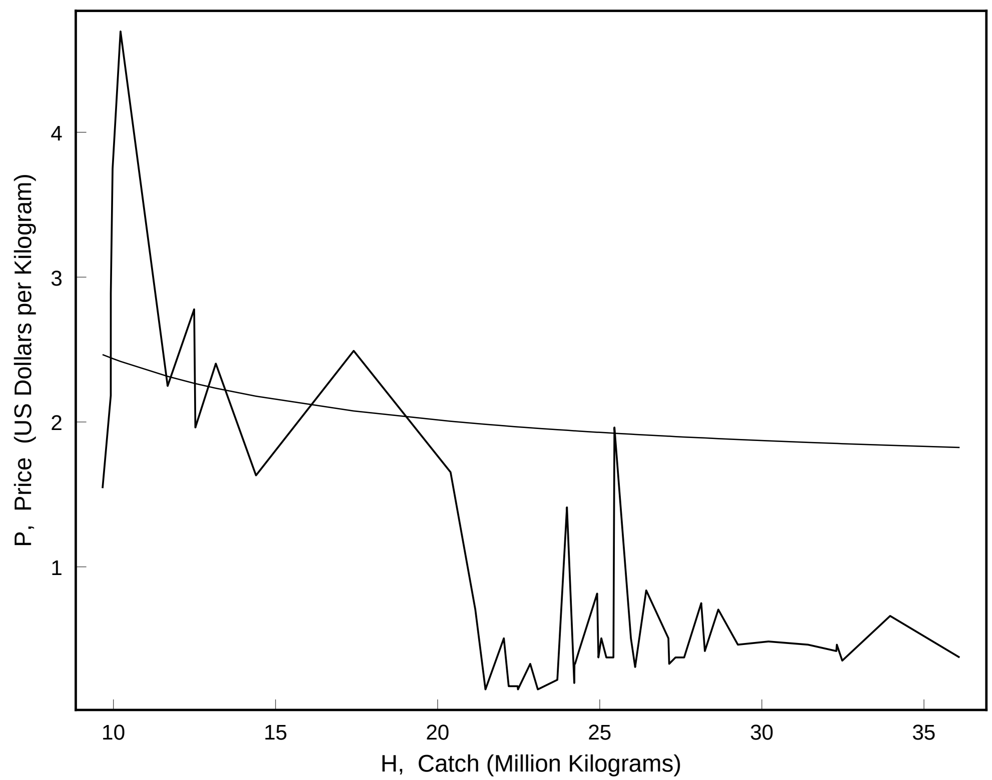

Figure 3: **Pacific halibut price in USdollars per kilogram versus catch in millions of kilograms for years from 1935 to 1985. Linear regression for price times catch as a function of catch from 1980 to 1985 displayed as smooth hyperbolic price curve. (Raw Data: IPHC 1984 and 1985 Annual Reports).**

2.3. Mean Quadratic Performance Index:

$$\overline{V}(\mathbf{x},\mathbf{y},\mathbf{u},t) = \mathbf{E}\left[\int_t^T e^{-\delta(s-t)}\left[(p_0 + p_1 q \mathbf{U}(s)\mathbf{X}(s))\mathbf{Y}(s)\right.\right.$$

$$\left.\left. - \; c(\mathbf{U}(s))\right] ds \mid \mathbf{X}(t) = \mathbf{x},\ \mathbf{Y}(t) = \mathbf{y},\ \mathbf{U}(t) = \mathbf{u}\right],$$

- $\overline{V}(\mathrm{x},\mathrm{y},\mathrm{u},t) =$ **Expected Current Value of Future Resources** (i.e., $\exp(\delta t)$ times Present Value);
- $\{\mathrm{x},\mathrm{y}\} =$ **2-Dim State of Inflationary Stochastic Dynamics** ;
- $T =$ **Time Horizon** $(T \geq t)$;
- $\delta =$ **Nominal Discount Rate** (NOT adjusted for inflation);

2.4. Stochastic Dynamic Programming:

- **Optimization Goal = Maximize Total Return:**

$$\mathbf{v}^*(\mathbf{x},\mathbf{y},t) = \overline{V}(\mathbf{x},\mathbf{y},\mathbf{u}^*,t) = \max_{u}\left[\overline{V}(\mathbf{x},\mathbf{y},\mathbf{u},t)\right] ;$$

