

# Optimal Portfolio and Consumption Policies Subject to Rishel's Important Jump Events Model: Computational Methods

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**Abstract**—At important events or announcements, there can be large changes in the value of financial portfolios. Events and their corresponding jumps can occur at random or scheduled times. However, the amplitude of the response in either case can be unpredictable or random. While the volatility of portfolios is often modeled by continuous Brownian motion processes, discontinuous jump processes are more appropriate for modeling the response to important external events that significantly affect the prices of financial assets. Discontinuous jump processes are modeled by compound Poisson processes for random events or by quasi-deterministic jump processes for scheduled events. In both cases, the responses are randomly distributed and are modeled in a stochastic differential equation formulation. The objective is the maximal, expected total discounted utility of terminal wealth and instantaneous consumption. This paper was motivated by a paper of Rishel (1999) concerning portfolio optimization when prices are dependent on external events. However, the model has been significantly generalized for more realistic computational considerations with constraints and parameter values. The problem is illustrated for a canonical risk-adverse power utility model. However, the usual explicit canonical solution is not strictly valid. Fortunately, iterations about the canonical solution result in computationally feasible approximations.

**Index Terms**—important jump events, optimal portfolio-consumption policies, Greenspan processes, canonical model approximate computations

## I. INTRODUCTION

A LARGE NUMBER of continuous time models of financial markets have been based upon continuous sample path geometric Brownian motion processes, such as Merton [12], [13], [15, Chapters 4-6] and Black and Scholes [1]. However, Merton [14], [15, Chapter 9], in the original jump diffusion finance model, applied discontinuous sample path Poisson processes, along with Brownian motion processes, to the problem of pricing options when the underlying asset returns are discontinuous. Several extensions of the classical diffusion theory of Black and Scholes [1] were derived by minimizing portfolio variance techniques to jump diffusion models similar to those techniques used to derive the classic Black and Scholes diffusion formulae.

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Earlier, Merton [13], [15, Chapters 5-6] treated optimal consumption and investment portfolios with either geometric Brownian motion or Poisson noise, and illustrated explicit solutions for constant risk-aversion in either the relative or the absolute forms. Karatzas et al. [10] pointed out that it is necessary to enforce non-negativity feasibility conditions on both wealth and consumption, deriving formally explicit solutions from a consumption investment dynamic programming model with a time-to-bankruptcy horizon, that qualitatively corrects the Merton's results. Sethi and Taksar [18] present corrections to certain formulae of Merton's [13], [15, Chapters 5-6] finite horizon consumption-investment model. Merton [15, Chapter 6] revisited the problem, correcting his earlier work by adding an absorbing boundary condition at zero wealth and using other techniques.

Rishel [17] introduced a optimal portfolio model for stock prices dependent on quasi-deterministic scheduled and stochastic unscheduled jump external events based on optimal stochastic control theory. The jumps can affect both the stock prices directly or indirectly through parameters. The quasi-deterministic jumps are deterministic only in the timing of the scheduled events, but the jump responses are random in magnitude. The response to an event can be unpredictable, being based on solid analysis, prefactored assessments, nuances or other factors external to the event. Rishel's theoretical paper is the motivation for this computational application paper. Much additional motivation comes from our extensive prior research on computational stochastic control models for jump environments, such as stochastic bioeconomic models with random disasters (see Hanson and Ryan [7], and Hanson and Tuckwell [8]) and stochastic manufacturing systems subject to jumps from failures, repairs and other events (see Westman and Hanson [19], [20], [21]). Here our model formulation is a modification on Rishel's [17] paper, with heavier reliance on stochastic differential equations, constrained control, more general utility objectives, generalized functions, and random Poisson measure. Many of the modifications make the model more realistic and computationally feasible. More realism has been implemented through modifications systematically relying on linear or geometric stochastic processes, while using control constraints on stock fractions and consumption policies.

The paper is arranged as follows. In Section II, the stochastic differential equation model for the underlying bond and more risky stock assets is formulated in terms of state and

control dependent marked Poisson processes and quasi-deterministic processes, with diffusion processes used for the less extreme background events. In Section III, the wealth equations, with asset fractions and consumption variables, are formulated for the portfolio. In Section IV, the portfolio and consumption optimization problem is formulated and the subsequent partial differential equation of stochastic dynamic programming is derived from a generalized Itô [9] chain rule. The computational complexity of the solution for the canonical risk-averse power utilities example in Section V is significantly reduced in dimension upon separating out the wealth and consumption dependencies. However, both scheduled and unscheduled event parameters are not separable from the time dependence as in the canonical case without these parameters, but the canonical solution still serves as a good leading order perturbation for the more complex problem here. Further computational considerations are discussed in Section VI concerning numerical procedures appropriate for jump process terms. The computational solutions for a numerical test model are presented in Section VII for various values of the parameters. The computations have been carried out in MATLAB<sup>TM</sup> to demonstrate the reasonableness of the calculations.

## II. STATE AND CONTROL DEPENDENT ASSET MODELS

A financial portfolio is selected from a less risky asset or bond and a number of risky assets or stocks. Let the bond earn a rate  $r$  of interest such that its price  $B(t)$  at time  $t$  satisfies the deterministic dynamical process with specified initial condition,

$$dB(t) = rB(t)dt, \quad B(0) = B_0. \quad (1)$$

The coefficient  $r$  is the constant interest rate for the bond, so the bond would strictly be zero-coupon bond. Payment of coupons could be accommodated, but that would be beyond the purpose of this paper. Jumps may occur in the bond rate directly related to announced changes in the federal interest rate, so then  $r = r(\vec{A}(t))$ , where  $\vec{A}(t)$  is an event parameter process described below, and for consistency with the assumptions on other financial parameters, this will be assumed in the analysis.

Let  $S_i(t)$  be the price of the  $i$ th stock satisfying the Markov geometric jump-diffusion and quasi-deterministic stochastic differential equation,

$$dS_i(t) = S_i(t) \left[ \mu_i(\vec{A}(t))dt + \sum_{j=1}^M \sigma_{i,j}(\vec{A}(t))dZ_j(t) + dP_i(t) + dQ_i(t) \right], \quad S_i(0) = S_{0,i}, \quad (2)$$

for  $i = 1, 2, \dots, N_1$  risky assets or stocks. Here, the argument  $\vec{A}(t) = (A_1(t), A_2(t))$  is an event parameter vector process consisting of unscheduled events  $A_1(t)$  and scheduled events  $A_2(t)$  where  $\vec{a} = (a_1, a_2)$  denotes an event or realization. First the terms in (2) will be briefly identified, but will be more thoroughly described later. The mean appreciation rate for the  $i$ th stock is  $\mu_i(\vec{a}) = \mu_i(a_1, a_2)$  with volatilities given

by  $\sigma_{i,j}(\vec{a})$ , where the  $j$  denotes the  $j$ th continuous Brownian motion processes is  $Z_j(t)$ , for  $j = 1, 2, \dots, M$ . The stochastic differential  $dP_i(t)$  is a discontinuous, random, space-time Poisson process representing important unscheduled events, for  $i = 1, 2, \dots, N_1$  stocks. Important scheduled events are modeled by an analogous quasi-deterministic differential process  $dQ_i(t)$  which has scheduled or deterministic jump times, but the amplitudes of the jumps are randomly distributed.

The  $Z_j(t)$ , for  $j = 1, 2, \dots, M$ , are independent, standard Brownian motion processes with zero mean and delta-correlations. The continuous sample path processes  $Z_j(t)$  model the less extreme background random events that affect the financial market. The  $P_i(t)$  are the  $i$ th component of a marked or space-time Poisson process. The discontinuous sample path processes  $dP_i(t)$  model the rare, extreme events that lead to large fluctuations in risk sensitive market assets. The space-time differential Poisson processes  $dP_i(t)$  are related to Poisson random measure,  $\mathcal{P}(dt, \vec{d}\hat{j}_1)$  (see Gihman and Skorohod [4]),

$$dP_i(t) = \int_{\mathcal{J}_1} J_{1,i}(t, \hat{j}_1; \vec{A}(t)) \mathcal{P}(dt, \vec{d}\hat{j}_1), \quad (3)$$

for  $i = 0, 1, \dots, N_1$ , where  $J_{1,i}$  is the  $i$ th Poisson jump amplitude function corresponding to the  $i$ th stock price when  $i > 0$  or to the parameter process  $A_1(t)$  when  $i = 0$ ,  $\vec{d}\hat{j}_1 = (d\hat{j}_{1,0}, \vec{d}\hat{j}_1) = [d\hat{j}_{1,i-1}]_{(N_1+1) \times 1}$  is the extended  $(N_1 + 1)$ -dimensional random mark vector for the composite stock and parameter unscheduled event mark space is  $\mathcal{J}_1$ . Each time the constituent Poisson counting process has a jump signifying a random unscheduled event, a random mark vector  $\vec{d}\hat{j}_1$  is generated which in turn generates the value of the extended vector jump amplitude

$$\begin{aligned} \vec{d}\hat{j}_1 &= \left( J_{1,0}, \vec{J}_1 \right) \left( t, \hat{j}_1; \vec{A}(t) \right) \\ &= \left[ J_{1,i-1} \left( t, \hat{j}_1; \vec{A}(t) \right) \right]_{(N_1+1) \times 1}, \end{aligned}$$

resulting in both the jump in the unscheduled events parameter process  $A_1(t)$  from  $J_{1,0}$  and in the jump in stock price  $S_i$  from  $J_{1,i}$  for  $i = 1, 2, \dots, N_1$ , respectively. The component  $dP_i(t)$  of the Poisson driven process has the conditional expectation:

$$\begin{aligned} E[dP_i(t)|\vec{A}(t) = \vec{a}] &= \lambda(t)E \left[ J_{1,i} | \vec{A}(t) = \vec{a} \right] dt \quad (4) \\ &\equiv \lambda(t) \int_{\mathcal{J}_1} J_{1,i}(t, \hat{j}_1; \vec{a}) \phi_1(\vec{d}\hat{j}_1) \vec{d}\hat{j}_1 dt, \end{aligned}$$

for  $i = 0, 1, \dots, N_1$ , where  $\lambda(t)$  is the rate for the common Poisson counting process, and  $\phi_1(\vec{d}\hat{j}_1)$  is the joint density of the unscheduled event amplitude marks. The  $dP_i(t)$  has conditional variance given by

$$\text{Var}[dP_i(t)|\vec{A}(t) = \vec{a}] = \lambda(t)E \left[ J_{1,i}^2 | \vec{A}(t) = \vec{a} \right] dt,$$

for  $i = 1, 2, \dots, N_1$ , provided there is component-wise independence. Given that there is an unscheduled event jump at  $T_{1,\ell}$ , the stock price  $S_i(t)$  jump magnitude is

$$\begin{aligned} [S_i](T_{1,\ell}) &\equiv S_i(T_{1,\ell}^+) - S_i(T_{1,\ell}^-) \\ &= J_{1,i} \left( T_{1,\ell}^-, \vec{d}\hat{j}_1; \vec{A}(T_{1,\ell}^-) \right) S_i \left( T_{1,\ell}^- \right), \end{aligned}$$

for  $\ell = 1, 2, 3, \dots$  jumps where  $\widehat{j}_{1,\ell}$  is a jump mark realization associated with jump time  $T_{1,\ell}$ .

