RISK-NEUTRAL OPTION PRICING FOR LOG-UNIFORM JUMP-AMPLITUDE JUMP-DIFFUSION MODEL

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Reduced European call and put option formulas by risk-neutral valuation are given. It is shown that the European call and put options for log-uniformjump-diffusion models are worth more than that for the Black-Scholes (diffusion) model with the common parameters. Due to the complexity of the jump-diffusion models, obtaining a closed option pricing formula like that of Black-Scholes is not tractable. Instead, a Monte Carlo algorithm is used to compute European option prices. Monte Carlo variance reduction techniques such as both antithetic and optimal control variates are used to accelerate the calculations by allowing smaller sample sizes. The numerical results show that this is a practical, efficient and easily implementable algorithm.

KEY WORDS: option pricing, jump-diffusion model, Monte Carlo method, antithetic variates, optimal control variates, variance reduction.

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1. INTRODUCTION

Two seminal papers, Black and Scholes (1973) and Merton (1973), were published in the Spring of 1973 on the celebrated Black-Scholes or Black-Scholes-Merton model on an option pricing formula for purely geometric diffusion processes with its associated log-normal distribution. Black and Scholes (1973) produced the model while Merton (1973) gave the mathematical justifications for the model, extensively exploring the underlying and more general assumptions. These papers led to the 1997 Nobel Prize in Economics for Scholes and Merton, since Black died in 1995 (see Merton and Scholes (1995)). The Black-Scholes formula is probably the most used financial formula of all time.

However, in spite of the practical usefulness of the Black-Scholes formula, it suffers from many defects, one defect is quite obvious during market crashes or massive buying frenzies which contradict the continuity properties of the underlying geometric diffusion process. In Merton's (1976) pioneering jump-diffusion option pricing model, he attempted to correct this defect in continuity and used log-normally distributed jump-amplitudes in a compound Poisson process. Merton argued that the portfolio volatility could not be hedged as in the Black-Scholes pure diffusion case, but that the risk-neutral property could preserve the no-arbitrage strategy by ensuring that the expected return grows at the risk-free interest rate on the average . Merton's (1976) solution is the expected value of an infinite set of Black-Scholes call option pricing formulas each one the initial stock price shifted by a jump factor depending on the number of jumps which have a Poisson distribution.

Beyond jumps there are other market properties that should be considered. Log-return market distributions are usually negatively skewed (provided the time interval for the data is sufficiently long), but Black-Schoes log-returns have a naturally skew-less normal distribution. Log-return market distributions are usually leptokurtic, i.e., more peaked than the normal distribution. Log-return market distributions have fatter or heavier tails than the normal distribution's exponentially small tails. For these defects, jump-diffusions offer some correction and more realistic properties. However, time-dependent rate coefficients are important, i.e., non-constant coefficients are important. Stochastic volatility can be just as important as jumps and this is demonstrated by Andersen, Benzoni and Lund (2002).

Several investigators have found statistical evidence that jumps are significant in financial markets. Ball and Torous (1985) studied jumps in stock and option prices. Jarrow and Rosenfeld (1984) investigated the connections between jump risk and the capital asset pricing model (CAPM). Jorion (1989) examined jump processes in foreign exchange and the stock market.

Kou (2002) and Kou and Wang (2004) derived option pricing results for jump-diffusion with log-double-exponentially distributed jump amplitudes. The double-exponential distribution uses one exponential distribution for the positive tail and another for the negative tail, back to back, in the log-return model. Kou and co-worker have done extensive analysis using this jump model. Cont and Tankov (2004) give a fairly extensive account of the theory of option pricing for Lévy processes which include finite variation jump-diffusions as well as generalizations to infinite variation processes. Also general incomplete markets are treated. In the recent literature, many other papers and several books have appeared or will soon appear on jump-diffusions. Øksendal and Sulem (2005) treat control problems for Lévy processes, including jump-diffusions. Hanson (2005) gives a more practical treatment of stochastic processes and control for jump-diffusions.

The purpose of this paper is to give a practical, reduced European call option formula by the risk-neutral valuation method for general jump-diffusions, including those with uniformly distributed jumps. For simplicity, constant coefficients are assumed, so stochastic volatility is also excluded. A collateral result shows that the European call and put based on the general jump-diffusion model are worth more than that based on the Black-Scholes (1973) model with the same common parameters. Since the analysis of the partial sum density for the independent identically distributed random variables (IIDs) is very complicated in the case of the uniform jump distribution, it is almost impossible to get a closed option pricing formula like that of Black-Scholes. Hence, we provide a Monte Carlo algorithm using variance reduction techniques such as antithetic variates and control variates, so that sample sizes can be reduced for a given Monte Carlo variance. The Monte Carlo method is used to compute risk-neutral valuations of European call and put option prices numerically with the aid of the obtained reduced formula. The numerical results show that this is a practical, efficient and easy to implement algorithm.

In Section 2, the jump-diffusion dynamics of the underlying risky asset and the risk-neutral formula for the European call are introduced. In Section 3, the risk-neutral formulation for the jump-diffusion SDE is derived. In Section 4, properties of sums of independent identically distributed random variables proved in Appendix A are used to show a reduced infinite expansion formula can be given, but it has not been possible to produce a simple closed formula like Black-Scholes. In Section 5, Monte Carlo methods with variance reduction techniques are introduced to compute otherwise intractable risk-neutral option prices and, in Section 6, Monte Carlo simulation results for call and put prices are given along with several comparisons. In Section 7, our conclusions are given. Finally, in Appendix A, properties of sums of uniformly distributed independent identically distributed random variables used in Section 4 are shown.