- **PDE of Stochastic Dynamic Programming:**

$$\begin{aligned}
0 &= \mathbf{v}_t^*(\mathbf{x},\mathbf{y},t) + r_1\mathbf{x}(1-\mathbf{x}/K)\mathbf{v}_x^*(\mathbf{x},\mathbf{y},t) - \delta\mathbf{v}^*(\mathbf{x},\mathbf{y},t) \\
&+ \frac{\sigma_1^2\mathbf{x}^2}{2}\mathbf{v}_{xx}^* + \sum_j f_j\left[\mathbf{v}^*\left((1+a_j)\mathbf{x},\mathbf{y},t\right) - \mathbf{v}^*(\mathbf{x},\mathbf{y},t)\right] \\
&+ r_2\mathbf{y}\mathbf{v}_y^* + \frac{\sigma_2^2\mathbf{y}^2}{2}\mathbf{v}_{yy}^* + \sum_j g_j\left[\mathbf{v}^*(\mathbf{x},(1+b_j)\mathbf{y},t) - \mathbf{v}^*(\mathbf{x},\mathbf{y},t)\right] \\
&+ S^*(\mathbf{x},\mathbf{y},t),
\end{aligned}$$

by General Itô Chain Rule;

- **Control Switching Term:**

$$S^*(\mathbf{x},\mathbf{y},t) = \max_{u}\left[p_0\mathbf{y} + \left(p_1\mathbf{y} - \mathbf{v}_x^*(\mathbf{x},\mathbf{y},t)\right)q\mathbf{u}\mathbf{x} - c_1\mathbf{u} - c_2\mathbf{u}^2\right];$$

2.4.1. More Stochastic Dynamic Programming:

- **Regular (Unconstrained) Control:**

$$\mathbf{u}_R(\mathbf{x},\mathbf{y},t) = \frac{(p_1\mathbf{y} - \mathbf{v}_x^*(\mathbf{x},\mathbf{y},t))q\mathbf{x} - c_1}{2c_2}\,, \quad c_2 > 0;$$

- **Optimal (Constrained) Control:**

$$\mathbf{u}^*(\mathbf{x},\mathbf{y},t) = \left\{ \begin{array}{ll} U_{\max}, & U_{\max} \le \mathbf{u}_R(\mathbf{x},\mathbf{y},t) \\ \mathbf{u}_R(\mathbf{x},\mathbf{y},t), & U_{\min} \le \mathbf{u}_R(\mathbf{x},\mathbf{y},t) \le U_{\max} \\ U_{\min}, & \mathbf{u}_R(\mathbf{x},\mathbf{y},t) \le U_{\min} \end{array} \right\};$$

- **Final Boundary Condition:** $\mathbf{v}^*(\mathbf{x},\mathbf{y},T) = 0;$

- **Extinction Natural Boundary Condition*:**

$$\mathbf{v}^*(0,0,t) = -\frac{(c_1 + c_2U_{\min})U_{\min}}{\delta}\left(1 - e^{-\delta(T-t)}\right)\,, \qquad \delta > 0.$$

* *see Kushner and Dupuis (1992) for proper handling of stochastic reflecting boundary conditions.*

Part 3. Numerical Approximations

3.1 Basic Hybrid Numerical Procedures.

- **Extrapolated, Predictor-Corrector for Nonlinear Iteration.**
- **Crank-Nicolson Implicit for 2nd Order in Time and State.**
- **Modifications for Poisson Functional Terms.**
- **Modifications for Optimization in Switching Term.**