The unscheduled events parameter  $A_1(t)$  is generated by the same space-time Poisson process  $\mathcal{P}$  as above, such that  $J_{1,0}$  is the jump amplitude and  $j_{1,0}$  is the jump mark of  $A_1(t)$ . The jump in  $A_1(t)$  is generated by the single underlying Poisson counting process of the unscheduled event as it is for the stocks in (2) at rate  $\lambda(t)$  for unscheduled events,

$$dA_1(t) = A_1(t)dP_0(t),$$

where  $dP_0(t)$  is given by (3) when  $i = 0$ . The infinitesimal, conditional expected parameter is

$$E_{j_{1,0}} \left[ dA_1(t) | \vec{A}(t) = \vec{a} \right] = \lambda(t) a_1 E_{j_{1,0}} \left[ J_{1,0} \left| \vec{A}(t) = \vec{a} \right. \right] dt,$$

where  $E_{j_{1,0}} [J_{1,0} | \vec{A}(t) = \vec{a}]$  is the conditional expected jump amplitude of the zeroth component of the space-time Poisson process. Again, a relative jump size is used here, rather than an absolute jump amplitude.

The last term is the semi- or quasi-deterministic term,

$$dQ_i(t) = \int_{\mathcal{J}_2} J_{2,i}(t, \widehat{j}_2; \vec{A}(t)) \mathcal{Q}(dt, d\widehat{j}_2) \quad (5)$$

on the right hand side of (2), and models the jumps resulting from scheduled events at certain specified times  $T_{2,\ell}$ , for  $\ell = 1, 2, \dots, N_2$  scheduled jumps. The jumps trigger random jump amplitudes of size  $J_{2,i}(t, \widehat{j}_2; \vec{A}(t))$  so that  $S_i(t)$  jumps by  $J_{2,i}S_i(T_{2,\ell}^-)$  if  $i = 1, \dots, N_1$ , while if  $i = 0$ , then  $A_1(t)$  jumps by  $J_{2,0}A_1(T_{2,\ell}^-)$ , assuming  $T_{2,\ell} < T_{2,\ell+1}$ . Here the extended random mark vector is  $\widehat{j}_2 = (j_{2,0}, \vec{j}_2) = [j_{2,i-1}]_{(N_1+1) \times 1}$  with quasi-deterministic space-time measure  $\mathcal{Q}(dt, d\widehat{j}_2)$  for scheduled jumps. The corresponding relative jump amplitude vector,  $\vec{J}_2$ , is such that

$$\begin{aligned} E[dQ_i(t) | \vec{A}(t) = \vec{a}] &= E[J_{2,i} \vec{A}(t) = \vec{a}] \\ &= \sum_{\ell=1}^{N_2} \delta_R(t - T_{2,\ell}) dt. \end{aligned}$$

Thus,  $\mathcal{Q}$  is similar to  $\mathcal{P}$ , except that the jump times of the former,  $T_{2,\ell}$ , are scheduled with certainty while those of the latter are random and thus unscheduled. The generalized function symbol  $\delta_R(t - T_{2,\ell})$  above defines a *right continuous delta function* by

$$\int_{-\infty}^{\infty} f(t) \delta_R(t - T_{2,\ell}) = f(T_{2,\ell}^-),$$

for some right-continuous function  $f$ , compatible with the right continuity (continuity from the right) of the Poisson process. Unlike the Dirac delta function,  $\delta_R(t - T_{2,\ell})$  is a bounded step function embodied in its constructive definition as the difference of corresponding right continuous step functions,  $\delta_R(t - T_{2,\ell}) dt = H_R(t + dt - T_{2,\ell}) - H_R(t - T_{2,\ell})$ , for infinitesimal  $dt$ , where  $H_R(t - T_{2,\ell})$  is the right-continuous unit step function that characterizes the simple Poisson counting

process. Thus, the scheduled jump amplitude for stock  $S_i$  at  $T_{2,\ell}$  is

$$\begin{aligned} [S_i](T_{2,\ell}) &\equiv S_i(T_{2,\ell}^+) - S_i(T_{2,\ell}^-) \\ &= J_{2,i}(T_{2,\ell}^-, \widehat{j}_{2,\ell}; \vec{A}(T_{2,\ell}^-)) S_i(T_{2,\ell}^-), \end{aligned}$$

for  $i = 1, \dots, N_1$  stocks, due to non-anticipating, right-continuity and such that stock  $i$  jumps from  $S_i(T_{2,\ell}^-)$  to  $J_{2,i}S_i(T_{2,\ell}^-)$  at the scheduled jump time  $T_{2,\ell}$  with jump mark vector realization  $\widehat{j}_{2,\ell}$ . It is further assumed that the final scheduled jump at  $T_{2,N_2}$  takes place before the terminal time  $T$ , i.e.,  $T_{2,N_2} < T \equiv T_{2,N_2+1}$ .

These scheduled jumps affect the market due to events such as changes in monetary policy, announcements of labor statistics, other economic announcements or eminent labor strikes, although the response magnitude of the jumps can be random, as described by Rishel [17]. An example (February 17, 2000) of large fluctuations caused by announced events is the semi-annual economic report of Chairman Alan Greenspan of the Federal Reserve Board to Congress that concerned the raising of interest rates and other matters. This was followed the next day (February 18, 2000) by a "double witching day" in which there was a simultaneous expiration of contracts on stock options and stock indices. Although these events and the market responses to them are quite complex, a strong motivation for these quasi-deterministic processes are the influential announcement events by Chairman Greenspan and thus they might be called "*Greenspan processes*."

For the continuous portion of the sample paths, the non-anticipating mean appreciation rate is  $\mu_i(\vec{A}(t))$  and the squared volatility is  $\sum_{j=1}^M \sigma_{i,j}^2(\vec{A}(t))$ , changing with scheduled or random jump events. The vector  $\vec{A}(t) = (A_1(t), A_2(t))$  represents the parametric arguments of the mean appreciation rates  $\mu_i$ , the volatilities  $\sigma_{i,j}$  and the jump amplitudes  $J_{k,i}$ , with  $k = 1$  denoting unscheduled events and  $k = 2$  denoting scheduled events.

The scheduled events parameter process  $A_2(t)$  is assumed to have jumps at the same times as that of the scheduled events,

$$dA_2(t) = A_2(t)dQ_0(t), \quad (6)$$

where  $dQ_0(t)$  is given in (5) when  $i = 0$ ,

$$[A_2](T_{2,\ell}) = J_{2,0}(T_{2,\ell}^-, \widehat{j}_{2,\ell}, \vec{A}(T_{2,\ell}^-)) A_2(T_{2,\ell}^-) \quad (7)$$

is the jump size for a given mark vector realization  $\widehat{j}_{2,\ell}$  at jump  $\ell$ , where the conditional expectation is

$$\begin{aligned} E_{\widehat{j}_2} \left[ A_2(T_{2,\ell}^+) | \vec{A}(T_{2,\ell}^-) = \vec{a}_\ell^- \right] \\ = \left( 1 + E_{\widehat{j}_2} [J_{2,0} | \vec{A}(T_{2,\ell}^-) = \vec{a}_\ell^-] \right) a_{2,\ell}^-, \end{aligned}$$

in terms of the conditional expected relative jump size for unscheduled parameter  $A_2$ . Here, a relative jump size is used, so that this relative size is added to one in the factor multiplying the old value, rather than an absolute size in Rishel [17] where the absolute size is added to the old value. Geometric or multiplicative noise is used here as being more appropriate than additive noise.

Our model for the underlying assets is similar to that of Rishel [17], except that more general and realistic distributions

are used here for the appreciation, volatilities and unscheduled jump parameters, systematically relying upon linear parameter processes, rather than the discrete random states used in Rishel [17]. Also, space-time Poisson processes are used extensively in the model here.

### III. PORTFOLIO WEALTH EQUATION

Let  $W(t)$  be the portfolio wealth process at time  $t$  that includes a bond asset at price  $B(t)$  and the  $N_1$  risky stocks  $S_i(t)$ . Let  $U_i(t)$  be the instantaneous fraction of the wealth  $W(t)$  invested in the  $i$ th risky asset at time  $t$  for  $i = 1, 2, \dots, N_1$  and  $U_0(t)$  denotes the fraction invested in bonds at time  $t$ , so that  $U_0(t) + \sum_{i=1}^{N_1} U_i(t) = 1$ , which serves as the defining constraint for the bond fraction  $U_0(t)$  in terms of the stock fraction vector  $\vec{U}(t) = [U_i(t)]_{N_1 \times 1}$ . Along with consumption of capital, the stock investment fractions will comprise the components of the stock fraction control policy vector,  $\vec{U}(t)$ , for this problem. The  $U_i(t)$  can take on arbitrary real values if  $i > 0$  in theory, since the fraction of stock  $i$  can be negative if the stock is sold short at time  $t$  in anticipation of a drop in prices making it profitable to buy back later, while the fraction invested in bonds can be negative if money is borrowed on the bond and invested in stock  $i$  with  $i > 0$ . The sum over the stock fractions can exceed unity and thus are unbounded above, in theory. However, for practical reasons, the stock fractions must be bounded or limited since borrowing and short selling would be limited. Further, if the jump model leads to singular control calculations, then the control space would need to be bounded. Thus for more realism, the control space will be assumed bounded, for example, by component-wise constraints,  $U_{\min,i} \leq u_i \leq U_{\max,i}$ , with  $U_{\min,i} \leq 0$  and  $U_{\max,i} > 0$  specified, defining a stock fraction control domain  $\mathcal{D}_u$  example.