2. RISKY ASSET PRICE DYNAMICS

The following constant rate, linear stochastic differential equation (SDE) is used to model the dynamics of the risky asset price, $S(t)$:

$$dS(t) = S(t)\left(\mu dt + \sigma dW(t) + J(Q)dN(t)\right), \tag{2.1}$$

where $S_0 = S(0) > 0$, μ is the expected rate of return in absence of asset jumps, σ is the diffusive volatility, $W(t)$ is the Wiener process, $J(Q)$ is the Poisson jump-amplitude, Q is an underlying Poisson amplitude mark process selected for convenience so that

$$Q = \ln(J(Q) + 1),$$

$N(t)$ is the standard Poisson jump counting process with joint mean and variance

$$\mathrm{E}[N(t)] = \lambda t = \mathrm{Var}[N(t)].$$

The jump term in (2.1) is a symbolic abbreviation for the stochastic sum

$$S(t)J(Q)dN(t) = \sum_{k=1}^{dN(t)} S(T_k^-)J(Q_k)\,,$$

where T_k is the kth Poisson jump, Q_k is the kth jump amplitude mark and the pre-jump asset value is $S(T_k^-) = \lim_{t\uparrow T_k} S(t)$, with the limit from left.

Let the density of the jump-amplitude mark Q be uniformly distributed:

$$\phi_Q(q) = \frac{1}{b-a}\left\{\begin{array}{ll} 1, & a \le q \le b \\ 0, & \text{else} \end{array}\right\}, \tag{2.2}$$

where $a < 0 < b$. The mark Q has moments, such that the mean is

$$\mu_j \equiv \mathrm{E}_Q[Q] = 0.5(b+a)$$

and variance is

$$\sigma_j^2 \equiv \mathrm{Var}_Q[Q] = (b-a)^2/12.$$

The original jump-amplitude J has mean

$$\bar{J} \equiv \mathrm{E}[J(Q)] = (\exp(b) - \exp(a))/(b-a) - 1. \tag{2.3}$$

The insufficient amount of jump data in the market make determining the best distribution for the jump amplitude statistically difficult. The uniform distribution has the advantage that it is the simplest distribution, has finite range and has the fattest tails, in fact it is all tail. The finite range property of the uniform distribution is consistent with the New York Stock Exchange (NYSE) *circuit breakers* on extreme market changes as described by Aouriri, Okuyama, Lott and Eglinton (2002). For more details on uniform distributions, see Hanson and Westman (2002a, 2002b), Hanson, Westman and Zhu (2004) and Hanson and Zhu (2004).

Note: in the following context, if absence of any special explanation, $\overline{X}$ will denote the mean of random variable X, that is, $\overline{X} = \mu_X = E[X]$.

According to the Itô stochastic chain rule for jump-diffusions (see Hanson (2005, Chapters 4-5)), the log-return process $\ln(S(t))$ satisfies the constant coefficient SDE

$$d\ln(S(t)) = (\mu - \sigma^2/2)dt + \sigma dW(t) + QdN(t)\ , \tag{2.4}$$

which can be immediately integrated and the logarithm inverted to yield the stock price solution with geometric noise properties,

$$S(t) = S_0 \exp((\mu - \sigma^2/2)t + \sigma W(t) + QN(t)), \tag{2.5}$$

where the jump part of the exponent denotes

$$QN(t) = \sum_{k=1}^{N(t)} Q_k$$

and the Q_k here are independent identically uniformly distributed jump-amplitude marks Q, subject to the notation that $\sum_{k=1}^{0} Q_k \equiv 0$ or else define $Q_0 \equiv 0$. See the jump-diffusion book of Hanson (2005, Chapter 5).

The objective of this paper is to derive a reduced formula and practical algorithm for the European call option price $\mathcal{C}(S_0, T)$, which is a function of the current stock price S_0 and the option expiration time T. There are also suppressed arguments like the strike price K, the stock volatility σ and the risk-free interest rate r, but for jump-diffusions also depends on parameters like the jump rate λ and the mean jump amplitude $\bar{J}$. In contrast to the Black-Scholes (1973) hedge for constructing a portfolio to eliminate the diffusion in the case of a pure diffusion process, Merton

(1976) argued that such hedging was not possible in the case of the jump-diffusion model, but the risk-neutral part of the Black-Scholes strategy could preserve the no arbitrage strategy to ensure that the the expected return grows at risk-free interest rate r on the average. This strategy can be formulated in terms of a change of the drift of jump-diffusion to a risk-neutral drift at rate r or more abstractly in terms of an equivalent change of measure to a risk-neutral measure, say $\mathcal{M}$. Consequently, the European call option price can be formulated as the discounted expectation of the terminal claim $\max[S(T)-K,0]$, so that

$$\mathcal{C}(S_0,T) \equiv e^{-rT}\mathrm{E}_{\mathcal{M}}[\max[S(T)-K,0]]\,. \tag{2.6}$$

It is sufficient to know that such a risk-neutral measure exists. For instance, see the readable accounts in Baxter and Rennie (1996) or Hull (2000) for the pure diffusions, else see Cont and Tankov (2004) for the more general jump-diffusion cases.