3.2. Numerical Discretizations:

- **State$_1$:** $X_i \equiv (i-1)\Delta x$, $i = 1, \cdots, N_x$, $\Delta x \equiv K/(N_x - 1)$;
- **State$_2$:** $Y_j \equiv (j-1)\Delta y$, $j = 1, \cdots, N_y$, $\Delta y \equiv e^{r_2 T}/(N_y - 1)$;
- **Time:** $T_k \equiv T - (k-1)\Delta t$, $k = 1, \cdots, N_t$, $\Delta t \equiv T/(N_t - 1)$;
- **Optimal Expected Value:** $\mathbf{v}^*(x_i, y_j, t_k) \longrightarrow V_{i,j,k}$;
- $\mathbf{v}^*_x(X_i, Y_j, T_k) \longrightarrow DVX_{i,j,k} \equiv 0.5(V_{i+1,j,k} - V_{i-1,j,k})/\Delta x$;
- $\mathbf{v}^*_y(X_i, Y_j, T_k) \longrightarrow DVY_{i,j,k} \equiv 0.5(V_{i,j+1,k} - V_{i,j-1,k})/\Delta y$;
- $\mathbf{v}^*_{xx}(X_i, Y_j, T_k) \longrightarrow DDVX_{i,j,k} \equiv$
  $(V_{i+1,j,k} - 2V_{i,j,k} + V_{i-1,j,k})/(\Delta x)^2$;
- $\mathbf{v}^*_{yy}(X_i, Y_j, T_k) \longrightarrow DDVY_{i,j,k} \equiv$
  $(V_{i,j+1,k} - 2V_{i,j,k} + V_{i,j-1,k})/(\Delta y)^2$;
- $\mathbf{v}^*_t(X_i, Y_j, T_{k+0.5}) \longrightarrow DVT_{i,j,k} \equiv -(V_{i,j,k+1} - V_{i,j,k})/\Delta t$;

with Error: $O(\Delta x)^2 + O(\Delta y)^2 + O(\Delta t/2)^2$;

3.2.1. More Numerical Discretizations:

- **X-Poisson Term:** $\mathbf{v}^*((1+a_l)X_i, Y_j, T_k) \longrightarrow ZV_{i,j,k,l}$ by 2nd order accurate interpolation between nearest nodes;
- **Y-Poisson Term:** $\mathbf{v}^*(X_i, (1+b_l)Y_j, T_k) \longrightarrow QV_{i,j,k,l}$ by 2nd order accurate interpolation between nearest nodes;
- **Regular Control:** $\mathbf{u}_R(X_i, Y_j, T_k) \longrightarrow \mathrm{UR}_{i,j,k} \equiv (p_1 Y_j - DVX_{i,j,k} \cdot q \cdot X_i - c_1)/(2c_2)$;
- **Optimal Control:** $\mathbf{u}^*(X_i, Y_j, T_k) \longrightarrow U_{i,j,k} \equiv$ same as exact composite expression;

3.3. Computational Stochastic Dynamic Programming:

For $k+1=2$ **to** N_t **while** $i=1$ **to** N_x **&** $j=1$ **to** N_y**:**

- **Accelerating Extrapolating Start:**

$$VE_{i,j,k} \equiv 0.5(3V^{(c,*)}_{i,j,k} - V^{(c,*)}_{i,j,k-1}) \simeq V_{i,j,k+0.5}, \quad \text{if} \quad k \le 2,$$

which are used to get components $DVXE$, $DVYE$, $DDVXE$, $DDVYE$, ZVE, QVE, URE, UE & SE, and where $V^{(c,*)}_{i,j,k}$) is the final correction from step k;

- **Extrapolated-Predictor Step:**

$$\begin{aligned}
V^{(p)}_{i,j,k+1} &= V^{(c,*)}_{i,j,k} + \Delta t\,[r_1 X_i(1 - X_i/K)DVXE_{i,j,k} \\
&+ \frac{1}{2}\sigma_1^2 X_i^2 DDVXE_{i,j,k} - \delta\, VE_{i,j,k} \\
&+ \Sigma_l f_l(ZVE_{i,j,k,l} - VE_{i,j,k}) \\
&+ r_2 Y_j DVYE_{i,j,k} + \frac{1}{2}\sigma_2^2 Y_j^2 DDVYE_{i,j,k} \\
&+ \Sigma_l g_l(QVE_{i,j,k,l} - VE_{i,j,k}) + SE_{i,j,k}]\,,
\end{aligned}$$

- **Predictor Evaluation (Crank-Nicolson Midpoint):**

$$VM^{(p)}_{i,j,k} \equiv 0.5(V^{(c,*)}_{i,j,k} + V^{(p)}_{i,j,k+1}) \simeq V_{i,j,k+0.5},$$

  which are used to get predicted components of $DVXM$, $DVYM$, $DDVXM$, $DDVYM$, ZVM, QVM, URM, UM & SM;