The relative change in wealth  $dW(t)/W(t)$  at time  $t$  due to the relative change in the bond price is  $U_0(t)dB(t)/B(t)$  and that due to the  $i$ th stock price is  $U_i(t)dS_i(t)/S_i(t)$ , but wealth also decreases due to instantaneous consumption  $C(t)$ . Thus using the dynamics in the stocks (2) coupled with the bonds, the wealth satisfies the stochastic differential equation (SDE),

$$dW(t) = -C(t)dt + W(t) \left[ r(\vec{A}(t))dt + \vec{U}^\top(t) \left\{ \left( \vec{\mu}(\vec{A}(t)) - r(\vec{a})\vec{1} \right) dt + \sigma(\vec{A}(t))d\vec{Z}(t) + d\vec{P}(t) + d\vec{Q}(t) \right\} \right], \quad (8)$$

with matrix-vector notation such that  $\vec{U}^\top(t) = [U_j(t)]_{1 \times N_1}$  denotes the transpose of  $\vec{U}(t)$ ,  $\vec{1} = [1]_{N_1 \times 1}$ ,  $\vec{\mu}(\vec{a}) = [\mu_i(\vec{a})]_{N_1 \times 1}$ ,  $\sigma(\vec{a}) = [\sigma_{i,j}(\vec{a})]_{N_1 \times M}$ ,  $d\vec{Z}(t) = [dZ_i(t)]_{M \times 1}$ ,  $d\vec{P}(t) = [dP_i(t)]_{N_1 \times 1}$  and  $d\vec{Q}(t) = [dQ_i(t)]_{N_1 \times 1}$ . The zeroth jump process components ( $dP_0(t), dQ_0(t)$ ) associated with the random jump parameters ( $A_1(t), A_2(t)$ ) do not directly appear in (8). The jump in wealth is given by

$$\begin{aligned} [W](T_{k,\ell}) &\equiv W(T_{k,\ell}^+) - W(T_{k,\ell}^-) \\ &= \sum_{i=1}^{N_1} U_i(T_{k,\ell}^-) J_{k,i}(T_{k,\ell}^-, \hat{J}_{k,\ell}; \vec{A}(T_{k,\ell}^-)) \end{aligned}$$

$$\cdot W(T_{k,\ell}^-),$$

at each jump time  $t = T_{k,\ell}$ , for  $\ell = 1, 2, 3, \dots$  when  $k = 1$  for unscheduled jumps or  $\ell = 1, 2, \dots, N_2$  when  $k = 2$  for scheduled jumps, and with the realized mark vector  $\hat{J}_{k,\ell}$  for each  $\ell$ th jump of type  $k$ , combining both types of jumps in a single formula.

### IV. CONSUMPTION AND PORTFOLIO OPTIMIZATION PROBLEM

Let  $\mathcal{U}_f(w; \vec{a})$  be the utility function of final wealth as well as of the events parameter vector  $\vec{a}$ , and let  $\mathcal{U}(c)$  be the utility of instantaneous consumption for the investor. Suppose the investor consumes  $c = C(t)$  at time  $t$  and ends up with wealth  $w = W(T)$  at the final time  $T$ . The investor seeks to maximize the conditional expected, *current value* at  $t$  of the discounted utility of the terminal wealth and instantaneous consumption, i.e.,

$$v^*(t, w; \vec{a}) = \max_{\{\vec{u}, c\}[t, T]} \left[ E \left[ e^{-\beta(\vec{a})(T-t)} \mathcal{U}_f(W(T); \vec{A}(T)) + \int_t^T e^{-\beta(\vec{a})(\tau-t)} \mathcal{U}(C(\tau)) d\tau \middle| \mathcal{S}_c(t) \right] \right], \quad (9)$$

where the conditioning set is  $\mathcal{S}_c(t) \equiv \{W(t) = w, \vec{U}(t) = \vec{u}, C(t) = c; \vec{A}(t) = \vec{a}\}$ , upon selecting the maximizing portfolio policies  $\vec{U}(t)$  and consumption  $C(t)$ , assuming the wealth process  $W(t)$  satisfies the stochastic dynamics specified by (8). Discounting at rate  $\beta = \beta(\vec{A}(t))$  is used here to account for opportunity costs due to potentially better alternative investments, in contrast to Rishel [17]. Here,  $\beta$  is the real (nominal less inflation) discount rate but could jump with the announced changes in the federal funds discount rate. The utility functions  $\mathcal{U}(c)$  and  $\mathcal{U}_f(w; \vec{a})$  are assumed to be increasing concave functions, i.e.,  $\mathcal{U}'(c) > 0$  and  $\mathcal{U}''(c) < 0$ , for example. The differences from Rishel's [17] paper are that events parameter vector  $\vec{a}$  is included in the terminal wealth utility making  $\vec{a}$  genuinely included in the model and also the cumulative discounted running utility for consumption of wealth is included as part of the optimal objective. The maximization in the case of constraints is over some specified feasible control domains, i.e.,  $\vec{u} \in \mathcal{D}_u$  and  $c \in \mathcal{D}_c$ , and is subject to the non-negative feasibility conditions on consumption  $C(t) \geq 0$  and on wealth  $W(t) \geq 0$  making zero wealth an absorbing state to avoid the possibility of arbitrage (Merton [15, Chapter 6]). Thus, The optimization problem is subject to the zero wealth absorbing boundary condition

$$v^*(t, 0^+; \vec{a}) = e^{-\beta(\vec{a})(T-t)} \mathcal{U}_f(0; \vec{a}) + \mathcal{U}(0) \int_t^T e^{-\beta(\vec{a})(\tau-t)} d\tau. \quad (10)$$

to account for the non-negative wealth condition  $W(t) \geq 0$  (Merton [15, Chapter 6]). It is assumed that consumption must be zero when wealth is zero. The bequest or terminal wealth condition  $v^*(T, w; \vec{a}) = \mathcal{U}_f(w; \vec{a})$ , must also be satisfied. The

events parameter vector  $\vec{a} = (a_1, a_2)$  forms an extension of the state space from the wealth state  $w$ .

Assuming that  $v^*(t, w; \vec{a}) = v^*(t, w; a_1, a_2)$  is continuously differentiable in  $t$ , twice continuously differentiable in  $w$  and continuous in the events parameter vector  $\vec{a}$  between scheduled jumps, plus sufficiently integrable, then stochastic dynamic programming equations between scheduled jumps (see Kushner [11], and Gihman and Skorohod [5] for the less familiar Poisson driven terms) is

$$\begin{aligned}
 0 &= v_t^*(t, w; \vec{a}) - \beta(\vec{a})v^*(t, w; \vec{a}) + \max_{\{\vec{u}, c\}} [\mathcal{U}(c) \\
 &+ \left( (r(\vec{a}) + \vec{u}^\top (\vec{\mu}(\vec{a}) - r(\vec{a})\vec{1})) w - c \right) v_w^*(t, w; \vec{a}) \\
 &\quad + \frac{1}{2} \vec{u}^\top \sigma(\vec{a}) \sigma^\top(\vec{a}) \vec{u} w^2 v_{ww}^*(t, w; \vec{a}) \\
 &\quad + \lambda(t) \int_{\mathcal{J}_1} \left[ v^* \left( t, (1 + \vec{J}_1^\top(t, \hat{j}_1; \vec{a}) \vec{u}) w; \right. \right. \\
 &\quad \left. \left. (1 + J_{1,0}(t, \hat{j}_1; \vec{a})) a_1, a_2 \right) \right. \\
 &\quad \left. - v^*(t, w; \vec{a}) \right] \phi_1(\hat{j}_1) d\hat{j}_1 \\
 &= v_t^*(t, w; \vec{a}) - \beta(\vec{a})v^*(t, w; \vec{a}) + \mathcal{U}(c^*) \\
 &+ \left( (r(\vec{a}) + (\vec{u}^*)^\top (\vec{\mu}(\vec{a}) - r(\vec{a})\vec{1})) w - c^* \right) v_w^*(t, w; \vec{a}) \\
 &\quad + \frac{1}{2} (\vec{u}^*)^\top \sigma(\vec{a}) \sigma^\top(\vec{a}) \vec{u}^* w^2 v_{ww}^*(t, w; \vec{a}) \\
 &\quad + \lambda(t) \int_{\mathcal{J}_1} \left[ v^* \left( t, (1 + \vec{J}_1^\top(t, \hat{j}_1; \vec{a}) \vec{u}^*) w; \right. \right. \\
 &\quad \left. \left. (1 + J_{1,0}(t, \hat{j}_1; \vec{a})) a_1, a_2 \right) \right] \phi_1(\hat{j}_1) d\hat{j}_1,
 \end{aligned} \tag{11}$$

where  $\vec{u}^* = \vec{u}^*(t, w; \vec{a}) \in \mathcal{D}_u$  and  $c^* = c^*(t, w; \vec{a}) \in \mathcal{D}_c$  are the optimal controls if they exist,  $v_w^*$  and  $v_{ww}^*$  are the partial derivatives with respect to wealth, when  $T_{2,\ell+1} > t > T_{2,\ell}$ , or in jump time notation  $T_{2,\ell+1}^- > t \geq T_{2,\ell}^+$ , for  $\ell = N_2, N_2 - 1, \dots, 1, 0$  by counting backward, given scheduled values at prejump times  $T_{2,\ell+1}^-$ . By right continuity, solution values at  $T_{2,\ell}^+$  and  $T_{2,\ell}$  are equivalent. Let  $T_{2,N_2+1} = T_{2,N_2+1}^- \equiv T$  finally and  $T_{2,0} = T_{2,0}^+ \equiv 0$  initially, for notational convenience to include the non-jump endpoints in the jump time accounting. Positivity of wealth when there is an unscheduled jump in wealth as in (11) requires the additional positivity condition that  $(1 + \vec{J}_1^\top(t, \hat{j}_1; \vec{a}) \vec{u}) \geq 0$ .