3. RISK-NEUTRAL FORMULATION FOR CONSTANT-COEFFICIENT SDE

By the equation (2.5), we can get the expected stock price at expiration time T as stated in the following theorem:

Theorem 3.1 *The* ***Expected Stock Price*** *is*

$$\mathrm{E}[S(T)] = S_0 e^{(\mu+\lambda\bar{J})T}. \tag{3.1}$$

Proof: Using the stock price solution (2.5),

$$\mathrm{E}[S(T)] \;=\; S_0 e^{(\mu-\sigma^2/2)T}\mathrm{E}\left[e^{\sigma W(T)}e^{\sum_{i=1}^{N(T)}Q_i}\right] = S_0 e^{(\mu-\sigma^2/2)T}\mathrm{E}_W\left[e^{\sigma W(T)}\right]\mathrm{E}_{N,Q}\left[\prod_{i=0}^{N(T)}e^{Q_i}\right]$$

$$= S_0 e^{(\mu-\sigma^2/2)T} e^{\sigma^2 T/2} \mathrm{E}_{N,Q}\left[\prod_{i=1}^{N(T)} e^{Q_i}\right] = S_0 e^{\mu T} \mathrm{E}_{N,Q}\left[\prod_{i=1}^{N(T)} e^{Q_i}\right].$$

However, since the marks are uniformly IID random variables and distributed independently of $N(t)$, by iterated expectations,

$$\begin{aligned}
\mathrm{E}\left[\prod_{i=1}^{N(T)} e^{Q_i}\right] &= \mathrm{E}_N\left[\mathrm{E}_{Q|N}\left[\prod_{i=1}^{N(T)} e^{Q_i}\middle| N(T)\right]\right] = \sum_{k=0}^{\infty} p_k(\lambda T)\mathrm{E}\left[\prod_{i=1}^{k} e^{Q_i}\right] \\
&= \sum_{k=0}^{\infty} p_k(\lambda T) \prod_{i=1}^{k} \mathrm{E}\left[e^{Q}\right] = \sum_{k=0}^{\infty} p_k(\lambda T)\mathrm{E}^k[J(Q)+1] \\
&= \sum_{k=0}^{\infty} p_k(\lambda T)\left(\bar{J}+1\right)^k = \sum_{k=0}^{\infty} e^{-\lambda T}\frac{(\lambda T)^k}{k!}\left(\bar{J}+1\right)^k = e^{\lambda T \bar{J}},
\end{aligned}$$

where the Poisson distribution

$$p_k(\lambda T) \equiv e^{-\lambda T}\frac{(\lambda T)^k}{k!}$$

has been used. Hence,

$$\mathrm{E}[S(T)] = S_0 e^{\left(\mu+\lambda\bar{J}\right)T}$$

and therefore the theorem is proved. □

Assume the source of the jumps is due to extraordinary changes in the firm's specifics, such as the loss of a court suit or bankruptcy, but not from external events such as war. Thus, such jump components in the jump-diffusion model represent only non-systematic risks. Hence, the *correlation beta* of the portfolio for non-systematic risk is constructed by *delta hedging* as in Black-scholes and is zero (see Merton (1976)). Further, under this assumption, the jump-diffusion model (2.1) is *arbitrage-free*. However, in the *risk-neutral world*, $\mathrm{E}[S(T)] = S_0 e^{rT}$, so $S_0 e^{(\mu+\lambda\bar{J})T} = S_0 e^{rT}$ and solving for μ, yields the risk-neutral appreciation rate,

(3.2) $$\mu = \mu_{\mathrm{rn}} \equiv r - \lambda\bar{J}.$$

In the the more general case with time-dependent coefficients, let the instant expected price grow rate as the risk-free rate $r(t)$, i.e.,

$$\mathrm{E}[dS(t)/S(t)] = (\mu(t) + \mathrm{E}[J(Q,t)]\lambda(t))dt = r(t)dt$$

thus obtaining the risk-neutral mean rate relationship $\mu(t) = \mu_{\mathrm{rn}}(t) \equiv r(t) - \mathrm{E}[J(Q,t)]\lambda(t)$.

Back to the constant coefficient case and substituting $\mu = r - \lambda\bar{J}$ into (2.1), we get the risk-neutral SDE under the risk-neutral measure $\mathcal{M}$ as the following:

$$\begin{aligned} \frac{dS(t)}{S(t)} &= \left(r - \lambda\bar{J}\right)dt + \sigma dW(t) + \sum_{k=1}^{dN(t)} J(Q_k) \\ &= rdt + \sigma dW(t) + \sum_{k=1}^{dN(t)} \left(J(Q_k) - \bar{J}\right) + \bar{J}\left(dN(t) - \lambda dt\right)\ , \end{aligned} \tag{3.3}$$

where the jump terms are separated into the zero-mean forms of the compound Poisson process for later convenience in statistical calculations.

Before computing the European call option price, several lemmas given in Appendix A on sums of uniformly distributed IID random variables are needed.