3.3.1 More Computational Dynamic Programming:

- $(L+1)$st **Corrector Step:**

$$
\begin{aligned}
V_{i,j,k+1}^{(c,L+1)} &= V_{i,j,k}^{(c,*)} + \Delta t \Big[r_1 x_i (1 - x_i/K) DVXM_{i,j,k}^{(c,L)} \\
&+ \frac{1}{2}\sigma_1^2 x_i^2 DDVXM_{i,j,k}^{(c,L)} - \delta\, VM_{i,j,k}^{(c,L)} \\
&+ \Sigma_l f_l \left(ZVM_{i,j,k,l}^{(c,L)} - VM_{i,j,k}^{(c,L)} \right) \\
&+ r_2 y_j DVYM_{i,j,k}^{(c,L)} + \frac{1}{2}\sigma_2^2 y_j^2 DDVYM_{i,j,k}^{(c,L)} \\
&+ \Sigma_l g_l \left(QVM_{i,j,k,l}^{(c,L)} - VM_{i,j,k}^{(c,L)} \right) + SM_{i,j,k}^{(c,L)} \Big],
\end{aligned}
$$

for $L+1 = 1$ to L^*, where $VM_{i,j,k}^{(c,0)} = VM_{i,j,k}^{(p)}$;

- **Corrector Evaluation:**

$$VM_{i,j,k}^{(c,L)} = 0.5(V_{i,j,k}^{(c,*)} + V_{i,j,k+1}^{(c,L)}),$$

which are used to get corrected components of $DVXM$, $DVYM$, $DDVXM$, $DDVYM$, ZVM, QVM, URM, UM & SM;

3.3.2 More Computational Dynamic Programming:

- **Corrector Relative Stopping Criterion:**

$$|V_{i,j,k+1}^{(c,L+1)} - V_{i,j,k+1}^{(c,L)}| < \varepsilon |V_{i,j,k+1}^{(c,L)}|$$

for all $\{i, j\}$ at fixed $k+1$ and some relative tolerance $\varepsilon > 0$ with $L+1 = L_k^*$ and $V_{i,j,k}^{(c,*)} = V_{i,j,k}^{(c,L_k^*)}$.

- **Mean Temporal-Spatial Mesh Corrector Convergence Condition:**

$$\Delta t < \frac{1}{2} \frac{1}{\sqrt{(\overline{2A/(\Delta\xi)^2})^2 + (\overline{B/\Delta\xi})^2}},$$

where for example $\overline{B/\Delta\xi} = 0.5(B_x/\Delta x + B_y/\Delta y)$ represents some mean reciprocal of state meshes weighted by respective linear comparison coefficients B_x and B_y. This condition is a combined Parabolic-Hyperbolic (CFL) Mesh Ratio Condition.