At the scheduled jumps, counting backward from  $t = T_{2,\ell}^+$  to  $t = T_{2,\ell}^-$ , the optimal, expected value function jumps due to the fact that the scheduled jump times are not averaged as are the unscheduled Poisson jump times (see Rishel [17] for a somewhat different formulation) and takes its value from the scheduled event jump in the amplitude  $A_2$  and the jump in wealth  $W$ ,

$$\begin{aligned}
 v^*(T_{2,\ell}^-, w; \vec{a}) &= \int_{\mathcal{J}_2} v^* \left( T_{2,\ell}^+, \left( 1 + \vec{J}_2^\top \left( T_{2,\ell}^-, \hat{j}_2; \vec{a} \right) \right. \right. \\
 &\quad \left. \left. \cdot \vec{u}_{2,\ell}^- \right) w; a_1, \right. \\
 &\quad \left. \left( 1 + J_{2,0} \left( T_{2,\ell}^-, \hat{j}_2; \vec{a} \right) \right) a_2 \right) \\
 &\quad \phi_2(\hat{j}_2) d\hat{j}_2,
 \end{aligned} \tag{12}$$

for  $\ell = N_2, N_2 - 1, \dots, 2, 1$  counting backward, where the optimal control at  $T_{2,\ell}^-$  is given by  $\vec{u}_{2,\ell}^- \equiv \vec{u}^*(T_{2,\ell}^-, w; \vec{a})$ , since  $W(T_{2,\ell}^+) = \left( 1 + \vec{J}_2^\top \left( T_{2,\ell}^-, \hat{j}_2; \vec{a} \right) \vec{u}_{2,\ell}^- \right) W(T_{2,\ell}^-)$ . The right continuity property and the instantaneous jump property have been used. Positivity of wealth when there is a scheduled jump in wealth as in (12) requires the additional positivity

condition that  $\left( 1 + \vec{J}_2^\top \left( t, \hat{j}_2; \vec{a} \right) \vec{u} \right) \geq 0$ . Since dynamic programming is a backward formulation in time, the jump condition (12) is an implicit condition for  $v^*(T_{2,\ell}^-, w; \vec{a})$  rather than  $v^*(T_{2,\ell}^+, w; \vec{a})$  which is found from (11). The implicitness is due to the argument of the maximum, the optimal control  $\vec{u}^*(T_{2,\ell}^-, w; \vec{a})$ . Equation (12) can be conceptualized as the dynamic programming equation for the artificial infinitesimal backward time step  $T_{2,\ell}^+ > t \geq T_{2,\ell}^-$ , given previously calculated values at  $T_{2,\ell}^+$ . There is no similar jump formula for the optimal consumption at  $T_{2,\ell}^-$  since the consumption does not satisfy a stochastic differential equation like  $W(t)$  and  $\vec{A}(t)$ . Jump condition (12) illustrates the fact that quasi-deterministic jumps are more difficult to treat than Poisson jumps, since the random jump times of the Poisson jumps are smoothed over during the expectation step in stochastic dynamic programming.

If the maximum in (11) is unconstrained and attained by the regular controls  $\vec{u}_{\text{reg}}(t, w; \vec{a})$  and  $c_{\text{reg}}(t)$ , given sufficient differentiability, then on  $T_{2,\ell-1} < t < T_{2,\ell}$  the regular controls implicitly satisfy the dual critical conditions,

$$\mathcal{U}'(c_{\text{reg}}(t, w; \vec{a})) = v_w^*(t, w; \vec{a}), \tag{13}$$

and

$$\begin{aligned}
 &w^2 v_{ww}^*(t, w; \vec{a}) \sigma(\vec{a}) \sigma^\top(\vec{a}) \vec{u}_{\text{reg}}(t, w; \vec{a}) \\
 &= -w v_w^*(t, w; \vec{a}) (\vec{\mu}(\vec{a}) - r(\vec{a})\vec{1}) \\
 &\quad - \lambda(t) w \int_{\mathcal{J}_1} \vec{J}_1 \left( t, \hat{j}_1; \vec{a} \right) v_w^*(t, \\
 &\quad \left( 1 + \vec{J}_1^\top \left( t, \hat{j}_1; \vec{a} \right) \vec{u}_{\text{reg}}(t, w; \vec{a}) \right) w; \\
 &\quad \left( 1 + J_{1,0} \left( t, \hat{j}_1; \vec{a} \right) \right) a_1, a_2 \right) \phi_1(\hat{j}_1) d\hat{j}_1,
 \end{aligned} \tag{14}$$

for the optimal consumption and portfolio policies with respect to the terminal wealth and instantaneous consumption utilities (9). Since these regular control relationships introduce nonlinearities in the dynamic programming equation (11), the solution for  $v^*(t, w; \vec{a})$  requires iteration, in general. Through (13), the regular consumption  $c_{\text{reg}}(t, w; \vec{a})$  inherits jump properties from  $v(t, w; \vec{a})$ .

At the scheduled jumps,  $t = T_{2,\ell}$ , the portfolio policy must jump when the optimal portfolio value jumps, but it may be practical to bring policy constraints into play since the first derivative critical condition for the regular control vector for the stock fractions is

$$\begin{aligned}
 &w \int_{\mathcal{J}_2} \vec{J}_2^- v_w^* \left( T_{2,\ell}^+, \left( 1 + \left( \vec{J}_2^- \right)^\top \vec{u}_{\text{reg},2,\ell}^- \right) w; a_1, \right. \\
 &\quad \left. \left( 1 + J_{2,0,\ell} \right) a_2 \right) \phi_2(\hat{j}_2) d\hat{j}_2 = \vec{0},
 \end{aligned} \tag{15}$$

for the optimal jump condition (12). Here,  $\vec{u}_{\text{reg},2,\ell}^- \equiv \vec{u}_{\text{reg}}(T_{2,\ell}^-, w; \vec{a})$ ,  $\vec{J}_2^- \equiv \vec{J}_2(T_{2,\ell}^-, \vec{j}_2; \vec{a})$ ,  $J_{2,0,\ell} \equiv J_{2,0}(T_{2,\ell}^-, j_{2,0}; \vec{a})$ . Under consumption and portfolio fraction constraints, the nonlinear effects in the optimal value are worsened, but leads to more realistic solutions. The iterative solution is similar to that of the regular control case. The regular policy set,  $\{c_{\text{reg}}, \vec{u}_{\text{reg}}\}$ , leads to new values for the constrained optimal policy set,  $\{c^*, \vec{u}^*\}$ , which in turn leads to new optimal values  $v^*(t, w; \vec{a})$  and new successive iterates (see, for example, Hanson [6])

### V. CONSTANT RELATIVE RISK-AVERSION UTILITY: CANONICAL MODEL

When the utility functions appearing in the objective functional (9) are power functions,

$$\begin{aligned} \mathcal{U}(c) &= c^\gamma / \gamma, \quad c \geq 0, \quad 0 < \gamma < 1; \\ \mathcal{U}_k(a_k) &= |a_k|^{\gamma_k}, \quad a_k \neq 0, \quad \gamma_k \neq 0; \quad k = 1, 2, \\ \mathcal{U}_f(w; \vec{a}) &= \mathcal{U}(w) \mathcal{U}_1(a_1) \mathcal{U}_2(a_2), \quad w \geq 0, \end{aligned} \quad (16)$$

then arbitrary powers of consumption and wealth imply that in order to enforce real values on the utility functions, consumption and wealth must be non-negative. For the jump event parameters,  $a_1$  and  $a_2$ , negative values are permitted to allow the parameters to have negative effects on the model, but the utility depends on the absolute value. The parameter utility also makes the influence of events stronger in the model since the parameters are genuinely in the optimization as well as in the dynamics here. This is the case of iso-elastic marginal utility or constant relative risk-aversion (CRRA). With these power utility functions, a good guess for the form of the canonical solution is by partial multiplicative separation of variables,

$$v^*(t, w; \vec{a}) = \mathcal{U}_f(w; \vec{a}) v_0(t; \vec{a}), \quad (17)$$

where the parameter-dependent, separated time function  $v_0(t; \vec{a})$  is to be determined. The absorbing boundary condition (10) is automatically satisfied with  $v^*(t, 0^+; \vec{a}) = 0$  by (17) since  $\mathcal{U}(0^+) = 0$  and  $\mathcal{U}_f(0^+; \vec{a}) = 0$  through (16).

Substitution of the solution form (17) yields an explicit linear dependence on the wealth for the regular control consumption values as in the canonical case, using (13),

$$\begin{aligned} c_{\text{reg}}(t, w; \vec{a}) &= w \cdot c_{0, \text{reg}}(t; \vec{a}) \\ &\equiv \frac{w}{[\mathcal{U}_1(a_1) \mathcal{U}_2(a_2) v_0(t; \vec{a})]^{1/(1-\gamma)}} \\ &= \frac{w q_2(\vec{a})}{v_0^{1/(1-\gamma)}(t; \vec{a})}, \end{aligned} \quad (18)$$

using  $\mathcal{U}'(c) = \gamma \mathcal{U}(c)/c$  and  $v_w^*(t, w; \vec{a}) = \gamma \mathcal{U}_f(w; \vec{a}) v_0(t; \vec{a})/w$ , provided  $v_0(t; \vec{a}) \neq 0$  and provided  $a_k \neq 0$  for each  $k$ , where

$$q_2(\vec{a}) \equiv 1 / [\mathcal{U}_1(a_1) \mathcal{U}_2(a_2)]^{1/(1-\gamma)}.$$

However, the regular consumption depends on a reciprocal nonlinear power of  $v_0(t; \vec{a})$  with the power in the range  $(-\infty, -1)$ . For the stock fractions, there is an implicit form, independent of wealth, found by (14), with

$$\begin{aligned} v_{ww}^*(t, w; \vec{a}) &= \gamma(\gamma - 1) \mathcal{U}_f(w; \vec{a}) v_0(t; \vec{a}) / w^2, \\ \mathcal{U}((1 + \vec{J}_1^\top \vec{u})w) &= \gamma \mathcal{U}(1 + \vec{J}_1^\top \vec{u}) \cdot \mathcal{U}(w), \\ \mathcal{U}_1((1 + J_{1,0}) \cdot a_1) &= \mathcal{U}_1(1 + J_{1,0}) \cdot \mathcal{U}_1(a_1), \end{aligned}$$

such that

$$\vec{u}_{\text{reg}}(t; \vec{a}) = \frac{1}{1-\gamma} (\sigma \sigma^\top)^{-1}(\vec{a}) \left[ \vec{\mu}(\vec{a}) - r(\vec{a}) \vec{1} \right. \\ \left. \vec{I}_1'(\vec{u}_{\text{reg}}(t; \vec{a}), t; \vec{a}) \right], \quad (19)$$

where the control gradient of  $I_1$  is denoted by

$$\begin{aligned} \vec{I}_1'(\vec{u}; t; \vec{a}) &= \gamma^2 \int_{\mathcal{J}_1} \vec{J}_1(t, \hat{j}_1; \vec{a}) \frac{\mathcal{U}(1 + \vec{J}_1^\top(t, \hat{j}_1; \vec{a}) \vec{u})}{(1 + \vec{J}_1^\top(t, \hat{j}_1; \vec{a}) \vec{u})} \\ &\quad \cdot \mathcal{U}_1(1 + J_{1,0}(t, \hat{j}_1; \vec{a})) \psi(t, \hat{j}_1; \vec{a}) \phi_1(\hat{j}_1) d\hat{j}_1, \end{aligned}$$

with the primary source of implicitness is expressed by

$$\psi(t, \hat{j}_1; \vec{a}) \equiv \frac{v_0(t; (1 + J_{1,0}(t, \hat{j}_1; \vec{a})) a_1, a_2)}{v_0(t; a_1, a_2)}, \quad (20)$$

provided that the diffusion matrix,  $\sigma(\vec{a}) \sigma^\top(\vec{a})$ , is invertible. Note the fact that the  $\vec{u}_{\text{reg}}(t; \vec{a})$  is independent of the wealth,  $w$ , a crucial property needed for partial separability. However,  $\vec{u}_{\text{reg}}$  is not independent of event parameter vector  $\vec{a}$ . The function  $\psi(t, \hat{j}_1; \vec{a})$  in (20) signifies the degree of nonseparability of the parameter vector  $\vec{a}$  from the time dependence. Hence the regular stock fraction policy depends on the separated value function  $v_0$  through  $\psi$ , but only through the relative dependence on the unscheduled events parameter  $a_1$ , between deterministic, scheduled jump events.