4. RISK-NEUTRAL OPTION PRICING SOLUTIONS

Using risk-neutral valuation of the payoff for the European call option in (2.6) with the stock price solution (2.5) and risk-neutral drift coefficient (3.2),

$$\begin{aligned} \mathcal{C}(S_0,T) &\equiv e^{-rT}\mathrm{E}_{\mathcal{M}}[\max(S(T)-K,0)] \\ &= \frac{e^{-rT}}{\sqrt{2\pi}}\sum_{k=0}^{\infty} p_k(\lambda T)\int_{ka}^{kb}\int_{Z_0(s_k)}^{\infty}\left(S_0 e^{(r-\lambda\bar{J}-\sigma^2/2)T+\sigma\sqrt{T}z+s_k} - K\right)e^{-z^2/2}\phi_{\mathcal{S}_k}(s_k)dz ds_k \\ &= \frac{1}{\sqrt{2\pi}}\sum_{k=0}^{\infty} p_k(\lambda T)\int_{ka}^{kb}\int_{Z_0(s_k)}^{\infty}\left(S_0 e^{-(\lambda\bar{J}+\sigma^2/2)T+\sigma\sqrt{T}z+s_k} - Ke^{-rT}\right)e^{-z^2/2}\phi_{\mathcal{S}_k}(s_k)dz ds_k \\ &= \frac{1}{\sqrt{2\pi}}\sum_{k=0}^{\infty} p_k(\lambda T)\mathrm{E}_{\mathcal{S}_k}\left[\int_{Z_0(\mathcal{S}_k)}^{\infty}\left(S_0 e^{-(\lambda\bar{J}+\sigma^2/2)T+\sigma\sqrt{T}z+\mathcal{S}_k} - Ke^{-rT}\right)e^{-z^2/2}dz\right], \end{aligned}$$

where

$$Z_0(s) \equiv \frac{\ln(K/S_0) - (r - \lambda\bar{J} - \sigma^2/2)T - s}{\sigma\sqrt{T}}$$

is the *at-the-money* value of the normal variable of integration z and $\mathcal{S}_k = \sum_{i=1}^{k} Q_i$ is the sum of k jump amplitudes, such that Q_i are uniformly distributed IID random variables over the interval $[a, b]$. In the above equation, the sum $\mathcal{S}_0 = \sum_{i=1}^{0} Q_i \equiv 0$ in reversed sum notation, consistent with the fact that there is no jump when $k = 0$.

Let

$$\begin{aligned}
A(s) &\equiv \frac{1}{\sqrt{2\pi}}\int_{Z_0(s)}^{\infty} S_0 e^{-(\lambda\bar{J}+\sigma^2/2)T+\sigma\sqrt{T}z+s} e^{-z^2/2}dz = \frac{1}{\sqrt{2\pi}} S_0 e^{-(\lambda\bar{J}+\sigma^2/2)T+s}\int_{Z_0(s)}^{\infty} e^{\sigma\sqrt{T}z} e^{-z^2/2}dz \\
&= \frac{1}{\sqrt{2\pi}} S_0 e^{s-\lambda\bar{J}T}\int_{Z_0(s)}^{\infty} e^{-(\sigma\sqrt{T}-z)^2/2}dz = \frac{1}{\sqrt{2\pi}} S_0 e^{s-\lambda\bar{J}T}\int_{-\infty}^{\sigma\sqrt{T}-Z_0(s)} e^{-\zeta^2/2}d\zeta \\
&= S_0 e^{s-\lambda\bar{J}T}\Phi\left(\sigma\sqrt{T} - Z_0(s)\right) = S_0 e^{s-\lambda\bar{J}T}\Phi\left(d_1\left(S_0 e^{s-\lambda\bar{J}T}\right)\right)
\end{aligned}$$

and

$$\begin{aligned}
B(s) &\equiv \frac{1}{\sqrt{2\pi}}\int_{Z_0(s)}^{\infty} K e^{-rT} e^{-z^2/2}dz = \frac{1}{\sqrt{2\pi}} K e^{-rT}\int_{Z_0(s)}^{\infty} e^{-z^2/2}dz \\
&= K e^{-rT}\Phi(-Z_0(s)) = K e^{-rT}\Phi\left(d_2\left(S_0 e^{s-\lambda\bar{J}T}\right)\right),
\end{aligned}$$

where

$$d_1(x) \equiv \frac{\ln(x/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}} \quad (4.1)$$

$$d_2(x) \equiv d_1(x) - \sigma\sqrt{T} \quad (4.2)$$

are the usual *Black-Scholes normal distribution argument functions*, while

$$\Phi(y) \equiv \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{y} e^{-z^2/2}dz \quad (4.3)$$

is the standardized normal distribution. Therefore, our infinite sum call option price formula is

$$
\begin{aligned}
\mathcal{C}(S_0,T) &= \sum_{k=0}^{\infty} p_k(\lambda T)\mathrm{E}_{\mathcal{S}_k}[A(\mathcal{S}_k)-B(\mathcal{S}_k)] \\
&= \sum_{k=0}^{\infty} p_k(\lambda T)\mathrm{E}_{\mathcal{S}_k}\Big[S_0 e^{\mathcal{S}_k-\lambda\bar{J}T}\Phi\left(d_1\left(S_0 e^{\mathcal{S}_k-\lambda\bar{J}T}\right)\right)-Ke^{-rT}\Phi\left(d_2\left(S_0 e^{\mathcal{S}_k-\lambda\bar{J}T}\right)\right)\Big]\,.
\end{aligned}
$$