Part 4. Numerical Results

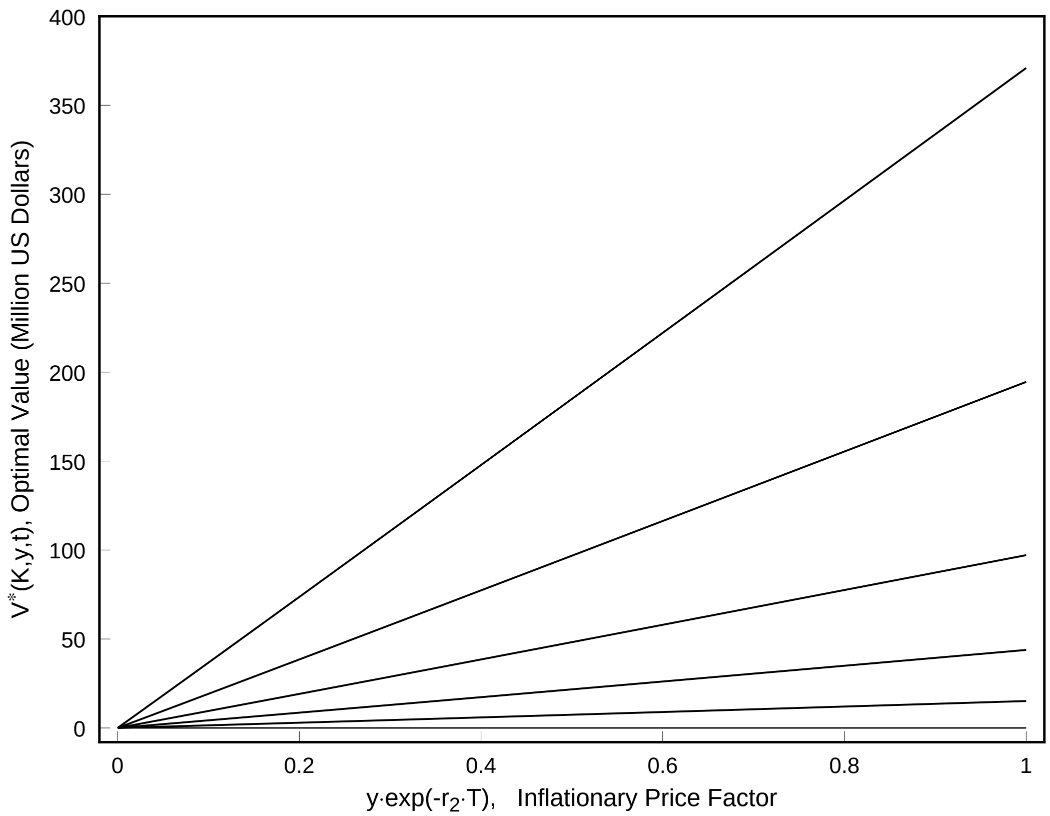

Figure 4: **Optimal current value, $V^*(K, y, t)$, in millions of USdollars versus scaled price factor, $y \cdot \exp(-r_2 \cdot T)$, with time parameter $t = 0.0$, 2.0, 4.0, 6.0, 8.0, 10.0 for each curve ordered from top to bottom, respectively, and with population size fixed at carrying capacity $x = K$.**