For more realism, constraints on the stock fraction control vector  $\vec{u}$  will be assumed,

$$u_i^*(t; \vec{a}) = \max[U_{\min, i}, \min[U_{\max, i}, u_{\text{reg}, i}(t; \vec{a})]],$$

for  $i = 1, 2, \dots, N_1$ , in the case of component-wise constraints for the stock fraction control domain  $\mathcal{D}_u$ , where  $U_{\min, i}$  and  $U_{\max, i}$  are the finite lower and upper bounds on the  $i$ th stock fraction, respectively. Similarly, constraints on consumption,  $c \in \mathcal{D}_c$ , may lead to the optimal consumption relative to wealth in the form,

$$\begin{aligned} c_0^*(t, w; \vec{a}) &= c^*(t, w; \vec{a}) / w \\ &= \min [c_{0, \text{reg}}(t, w; \vec{a}), C_{0, \text{max}}], \end{aligned}$$

where  $C_{0, \text{max}}$  is the spending cap relative to wealth. In view of the vanishing denominator problem in the canonical solution for CRRA model  $c_{\text{reg}}(t, w; \vec{a})$  in (18), a consumption cap is essential to avoid infinite consumption whenever  $a_1 = 0$ ,  $a_2 = 0$  or  $v_0(t; \vec{a}) = 0$ .

Substitution of the power solution form (17) and the constrained optimal control vector  $\vec{u}^*(t; \vec{a})$  corresponding to the regular control vector  $\vec{u}_{\text{reg}}(t; \vec{a})$  in (18-19) into the stochastic dynamic programming equation (11) leads to an ordinary differential equation depending on the vector parameter  $\vec{a}$ . This equation can be viewed as an implicit Bernoulli equation with variable coefficients for sufficiently small parameter values,

$$\begin{aligned} 0 &= v_0'(t; \vec{a}) \\ &+ (1 - \gamma) \left( q_1'(t, \vec{u}^*(t; \vec{a}); \vec{a}) v_0(t; \vec{a}) \right. \\ &\quad \left. + \hat{q}_2(t; \vec{a}) v_0^{\frac{\gamma}{\gamma-1}}(t; \vec{a}) \right), \end{aligned} \quad (21)$$

$$\begin{aligned} q_1'(t, \vec{u}; \vec{a}) &\equiv \frac{\partial q_1(t, \vec{u}; \vec{a})}{\partial t} \\ &= \frac{1}{1-\gamma} \left[ -\beta(\vec{a}) + \gamma \left( r(\vec{a}) + \vec{u}^\top (\vec{\mu}(\vec{a}) - r(\vec{a}) \vec{1}) \right) \right. \\ &\quad \left. - \frac{\gamma(1-\gamma)}{2} [\vec{u}^\top (\sigma(\vec{a}) \sigma^\top(\vec{a}))^{-1} \vec{u}] \right. \\ &\quad \left. + \lambda(t) (I_1(\vec{u}, t; \vec{a}) - 1) \right], \end{aligned} \quad (22)$$

$$\widehat{q}_2(t; \vec{a}) \equiv \frac{1}{1-\gamma} \left[ \left( \frac{c_0^*(t; \vec{a})}{c_{0,\text{reg}}(t; \vec{a})} \right)^\gamma - \gamma \left( \frac{c_0^*(t; \vec{a})}{c_{0,\text{reg}}(t; \vec{a})} \right) \right] q_2(\vec{a}), \quad (23)$$

$$I_1(\vec{u}, t; \vec{a}) \equiv \gamma \int_{\mathcal{J}_1} \mathcal{U} \left( 1 + \bar{J}_1^\top \left( t, \widehat{j}_1; \vec{a} \right) \vec{u} \right) \mathcal{U}_1 \left( 1 + J_{1,0} \left( t, \widehat{j}_1; \vec{a} \right) \right) \psi \left( t, \widehat{j}_1; \vec{a} \right) \phi_1 \left( \widehat{j}_1 \right) d\widehat{j}_1, \quad (24)$$

for  $t$  on  $[T_{2,\ell-1}^+, T_{2,\ell}^-]$  for  $\ell = N_2+1, N_2, \dots, 2, 1$  subintervals with  $T_{2,0} \equiv 0$  and  $T_{2,N_2+1} \equiv T$ . The formula defining  $I_1'(\vec{u}, t; \vec{a})$  is the control gradient of  $I_1(\vec{u}, t; \vec{a})$  using the facts that  $\mathcal{U}'(w) = \gamma \mathcal{U}(w)/w$  and  $\mathcal{U}(b \cdot w) = \gamma \mathcal{U}(b) \cdot \mathcal{U}(w)$ . In the presence of control constraints, constrained perturbations of  $q_1(t, \vec{u}; \vec{a})$ , upon replacing the unconstrained  $\vec{u}_{\text{reg}}$  with the constrained optimal  $\vec{u}^*$ , force iterative perturbations on  $v_0(t; \vec{a})$  to yield approximations of the constrained, scaled optimal value  $v_0(t; \vec{a})$ . Similarly, the dependence of the modified function  $\widehat{q}_2(t; \vec{a})$  on the optimal consumption functions  $c_0^*(t; \vec{a})$  and  $c_{0,\text{reg}}(t; \vec{a})$  forces iterative perturbations on  $v_0(t; \vec{a})$ . The advantage is that the perturbation is still independent of the state of the wealth.

The partial separability assumption where  $\vec{a}$  still appears in the time function  $v_0(t; \vec{a})$  is mainly due to the jumps in the event parameters, but also due to the  $\vec{a}$ -dependence of the coefficients of the variance  $\sigma(\vec{a})$ , the mean return  $\mu(\vec{a})$ , and the utilities  $\mathcal{U}_k(a_k)$ . If these coefficients were independent of  $\vec{a}$ , then the time function could be replaced by just  $v_0(t)$ . Otherwise, the implicit dependence on  $\psi(t, \widehat{j}_1; \vec{a})$  in the integrals  $I_1'$  and  $I_1$  will require iterative, interpolation, or other approximate solutions.

The implicit Bernoulli equation (21) can be formally transformed to an easily integrable formal linear differential equation by the change of variables

$$\theta(t) = v_0^{1-\gamma/(\gamma-1)}(t; \vec{a}) = v_0^{1/(1-\gamma)}(t; \vec{a}),$$

$$0 = \theta'(t) + q_1'(t, \vec{u}^*(t; \vec{a}); \vec{a})\theta(t) + \widehat{q}_2(t; \vec{a}), \quad (25)$$

which has a general solution that can easily be converted to the general solution for the desired time function,

$$v_0(t; \vec{a}) = \theta^{1-\gamma}(t; \vec{a}) = \left[ e^{-\widehat{q}_1(t; \vec{a})} K_0 - \int_t^T \widehat{q}_2(\tau; \vec{a}) e^{\widehat{q}_1(\tau; \vec{a})} d\tau \right]^{1-\gamma}, \quad (26)$$

where  $K_0$  is a constant of integration and

$$\widehat{q}_1(t; \vec{a}) \equiv \int_t^T q_1'(\tau, \vec{u}^*(\tau; \vec{a}); \vec{a}) d\tau,$$

is the cumulative growth rate exponent on  $[t, T]$  for the linear system  $\theta(t)$ . Since  $v_0(t; \vec{a})$  will be only piece-wise continuous and have jumps at scheduled jump times, the constant of integration  $K_0$  will be different on different inter-jump intervals between scheduled jumps, i.e.,

$$v_0(t; \vec{a}) = \left\{ \begin{array}{ll} V_{0,\ell}(t; \vec{a}), & T_{2,\ell-1}^+ \leq t < T_{2,\ell}^-, \\ & \ell = N_2 + 1, \dots, 2, 1 \\ V_{0,\ell}(T_{2,\ell}^-; \vec{a}), & t = T_{2,\ell}^-, \\ & \ell = N_2 + 1, \dots, 2, 1 \end{array} \right\}, \quad (27)$$

with semi-open intervals appropriate for right continuous limits, where  $T_{2,0} = T_{2,0}^+ \equiv 0$  and  $T_{2,N_2+1} = T_{2,N_2+1}^- \equiv T$  are taken as the starting and stopping times, respectively, for notational convenience.