Alternatively, this can be written more compactly as

$$
\mathcal{C}(S_0,T) \;=\; \sum_{k=0}^{\infty} p_k(\lambda T)\mathrm{E}_{\mathcal{S}_k}\left[C^{(\mathrm{bs})}\left(S_0 e^{\mathcal{S}_k-\lambda\bar{J}T}, T; K, \sigma^2, r\right)\right], \tag{4.4}
$$

where

$$
C^{(\mathrm{bs})}(x,T;K,\sigma^2,r) = x\Phi(d_1(x)) - Ke^{-rT}\Phi(d_2(x)) \tag{4.5}
$$

is the Black-Scholes (1973) formula, but with the stock price argument shifted by a jump factor $e^{\mathcal{S}_k-\lambda\bar{J}T}$ in (4.4). The above equation agrees with Merton's formula (16) in Merton (1976).

The next step is to compute

$$
\mathrm{E}_{\mathcal{S}_k}\left[C^{(\mathrm{bs})}\left(S_0 e^{\mathcal{S}_k-\lambda\bar{J}T}, T; K, \sigma^2, r\right)\right].
$$

However, it will be difficult to produce a simple analytical solution, since the probability density of the partial sums $\mathcal{S}_k$ for the log-uniform jump-diffusion model is very complicated which is shown in Corollary A.1. The approximation to the solution for this problem will be computed by high-level simulation techniques.

4.1 Put-Call Parity

The *put-call parity* is founded on basic maximum function properties (Merton (1973), Hull (2000) and Higham (2004)), so is independent of the particular process, so

$$\mathcal{C}(S_0, T) + Ke^{-rT} = \mathcal{P}(S_0, T) + S_0 \tag{4.6}$$

or solving for the European put option price,

$$\mathcal{P}(S_0, T) = \mathcal{C}(S_0, T) + Ke^{-rT} - S_0, \tag{4.7}$$

in absence of dividends. Also, a risk-neutral argument is given as the following:

$$\begin{aligned}
\mathcal{C}(S_0, T) - \mathcal{P}(S_0, T) &= e^{-rT}\mathrm{E}_{\mathcal{M}}[\max(S(T) - K, 0)] - e^{-rT}\mathrm{E}_{\mathcal{M}}[\max(K - S(T), 0)] \\
&= e^{-rT}\mathrm{E}_{\mathcal{M}}[\max(S(T) - K, 0) - \max(K - S(T), 0)] \\
&= e^{-rT}\mathrm{E}_{\mathcal{M}}[S(T) - K] = e^{-rT}\mathrm{E}_{\mathcal{M}}[S(T)] - Ke^{-rT} \\
&= e^{-rT}(S_0 e^{rT}) - Ke^{-rT} = S_0 - Ke^{-rT}.
\end{aligned}$$

4.2 A Special Non-Jump Case

If $\lambda = 0$, then there are no jump and the model is just a pure diffusion model. In this case, $p_k(\lambda T) = \exp(-\lambda T)(\lambda T)^k/k! = 0$ for $k > 0$ and $p_0(t) = 1$. Also, by the definition, $\mathcal{S}_0 = \sum_{i=1}^{0} Q_i \equiv 0$, a constant. Hence, expectations with respect to $\mathcal{S}_0$ is

$$\mathrm{E}_{\mathcal{S}_0}\left[C^{(\mathrm{bs})}\left(S_0 e^{\mathcal{S}_0 - \lambda \bar{J} T}, T; K, \sigma^2, r\right)\right] = C^{(\mathrm{bs})}\left(S_0, T; K, \sigma^2, r\right)$$

which is the standard Black-Scholes (1973) formulas (4.5, 4.1, 4.2) for the pure diffusion model.

5. MONTE CARLO WITH VARIANCE REDUCTION

From (4.4), the European call option price formulae can be equivalently written as

(5.1) $$\mathcal{C}(S_0, T) \quad = \quad \mathrm{E}_{\gamma(T)}\left[C^{(\mathrm{bs})}\left(S_0 e^{\gamma(T)-\lambda \bar{J} T}, T; K, \sigma^2, r\right)\right],$$

where

(5.2) $$\gamma(T) = \sum_{i=1}^{N(T)} Q_i,$$

Q_i are uniformly distributed IID random variables from $[a, b]$. Note if $\widehat{\gamma}(T) \equiv \gamma(T) - \lambda T \bar{J}$, where $\exp(\lambda T \bar{J}) = \mathrm{E}[\exp(\gamma(T)]$, then $\exp(\widehat{\gamma}(T))$ is an exponential compound Poisson process with the *exponential martingale* property on $[0, T]$ such that

$$\mathrm{E}[\exp(\widehat{\gamma}(T))] = \exp(\widehat{\gamma}(0)) = \exp(0) = 1.$$

So, the Monte Carlo approach may be a good choice to compute risk-neutral option prices numerically. For some treatments of Monte Carlo methods, please see Hammersley and Handscomb (1964), Boyle, Broadie and Glasserman (1997) and more recent compendium of Glasserman (2004).