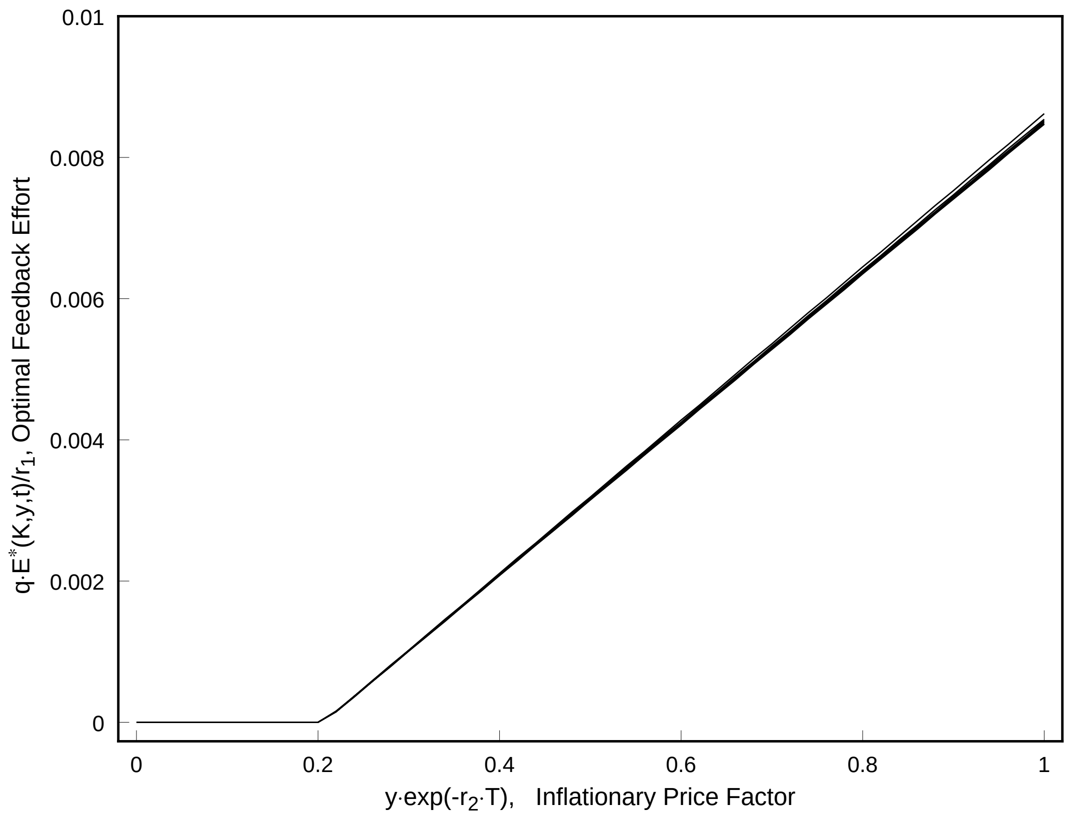

Figure 5: **Optimal feedback effort, $q \cdot E^*/r_1(K, y, t)$, in dimensionless form versus scaled price inflation factor, $y \cdot \exp(-r_2 \cdot T)$, with time parameter covering $t = 0.0$, 2.0, 4.0, 6.0, 8.0, 10.0 for each curve closely spaced from bottom to top, respectively, and with population size fixed at carrying capacity $x = K$.**

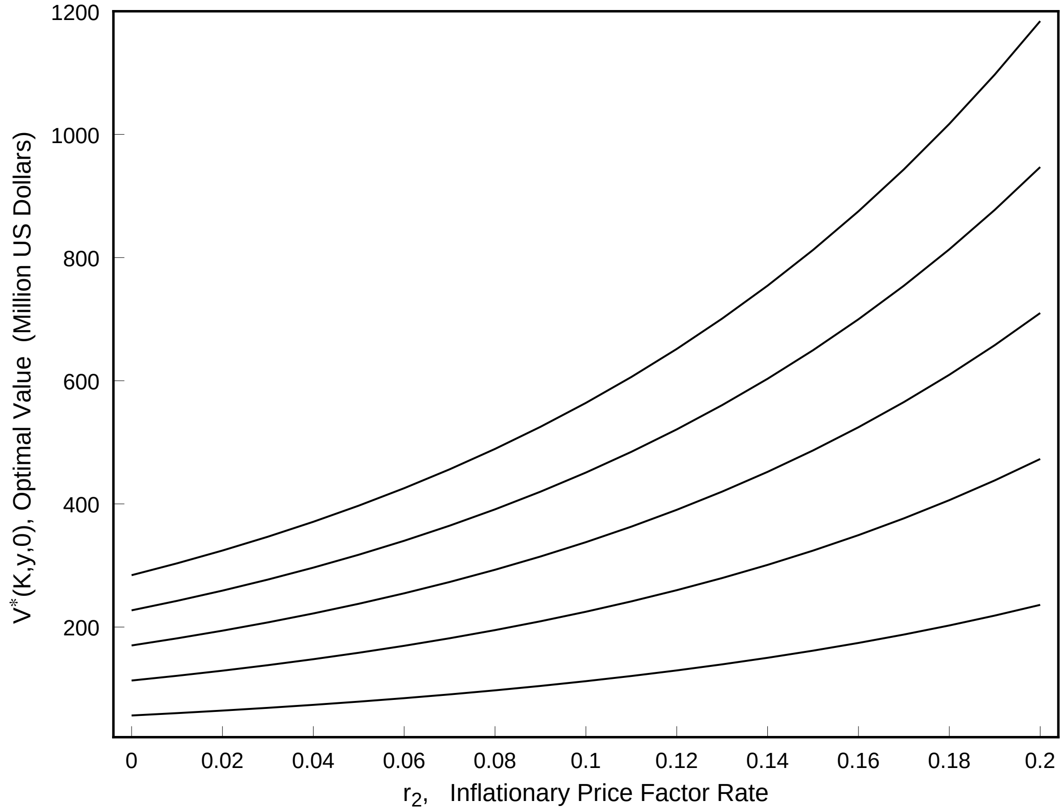

Figure 6: **Sensitivity of optimal current value, $V^*(K, y, 0)$, to inflation price factor rate r_2, with curves parameterized by scaled inflation price factor, $y \cdot \exp(-r_2 \cdot T)$, ranging from 1.0 at top to 0.2 at bottom in steps of 0.2, with time fixed at initial value $t = 0.0$, and with population size fixed carrying capacity $x = K$.**

Part 5. Conclusions

- **Examined Effects of Random Price Fluctuations on Optimal Policy and Optimal Return.**
- **Successfully Applied Computational Stochastic Dynamic Programming.**
- **Random Price Jumps Strongly Affect Optimal Return.**
- **Random Price Jumps have Less Impact on Optimal Policy.**
- **Random Price Jumps needed as Serious Consideration as Hazardous Environments and other Environmental Effects.**