On the final time step,  $[T_{2,N_2}, T)$ , the optimal utility value function final condition is  $v^*(T, w; \vec{a}) = \mathcal{U}_f(w; \vec{a})$ , so the partially separated time function satisfies the reduced final condition,  $v_0(T; \vec{a}) = 1$ . Thus, using (26),

$$v_0(t; \vec{a}) = V_{0,N_2+1}(t; \vec{a}) \equiv \left[ e^{-\widehat{q}_1(t; \vec{a})} \left( 1 + \int_t^T \widehat{q}_2(\tau; \vec{a}) e^{\widehat{q}_1(\tau; \vec{a})} d\tau \right) \right]^{1-\gamma},$$

where  $\widehat{q}_1(T; \vec{a}) \equiv 0$  defines  $q_1$ 's constant of integration, and the solution for the optimal value function is  $v^*(t, w; \vec{a}) = \mathcal{U}_f(w; \vec{a}) V_{0,N_2+1}(t; \vec{a})$  with optimal controls,  $\vec{u}^*(t; \vec{a})$ , in presence of control constraints, using solutions from (19). On earlier time steps  $[T_{2,\ell-1}^+, T_{2,\ell}^-]$  between scheduled jumps, for  $\ell = N_2, \dots, 2, 1$ , in the natural backward time of dynamic programming, the  $v_0(t; \vec{a})$  using (12) and (17) must satisfy the *local final jump condition* for that interval,

$$V_{0,\ell}(T_{2,\ell}^-; \vec{a}) = I_2(\vec{u}_{2,\ell}^-, T_{2,\ell}^-; \vec{a}),$$

where

$$I_2(\vec{u}, T_{2,\ell}^-; \vec{a}) \equiv \gamma \int_{\mathcal{J}_2} \mathcal{U}(1 + (\bar{J}_{2,\ell}^-)^\top \vec{u}) \mathcal{U}_2(1 + J_{2,0,\ell}^-) \cdot V_{0,\ell+1}(T_{2,\ell}^+; a_1, (1 + J_{2,0,\ell}^-) a_2) \phi_2(\widehat{j}_2) d\widehat{j}_2. \quad (28)$$

and the corresponding control jump condition is given by the optimal control,

$$\vec{u}_{2,\ell}^-(\vec{a}) \equiv [u_i^*(T_{2,\ell}^-; \vec{a})]_{N_1 \times 1} = \underset{\vec{u} \in \mathcal{D}_u}{\text{argmax}} [I_2(\vec{u}, T_{2,\ell}^-; \vec{a})],$$

taking the regular control vector when the constraints are satisfied. Also recall that positivity constraint on the wealth multiplying factor  $(1 + \bar{J}_{2,\ell}^- \vec{u})$  must be satisfied. The argument  $(1 + J_{2,0,\ell}^-) a_2$  of  $V_{0,\ell+1}$  in the integral  $I_2$  above requires interpolation of  $V_{0,\ell+1}$  over the scheduled event parameter  $a_2$  to fit the finite difference representation. With the local final condition and the general solution in (26), the time dependent solution is written

$$v_0(t; \vec{a}) = V_{0,\ell}(t; \vec{a}) \equiv \left[ e^{-\widehat{q}_{1,2,\ell}(t; \vec{a})} \left( \left( V_{0,\ell}^- \right)^{1/(1-\gamma)} + \int_t^{T_{2,\ell}^-} \widehat{q}_2(\tau; \vec{a}) e^{\widehat{q}_{1,2,\ell}(\tau; \vec{a})} d\tau \right) \right]^{1-\gamma}, \quad (29)$$

for the interval  $[T_{2,\ell-1}^+, T_{2,\ell}^-]$  when  $\ell = N_2 + 1, \dots, 2, 1$  subintervals between scheduled jumps, where

$$\widehat{q}_{1,2,\ell}(t; \vec{a}) \equiv \int_t^{T_{2,\ell}^-} q_1'(\tau, \vec{u}^*(\tau; \vec{a}); \vec{a}) d\tau,$$

$V_{0,\ell}^- \equiv V_{0,\ell}(T_{2,\ell}^-; \vec{a})$  is given by the local final condition with  $V_{0,\ell+1}(T_{2,\ell-1}^+; \vec{a}) = V_{0,\ell+1}(T_{2,\ell}^-; \vec{a})$  by piece-wise continuity and right continuous limits to supply the value under the integral  $I_2$  with  $\ell + 1$  replaced by  $\ell$ . Note that (29) defines

$V_{0,\ell}(t; \vec{a})$  only implicitly, since  $\hat{q}_2(t; \vec{a})$  depends on  $c_{0,\text{reg}}$  and  $c_{0,\text{reg}}^*$ , which in turn depend on  $v_0(t; \vec{a}) = V_{0,\ell}(t; \vec{a})$ .

The corresponding optimal consumption is

$$\begin{aligned} c_{0,\text{reg}}^*(T_{2,\ell}^-; \vec{a}) &= \min \left[ c_{0,\text{reg}}(T_{2,\ell}^-; \vec{a}), C_{0,\text{max}} \right] \\ &= \min \left[ q_2(\vec{a})/V_{0,\ell}^{1/(1-\gamma)}(T_{2,\ell}^-; \vec{a}), C_{0,\text{max}} \right], \end{aligned} \quad (30)$$

which is piece-wise continuous with jumps at the scheduled jump times due to the jumps of  $v_0(t; \vec{a})$ . The optimal control on  $[T_{2,\ell-1}^+, T_{1,\ell}^-]$  has the same composite form as for  $[T_{2,N_2}^+, T]$ , since it depends only on the diffusive volatility matrix  $\sigma(\vec{a})$ , the mean appreciation rate  $\mu(\vec{a})$  less the interest rate  $r(\vec{a})$ , the jump amplitudes and their distributions.

## VI. FURTHER COMPUTATIONAL CONSIDERATIONS

In Section V, the optimal, expected instantaneous consumption and terminal wealth investment portfolio problem using power utilities reduces the computational complexity of the problem to a much more feasible level than that for the more general problem in Section IV. The main computational difficulty compared to the more standard Gaussian noise problem is the numerical treatment of the marked Poisson process related integrals that appear in the reduced equations for the optimal control  $\vec{u}^*(t, w; \vec{a})$  in both regular form (19) as well as the corresponding jumps, the cumulative growth rate of the linear form,  $\hat{q}_1(t; \vec{a})$ , in (22) for the separated time function  $v_0(t; \vec{a})$  in (21) and the jump conditions for  $v_0(t; \vec{a})$  in the local final condition. Westman and Hanson [21] have developed numerical procedures for treating these marked Poisson jump integrals that are valid for arbitrary jump densities. The procedure generalizes Gaussian quadrature rules using an arbitrary density as the integral weighting function. Given a continuous density  $\phi(z)$ , the *Gaussian-Statistics quadrature* for jump integrals approximates the integrals over continuous functions  $f(z)$  as

$$\bar{f} \equiv \int_{\mathcal{J}} f(z)\phi(z)dz \simeq \sum_{k=1}^n w_k f(z_k),$$

where the  $n$  nodes  $z_k \in \mathcal{J}$  and corresponding  $n$  weights  $w_k$  are related to the first few moments of the density  $\phi(z)$ , which for the two point rule has cubic moment accuracy up to and including skewness. Piece-wise rules, with piece-wise renormalization, were also constructed in Westman and Hanson [21].

The implicit equations governing the regular or optimal controls in (19) and the scheduled jump controls require some iterative procedure such as Newton's method to find the regular optimal controls when they exist using the gradient and Hessian matrix of the integral  $I_1(\vec{u}, t; \vec{a})$  with respect to the stock fraction control  $\vec{u}$ . Since (19) has the functional form  $\vec{u}_{\text{reg}} = K_1 + K_2 \vec{I}_1'(\vec{u}_{\text{reg}})$ , suppressing the  $(t; \vec{a})$  dependence for all quantities, where  $K_1$  and  $K_2$  are functions independent of control  $\vec{u}_{\text{reg}}$ , then the  $(k+1)$ st Newton iterate between jumps is

$$\begin{aligned} \vec{u}_{\text{reg}}^{(k+1)} &\simeq \vec{u}_{\text{reg}}^{(k)} - \left[ K_2 I_1''(\vec{u}_{\text{reg}}^{(k)}) - I_N \right]^{-1} \\ &\quad \cdot \left[ K_1 + K_2 \vec{I}_1'(\vec{u}_{\text{reg}}^{(k)}) - \vec{u}_{\text{reg}}^{(k)} \right], \end{aligned}$$

where  $I_1''(\vec{u})$  is the Hessian matrix of second  $\vec{u}$ -derivatives of  $I_1$  and  $I_N$  is the  $N$ th order identity matrix. Also, the separation imperfection function  $\psi(t, \hat{j}_1; \vec{a})$  in (20) in general requires linear interpolation to evaluate the value function when  $(1 + J_{1,0})a_1$  is not an  $a_1$ -node, so that the Newton's iteration for  $\vec{u}_{\text{reg}}$  is coupled with the iterations for  $\psi$ .

In the case of the control jump condition and if regular controls exist, then the Newton's  $(k+1)$ th iterate for the critical points yields,

$$\vec{u}_{\text{reg}}^{(k+1)} \simeq \vec{u}_{\text{reg}}^{(k)} - \left[ I_2''(\vec{u}_{\text{reg}}^{(k)}) \right]^{-1} \vec{I}_2'(\vec{u}_{\text{reg}}^{(k)}),$$

at the jump time  $t = T_{2,\ell}^-$ . For this case, an additional approximation is required which is the linear interpolation over the scheduled events parameter  $a_2$  to convert the argument  $(1 + J_{2,0,\ell}^-)a_2$  to a proper discrete node of  $V_{0,\ell+1}$  needed for evaluating the integrand of  $I_2$ . This linear interpolation is coupled with the Newton iteration for the critical points of  $I_2$ .

### A. Algorithm Summary

Since there is very little literature about numerical procedures for stochastic dynamic programming with jump processes as compared to those for Brownian motion processes, the numerical procedure for the current problem will be outlined here.

- 1) Initialize stochastic dynamic and financial model parameters, using as realistic values as possible from available data.
- 2) Check validity of model parameters for satisfaction of wealth positivity conditions with respect to the range of control fraction constraints.
- 3) Set up finite numerical grids for  $t$ ,  $w$ ,  $a_1$ , and  $a_2$ , subject to problem conditions:
  - Initialize stochastic dynamic and financial model parameters,
  - Note that the wealth  $w$  dependence has been separated from the  $(t, a_1, a_2)$  dependence in the canonical solution so that the wealth grid is only needed in the final assembly of the final solution upon appending the wealth factors to  $v^*(t, w; \vec{a})$  and  $c^*(t, w; \vec{a})$ .
  - The time grid is structured with scheduled jump times and an inter-jump subgrid between the scheduled jump times.
  - Each scheduled jump time has two representations in the inter-jump accounting: the post-jump at  $T_{2,\ell}^+$  and the prejump at  $T_{2,\ell}^-$ , numerically the same times, but the solutions will have different values due to the scheduled jumps. The jump conditions specified by the integral  $I_2$  are handled as a special loop. Thus, for example, the total number of points  $N_{\text{tot}} = N_2 \cdot (N_{2D} + 1) + 1$ , where  $N_2$  is the number of scheduled jumps and  $N_{2D}$  is the number of divisions per scheduled jump, including two points for each scheduled jump time to represent both pre-jump and post-jump values.
  - For the  $(a_1, a_2)$  grids, avoid the singular nodes of  $c_{\text{reg}}$  in (18) at  $a_1 = 0$  and  $a_2 = 0$ .