Let N_i be a sample point taken from the same Poisson distribution as $N(T)$, so that the N_i for $i = 1 : n$ sample points form a set of IID Poisson variates. Given N_i jumps, let the $U_{i,j}$ for $j = 1 : N_i$ jump amplitude sample points, so that they are IID generated uniformly distributed random variables on [0, 1], then the

$$\gamma_i = \sum_{j=1}^{N_i}(a + (b-a)U_{i,j}) = aN_i + (b-a)\sum_{j=1}^{N_i} U_{i,j}$$

for $i = 1 : n$ will be a set of IID random variables on [a, b] having the same compound Poisson distribution with uniformly distributed jump amplitudes as does $\gamma(T)$ in (5.2). Thus, based upon (5.1), an *elementary Monte Carlo estimate* (EMCE) for $\mathcal{C}(S_0, T)$ is the following

$$(5.3) \qquad \widehat{\mathcal{C}}_n = \frac{1}{n}\sum_{i=1}^{n} C^{(\text{bs})}\left(S_0 e^{\gamma_i - \lambda \bar{J} T}, T; K, \sigma^2, r\right) \equiv \frac{1}{n}\sum_{i=1}^{n} C_i^{(\text{bs})} ,$$

such that the $C_i^{(\text{bs})}$ are IID random variables with the γ_i. Then, by the strong law of large numbers,

$$\widehat{\mathcal{C}}_n \to \mathcal{C}(S_0, T) \quad \text{with probability one as} \quad n \to \infty,$$

and by the IID property of $C_i^{(\text{bs})}$ the standard deviation $\sigma_{\widehat{\mathcal{C}}_n}$ is equal to $\sigma^{(\text{bs})}/\sqrt{n}$, where

$$\sigma^{(\text{bs})} = \sqrt{\text{Var}\left[C^{(\text{bs})}(S_0 e^{\gamma(T) - \lambda \bar{J} T}, T; K, \sigma^2, r)\right]} = \sqrt{\text{Var}\left[C_i^{(\text{bs})}\right]},$$

but may be estimated by the sample variance

$$(5.4) \qquad \sigma^{(\text{bs})} \simeq s^{(\text{bs})} = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}\left(\widehat{\mathcal{C}}_n - C_i^{(\text{bs})}\right)^2}.$$

In order to reduce the standard deviation for the elementary Monte Carlo estimate $\sigma_{\widehat{\mathcal{C}}_n}$ by a factor of ten, the number of simulations n has to be increased one hundredfold. However, there are alternative Monte Carlo methods which can have smaller variance than that of EMCE by variance reduction techniques.

5.1 Antithetic Variate and Control Variate Variance Reduction Techniques

Notice that if $U_{i,j}$ is uniformly distributed from $[0, 1]$, then $\widehat{Q}_{i,j} = a + (b - a)U_{i,j}$ is an uniformly distributed (thetic) random variate from $[a, b]$ and so also is the *antithetic random variate*

$$\widehat{Q}_{i,j}^{(a)} = a + (b - a)(1 - U_{i,j}),$$

the counterpart to the *thetic random variate* $\widehat{Q}_{i,j}$. Hence,

$$\gamma_i^{(a)} = \sum_{j=1}^{N_i}(a + (b-a)(1-U_{i,j})) = bN_i - (b-a)\sum_{j=1}^{N_i} U_{i,j} = (b+a)N_i - \gamma_i, \tag{5.5}$$

for $i = 1 : n$, are IID random variables having the same compound Poisson distribution with uniformly distributed jump amplitudes as does $\gamma(T)$ in (5.2). So, the antithetic variates method, first applied to finance by Boyle (1977) (see also Boyle, Broadie and Glasserman (1997) and Glasserman (2004) for more recent and expanded treatments), can be used.

Furthermore, we notice that the variable $\exp(\gamma(T))$ has the expectation $\exp(\lambda T\bar{J})$ known from the proof of Theorem 3.1 and has positive correlation with $C^{(\mathrm{bs})}\left(S_0 e^{\gamma(T)-\lambda\bar{J}T}, T; K, \sigma^2, r\right)$. Therefore, the *control variates technique* can be used to further reduce the variance of Monte Carlo estimation since the technique works faster the higher the correlation between the paired target and control variates, provided that the mean of the control variate is known (Glasserman (2004)). The control variates technique was also first used by Boyle (1977) for financial applications.

From the above analysis, we can get the Monte Carlo simulations with antithetic and control variate techniques. Let

$$\begin{aligned} X_i &= 0.5\left(C^{(\mathrm{bs})}(S_0 e^{\gamma_i-\lambda\bar{J}T}, T; K, \sigma^2, r) + C^{(\mathrm{bs})}(S_0 e^{\gamma_i^{(a)}-\lambda\bar{J}T}, T; K, \sigma^2, r)\right) && (5.6)\\ &\equiv 0.5(C_i^{(\mathrm{bs})} + C_i^{(\mathrm{abs})}), && (5.7)\end{aligned}$$

for $i = 1 : n$ represent the *thetic-antithetic averaged*, Black-Scholes risk-neutral discounted payoffs and

$$Y_i = 0.5\left(\exp(\gamma_i) + \exp\left(\gamma_i^{(a)}\right)\right). \tag{5.8}$$

represents the average of thetic or original and antithetic jump factors that will be used as a variance reducing control variate.