- Set up arrays for drift vector  $\vec{\mu}$  and volatility  $\sigma$  since they depend only on the  $(a_1, a_2)$  in the dynamic programming formulation.
- 4) Initialize the wealth independent dynamic programming problem with the time factor  $v_0(t; \vec{a})$ , and related solutions needed to initialize the iterations, at the final time,  $t = T$  or  $\ell = N_2 + 1$ , over the  $(a_1, a_2)$  grids.
  - 5) Loop backward over the scheduled jumps from the last jump ( $\ell = N_2$ ) to the initial time (artificial jump at  $\ell = 0$ ).
  - 6) Loop backward over the subset of the inter-jump times from the prior scheduled pre-jump time to the next post-jump time, i.e., from  $T_{2,\ell+1}^-$  to  $T_{2,\ell}^+$ .
  - 7) Loop over the scheduled parameter grid for  $a_2$ .
  - 8) Loop over an iteration loop for  $\psi$ ,  $\vec{u}_{\text{reg}}$ ,  $\vec{u}^*$  and  $v_0$ , using the inter-jump dynamic programming equations (20, 19, 29).
  - 9) Loop over the unscheduled parameter grid for  $a_1$  within each iteration.
  - 10) If the absolute changes in successive iterations of  $\vec{u}_{\text{reg}}$  are sufficiently less than some prescribed tolerance, then end current iteration loop. Typically, control variables are the slowest to converge, but other variables could be included.
  - 11) End nested  $a_2$  grid and backward inter-jump time loops.
  - 12) Repeat nested  $a_2$  grid loop, iteration loop and  $a_1$  grid loop, but with the jump conditions for wealth independent forms of  $\vec{u}_{\text{reg}}$ ,  $\vec{u}^*$  and  $v_0$  using post-jump  $T_{2,\ell}^+$  values to get pre-jump  $T_{2,\ell}^-$  values using iteration procedures appropriate for the  $I_2$  jump conditions,
  - 13) Loop over the wealth grid with nested  $a_1$ ,  $a_2$  and  $t$  grid loops to assemble the wealth dependent final results:  $v^*(t, w; \vec{a})$ ,  $c_{\text{reg}}(t, w; \vec{a})$  and  $c^*(t, w; \vec{a})$ .
  - 14) Output results:  $v^*(t, w; \vec{a})$ ,  $c^*(t, w; \vec{a})$  and  $\vec{u}^*(t; \vec{a})$ .

Due to the large complexity of this important event model, there are no available convergence proofs, but the heuristic convergence of the algorithm procedure has been computationally demonstrated with a numerical test described in the next section. All approximations in the implementation of the algorithmic procedure are based on sound procedures to the best of our knowledge and wide experience.

## VII. NUMERICAL TEST MODEL

As a numerical test, a simple jump model is considered, bearing in mind that jump processes lead to more analytical complexity than diffusion processes. It is assumed the state space has the dimension of one stock ( $N_1 = 1$ ). Both unscheduled (random) and scheduled (deterministic) jumps result in two equally likely discrete random jump amplitude marks at each jump time, respectively. The reduced model still retains much of the analytical and numerical complexity of the full model, and is represented by the log-return stochastic differential equation,

$$\begin{aligned}
 d \ln(S(t)) &= [\mu(a_1, a_2) - 0.5\sigma^2]dt + \sigma dZ(t) \\
 &+ \int_{\mathcal{J}_1} \ln(1 + J_{1,1}(j_1))\mathcal{P}(dt, dj_1) \quad (31) \\
 &+ \int_{\mathcal{J}_2} \ln(1 + J_{2,1}(j_2))\mathcal{Q}(dt, dj_2)
 \end{aligned}$$

where both the drift  $\mu(a_1, a_2)$  and parameter-less volatility  $\sigma$  are scalar processes. The jump amplitudes are discrete in the marks,  $j_{k,1}$ ,  $\ln(1 + J_{k,1}(j_{k,1})) = q_{k,1}$ , for each jump type,  $k = 1$  when unscheduled or  $k = 2$  when scheduled, and are realized through two possible integer marks  $j_{k,1} = 1$  or  $2$ , with probabilities  $p_{k,1}(j_{k,1}) = 0.5$  and values  $q_{k,1}$  each at each jump time  $T_{k,\ell}$ . For the scheduled process, when  $k = 2$ , only the jump times are scheduled, but the amplitudes or responses are random.

For realistic values for the coefficients, the daily closings of the S&P500 stock index from 1995-1999 (Financial Forecast Center, [3]) are used as a large sample composite estimate of a stock market mutual fund. The S&P500 data has been transformed into changes in the natural logarithm of the index closings from day to day. The use of higher order moments for determining the model coefficients are avoided due to the high ill-conditioning when using nonlinear curve fitting. The Poisson rate is taken as  $\lambda = 3$  per year as a rough estimate of the number of extreme outliers in the data corresponding to the day to day changes in the logarithm of the S&P500 stock index. For unscheduled jumps ( $k=1$ ), approximate extreme values in the logarithmic changes  $q_{1,1,a} = -0.07$  and  $q_{1,1,b} = +0.05$  will be assumed. The extreme values for scheduled jumps ( $k=2$ ) are assumed to be  $q_{2,1,a} = -0.05$  and  $q_{2,1,b} = +0.03$ . With the 1995-1999 S&P500 sample standard deviation 0.010027 (very close to 0.01), the volatility  $\sigma$  can be found, since all jump process parameters have been specified. The corresponding sample mean change in the logarithm of the S&P500 index between trading days is nearly zero or  $9.22 \times 10^{-4}$ . The sample time step has been taken as  $\Delta t = 1/252.6$  years, using the average number of trading days per year in 1995-1999. Finally with the volatility determined, the leading drift  $\mu(0, 0)$  coefficient follows from the expected first moment. The parameter processes,  $(a_1, a_2)$ , are assumed to effect only the drift, so that

$$\mu(a_1, a_1) = \mu(0, 0)(1 - 0.1(a_1 + a_2)),$$

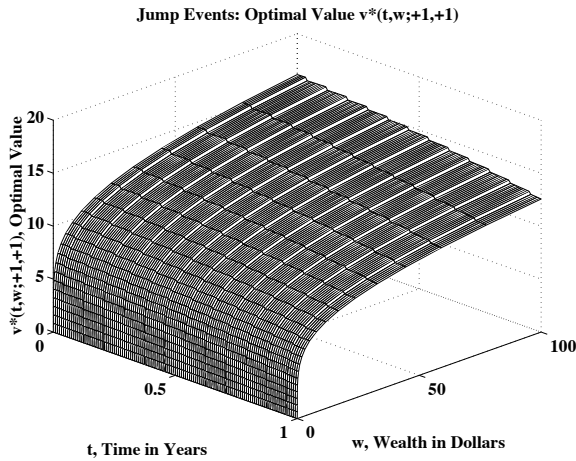
selecting a decreasing linear function in the parameters  $a_1$  and  $a_2$ .

Economic parameters are taken to be constant for this test with  $r = 0.070537$  using the average rate for Moody AAA bonds and  $\beta = 0.046167$  using the average discount rate, both from the Federal Reserve Bank (Federal Reserve System, [2]) for 1995-1999. The powers of the utility functions were taken as  $\gamma = 0.20$  and  $\gamma_1 = 0.10 = \gamma_2$  with  $N_2 = 12$  scheduled jumps per year in the middle of the month. Other parameters of the parameter processes  $(a_1, a_2)$  are taken to be  $J_{1,0} = -0.05$  and  $J_{2,0} = -0.05$ , similar to other jump amplitudes. Control constraints are  $U_{\min} = -2.0$  and  $U_{\max} = +2.0$  for stock fractions, while  $C_{\max} = +400.0$  for consumption.

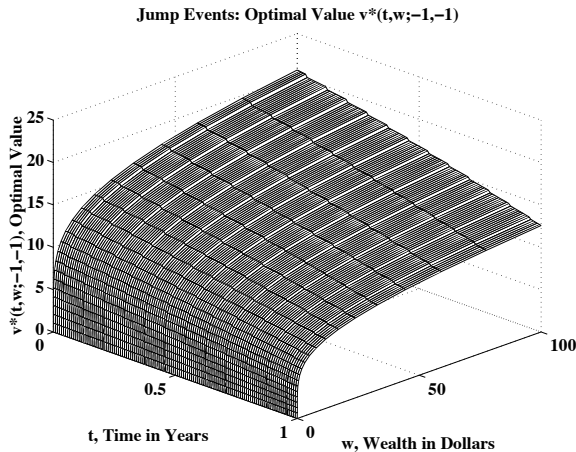
The numerical and graphical results were generated using the MATLAB<sup>TM</sup> matrix laboratory system Full Version 5.3.1R11.1 (Moler et al. [16]). The use of MATLAB<sup>TM</sup> was motivated by the usual preference in financial engineering work to keep the computational demands reasonable.

The optimal value,  $v^*(t, w; \vec{a})$ , is exhibited in Fig. 1(a) for events parameter vector fixed at  $\vec{a} = (+1, +1)$  and in Fig. 1(b) at fixed  $\vec{a} = (-1, -1)$ . In both cases the optimal value

appears to be nearly linearly decreasing with time, except for small jump decrements at scheduled jump times while mainly following the  $\mathcal{U}(w) = w^\gamma/\gamma$  power utility for wealth as a template between jumps as in the canonical solution (17). Although the data used for the scheduled events allow either a negative or a positive jump, the expected jump is a decrement since the magnitude of the negative jump is greater as it is in the S&P500 data. The nearly linear decrease with time is due to the decreasing cumulation of instantaneous consumption as the time horizon  $T$  is approached, provided that the discount rate is sufficiently small in the original objective formulation (9). This perspective is that of stochastic dynamic programming, such that starting at  $(t, w; \vec{a})$  then  $v^*(t, w; \vec{a})$  is interpreted as the optimal expected current value. The time-dependence is much stronger for the event parameter vector  $\vec{a} = (-1, -1)$  in Fig. 1(b) than for  $\vec{a} = (+1, +1)$  in Fig. 1(a).