Next, form the control variate adjusted payoff

$$Z_i(\alpha) \equiv X_i - \alpha \cdot (Y_i - \exp(\lambda T \bar{J}))\ ,$$

where $(Y_i - \exp(\lambda T \bar{J}))$ is the control deviation and α is an adjustable control coefficient. Then taking the sample mean of $Z_i(\alpha)$ produces the Monte Carlo estimator for $\mathcal{C}(S_0, T)$:

$$\overline{Z}_n(\alpha) \equiv \frac{1}{n}\sum_{i=1}^{n} Z_i(\alpha) = \frac{1}{n}\sum_{i=1}^{n} X_i - \alpha\frac{1}{n}\sum_{i=1}^{n}\left(Y_i - \exp(\lambda T \bar{J})\right) \equiv \overline{X}_n - \alpha(\overline{Y}_n - \exp(\lambda T \bar{J})),$$

which is an unbiased estimation since $\mathrm{E}[\overline{Z}_n(\alpha)] = \mathcal{C}(S_0, T)$ using IID mean properties $\mathrm{E}[\overline{X}_n] = \mathrm{E}[X_i] = \mathcal{C}(S_0, T)$ by (5.1) and $\mathrm{E}[\overline{Y}_n] = \mathrm{E}[Y_i] = \exp(\lambda T \bar{J})$ from the proof of Thm. 3.1.

The variance of the sample mean $\overline{Z}_n(\alpha)$ is

$$\sigma^2_{\overline{Z}_n(\alpha)} \equiv \mathrm{Var}\left[\overline{Z}_n(\alpha)\right] = \frac{1}{n}\mathrm{Var}[Z_i(\alpha)]\ , \tag{5.9}$$

following from the inherited IID property of the $Z_i(\alpha)$. However,

$$\mathrm{Var}[Z_i(\alpha)] = \mathrm{Var}[X_i] - 2\alpha\mathrm{Cov}[X_i, Y_i] + \alpha^2\mathrm{Var}[Y_i].$$

So, the *optimal control parameter* α^* to minimize $\mathrm{Var}[Z_i(\alpha)]$ is

$$\alpha^* = \frac{\mathrm{Cov}[X_i, Y_i]}{\mathrm{Var}[Y_i]}\ , \tag{5.10}$$

since $\mathrm{Var}[Y_i] > 0$, the Y_i in (5.8) being the positive sum of exponentials. Using this optimal parameter α^*,

$$\mathrm{Var}[Z_i^*] \equiv \mathrm{Var}[Z_i(\alpha^*)] \ = \ \mathrm{Var}[X_i] - \frac{\mathrm{Cov}^2[X_i, Y_i]}{\mathrm{Var}[Y_i]} = \left(1 - \rho^2_{X_i,Y_i}\right)\mathrm{Var}[X_i],$$

where ρ_{X_i,Y_i} is the correlation coefficient between X_i and Y_i. We also know that

$$\text{(5.11)} \quad \mathrm{Var}[X_i] = \frac{1}{4}\left(\mathrm{Var}\left[C_i^{(\mathrm{bs})}\right] + 2\mathrm{Cov}\left[C_i^{(\mathrm{bs})}, C_i^{(\mathrm{abs})}\right] + \mathrm{Var}\left[C_i^{(\mathrm{abs})}\right]\right)$$

$$\text{(5.12)} \quad = \frac{1}{2}\left(1 + \rho_{C_i^{(\mathrm{bs})}, C_i^{(\mathrm{abs})}}\right) \mathrm{Var}\left[C_i^{(\mathrm{bs})}\right]$$

because $\mathrm{Var}\left[C_i^{(\mathrm{abs})}\right] = \mathrm{Var}\left[C_i^{(\mathrm{bs})}\right]$. Therefore,

$$\text{(5.13)} \quad \mathrm{Var}[Z_i^*] = \frac{1}{2}\left(1 - \rho_{X_i,Y_i}^2\right)\left(1 + \rho_{C_i^{(\mathrm{bs})}, C_i^{(\mathrm{abs})}}\right) \mathrm{Var}\left[C_i^{(\mathrm{bs})}\right]$$

$$\text{(5.14)} \quad \leq \frac{1}{2}\left(1 + \rho_{C_i^{(\mathrm{bs})}, C_i^{(\mathrm{abs})}}\right) \mathrm{Var}\left[C_i^{(\mathrm{bs})}\right]$$

$$\text{(5.15)} \quad \leq \frac{1}{2}\mathrm{Var}\left[C_i^{(\mathrm{bs})}\right],$$

because $\rho_{X_i,Y_i}^2 \geq 0$ and provided $\rho_{C_i^{(\mathrm{bs})}, C_i^{(\mathrm{abs})}} \leq 0$. From (5.13-5.14), $\mathrm{Var}[Z_i^*] \leq \mathrm{Var}[X_i]$, which says that the variance of the Monte Carlo estimate with antithetic variates and optimal control variates techniques (AOCV) will be less or equal to the Monte Carlo estimate with antithetic variates (AV) only, where X_i is given in (5.6) and $\mathrm{Var}[X_i]$ is given in (5.11) . By (5.9) and (5.13),

$$\text{(5.16)} \quad \mathrm{Var}[\overline{Z_n^*}] \equiv \mathrm{Var}[\overline{Z}_n(\alpha^*)] = \frac{1}{2n}\left(1 - \rho_{X_i,Y_i}^2\right)\left(1 + \rho_{C_i^{(\mathrm{bs})}, C_i^{(\mathrm{abs})}}\right) \mathrm{Var}\left[C_i^{(\mathrm{bs})}\right],$$