(a) Optimal expected value approximation  $v^*(t, w; +1, +1)$

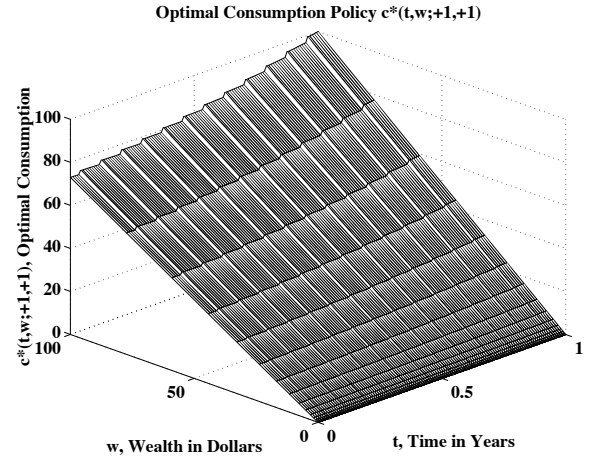


(b) Optimal expected value approximation  $v^*(t, w; -1, -1)$

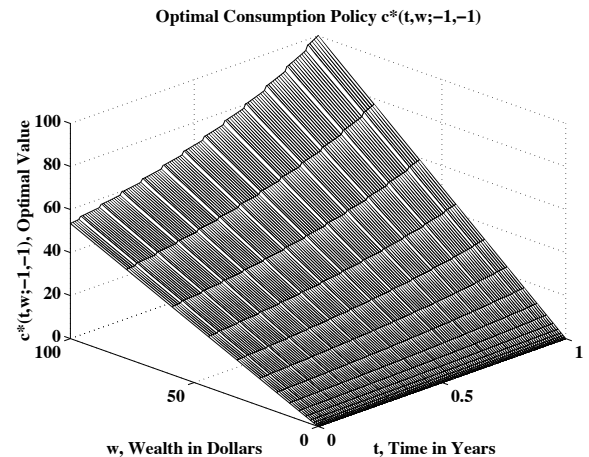
Fig. 1. Optimal expected value approximations versus time  $t$  and wealth  $w$  in numerical results for test model.

The optimal consumption appears to decrease slowly with time, except for small jump increments at the scheduled jump times in Fig. 2(a) for  $c^*(t, w; +1, +1)$  versus time  $t$  and state of wealth  $w$  at fixed events parameter vector  $\vec{a} = (+1, +1)$  while in Fig. 2(b) for  $c^*(t, w; -1, -1)$  for  $\vec{a} = (-1, -1)$ .

The linear variation in (18) of  $c_{\text{reg}}(t, w; \vec{a})$  with the wealth  $w$  is quite clear. Since from (18),  $c_{\text{reg}}(t, w; \vec{a})$  has a time dependence proportional to  $v_0^{-1/(1-\gamma)}(t; \vec{a})$  while  $v^*(t, w; \vec{a})$  is proportional to  $v_0(t; \vec{a})$ ,  $c_{\text{reg}}(t, w; \vec{a})$  and  $v^*(t, w; \vec{a})$  have time rates of change of opposite sign, so  $c_{\text{reg}}(t, w; \vec{a})$  is generally increasing while  $v^*(t, w; \vec{a})$  is decreasing in time. For the opposite extremes in parameter values,  $\vec{a} = (-1, -1)$ , the values of  $c^*(t, w; -1, -1)$  are generally smaller, decreasing in time, except for small jumps at scheduled jumps.



(a) Optimal expected consumption policy approximation  $c^*(t, w; +1, +1)$

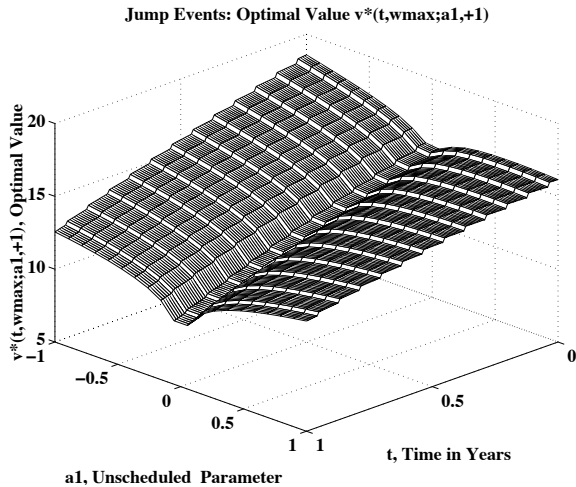


(b) Optimal consumption policy approximation  $c^*(t, w; -1, -1)$

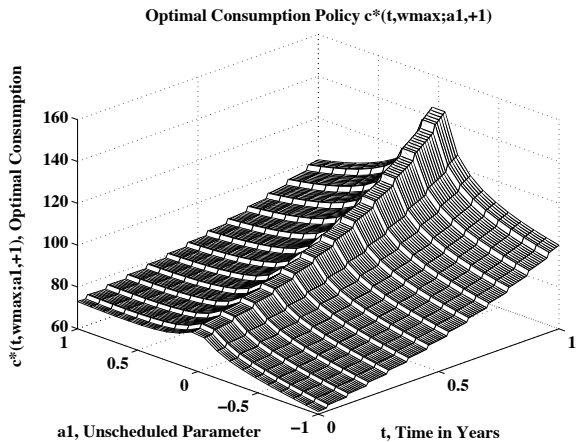
Fig. 2. Optimal consumption policy approximations versus time  $t$  and wealth in numerical results for test model.

The variation of the approximate optimal quantities with the events parameters  $a_1$  corresponding to unscheduled events and  $a_2$  for scheduled events is also quite interesting. In Fig. 3(a) the optimal value,  $v^*(t, w_{\text{max}}; a_1, +1)$  and in Fig. 3(b) the optimal consumption policy  $c^*(t, w_{\text{max}}; a_1, +1)$  illustrated, at the maximal constraint on wealth,  $w_{\text{max}} = 100$ , versus time  $t$  and  $a_1$  at fixed  $a_2 = +1$ . The dependence on the parameter  $a_1$  is very strong reflecting the direct economic dependence on the utility  $\mathcal{U}_1(a_1) = |a_1|^{\gamma_1}$  with  $\gamma_1 = 0.1$  for fixed  $t$ , but with additional dynamical effects from  $v_0(t; \vec{a})$ . There is a step-like character at scheduled jumps, also from  $v_0(t; \vec{a})$ . The variation of the value and consumption policy are complementary in that

they are changing in opposite directions with respect to the parameter  $a_1$ . Note that only 20 discrete values were used for the jump event parameters, so the lack of sufficient smoothness in the  $a_1$  is due to the economy of the discrete representation and not a real effect due to the model.



(a) Optimal expected value approximation  $v^*(t, w_{\max}; a_1, +1)$



(b) Optimal consumption policy approximation  $c^*(t, w_{\max}; a_1, +1)$

Fig. 3. Optimal expected value approximation and the optimal consumption policy versus time  $t$  and unscheduled events parameter  $a_1$  at  $w_{\max} = 100$  in numerical results for test model.

The optimal control policy for stock 1 fraction  $u^*(T/2; a_1, a_2)$  at the midpoint of the time horizon interval,  $t = T/2$ , versus both event parameters  $a_1$  and  $a_2$  is shown in Fig. 4. With respect to the dependence on the event parameters, the stock fraction optimal control policy surface appears to be an increasing function of the parameter values  $a_1$  and  $a_2$ . Further, quantitative interpretation is difficult since  $u_{\text{reg}}(t; \vec{a})$  satisfies a complicated implicit equation in (19-20). However, since the main coupling between the parameters and the stock fraction  $u_1(t; \vec{a})$  is in the fraction-drift product  $u_1(t; \vec{a}) \cdot \mu(\vec{a})$  which has an optimum when  $\partial u_1 / \partial a_k = -(u_1 / \mu) \partial \mu / \partial a_k$  and  $\partial \mu / \partial a_k < 0$  in the linear drift model chosen here, this leads to the intuitive interpretation that  $\partial u_1 / \partial a_k > 0$ .

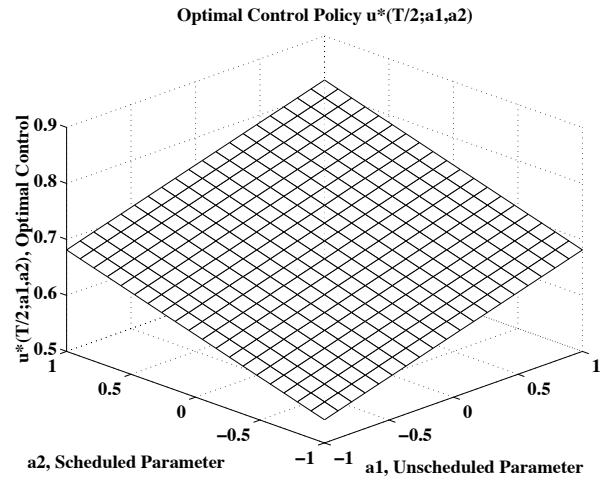


Fig. 4. Optimal control policy approximation  $u^*(T/2; a_1, a_2)$  versus unscheduled events parameter  $a_1$  and scheduled events parameter  $a_2$  in numerical results for test model.

## VIII. CONCLUSIONS

The portfolio optimization model for investment wealth dependent on external jump events introduced by Rishel [17] has been improved and generalized. The underlying stock price, as well as random scheduled and quasi-deterministic unscheduled event jump processes have been modeled consistently by Markov noise in continuous time and deterministic processes have been modeled by the generalized functions corresponding to the stochastic jump processes. The Markov noise includes both Brownian motion for background noise and marked, space-time Poisson processes for rare random jump events, while the analogous quasi-deterministic processes are modeled by differentials of right continuous step functions for scheduled events.

The expected terminal wealth utility objective of Rishel [17] has been extended by including the scheduled and unscheduled events jump parameters in a genuine way by including them in the terminal utility, while consumption has been added in terms of a cumulative instantaneous utility. Linear parameter stochastic quasi-deterministic processes are used systematically for more realism. Discounting has been included in the terminal objective and the instantaneous or running objective, as it would be in any policy strategy sensitive to other opportunities that might produce higher gains, lower costs or more returns. Also, constraints are placed on the stock fraction controls to make optimal control computation for the power utility, jump model finite, better-posed and more realistic.

Formulae are carefully worked out for the piece-wise continuous solutions with jump conditions for the power utility models of the constant relative risk aversion type. Overall, the modifications make the optimal portfolio jump model more realistic and computationally feasible. The canonical power utility model solutions are not exactly separable with respect to parameter arguments due to the presence of distributed Poisson jump amplitudes and jump event parameters for both scheduled and unscheduled events. Computational techniques are given to handle iterations about the canonical power utility model solutions for complications due to implicitly defined

stock fraction control policies and due to jump perturbations in the jump event parameter arguments of the optimal, expected value. The approximate canonical model approach greatly reduces the computational demands over the conventional computational stochastic dynamic programming approach. Optimal, expected value, stock fraction and consumption results are illustrated for a numerical model test problem with two discrete random jump amplitudes in each of scheduled and unscheduled type jump events. Computational feasibility has been demonstrated using the matrix laboratory system MATLAB<sup>TM</sup> for the numerical solution development, rather than a large scale programming code. The major contribution of this paper is the successful computation of the results considering the complexity of the jump processes used in this application.

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