which together with (5.13-5.15), gives $\mathrm{Var}[\overline{Z_n^*}] \leq \frac{1}{2n}\mathrm{Var}[C_i^{(\mathrm{bs})}] = \frac{1}{2}\mathrm{Var}[\widehat{\mathcal{C}}_n]$. This says the variance of the Monte Carlo estimate with AOCV (5.16) is at most half the variance of the elementary Monte Carlo estimate (EMCE) (5.3-5.4) if $\rho_{C_i^{(\mathrm{bs})}, C_i^{(\mathrm{abs})}} \leq 0$. In general, $\rho_{C_i^{(\mathrm{bs})}, C_i^{(\mathrm{abs})}}$ is not always negative, since it also can be positive as demonstrated in the following Figure 1.

However, if the jump amplitude bounds a and b satisfy $a + b = 0$, then $\rho_{C_i^{(\mathrm{bs})}, C_i^{(\mathrm{abs})}} < 0$. We state it in the following Proposition 5.1.

Proposition 5.1 *If* $b/a = -1$, *then* $\rho_{C_i^{(\mathrm{bs})}, C_i^{(\mathrm{abs})}} < 0$.

In order to prove the Proposition 5.1, we need the following Lemma 5.1 which is given in Higham (2004) but with a little modification.

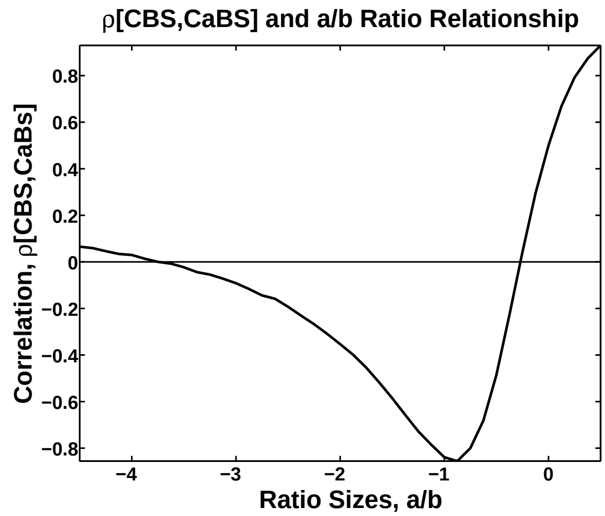

Figure 1: The Monte Carlo estimate of $\rho_{C_i^{(\mathrm{bs})},C_i^{(\mathrm{abs})}}$ is not always negative, but is negative in the region of interest near $a/b = -1$.

Lemma 5.1 *If $f(X)$ and $g(X)$ are both monotonic strictly increasing or both monotonic decreasing functions then, for any random variable X, $\mathrm{Cov}[f(X), g(X)] > 0$.*

Proof: Let Y be a random variable that is independent of X with the same distribution, then $(f(X) - f(Y))(g(X) - g(Y)) > 0$ if $X \neq Y$, otherwise 0. Hence,

$$\begin{aligned} 0 &< E\left[(f(X) - f(Y))(g(X) - g(Y))\right] \\ &= E[f(X)g(X)] - E[f(X)g(Y)] - E[f(Y)g(X)] + E[f(Y)g(Y)]. \end{aligned}$$

Since X and Y are IID, that last right-hand side simplifies to $2E[f(X)g(X)]-2E[f(X)]E[g(X)]$, which is $2\mathrm{Cov}[f(X), g(X)]$, and the result follows. $\square$

Now we prove Proposition 5.1:

Proof: If $b/a = -1$, that is $a + b = 0$, from (5.5), $\gamma_i^{(a)} = -\gamma_i$. So, $C_i^{(\mathrm{bs})} = C_i^{(\mathrm{bs})}(S_0 \exp(-rT\overline{J} + \gamma_i), T; K, \sigma^2, r)$ and $-C_i^{(\mathrm{abs})} = -C_i^{(\mathrm{bs})}(S_0 \exp(-rT\overline{J} - \gamma_i), T; K, \sigma^2, r)$ are strictly increasing functions with respect to γ_i. Hence, $\mathrm{Cov}\left[C_i^{(\mathrm{bs})}, -C_i^{(\mathrm{abs})}\right] > 0$ by the Lemma 5.1. Therefore,

$\mathrm{Cov}\left[C_i^{(\mathrm{bs})}, C_i^{(\mathrm{abs})}\right] < 0$. So, $\rho_{C_i^{(\mathrm{bs})}, C_i^{(\mathrm{abs})}} < 0$. ∎

Remark: In a real market, the ratio b/a will be close to -1, that is $b+a$ will be close to 0 since the skewness of the daily return distribution is not far away from 0 and the skewness is generated by the jump part of the jump-diffusion model. Hence, in general $\rho_{C_i^{(\mathrm{bs})}, C_i^{(\mathrm{abs})}} \leq 0$ in the real market by Proposition 5.1 and the continuity of the function $\rho_{C_i^{(\mathrm{bs})}, C_i^{(\mathrm{abs})}}$ about b/a. For example, the skewness is -0.1952 for 1988-2003 S&P 500 daily return market data and $b/a = -0.9286$ as found by Zhu and Hanson (2005). In fact, in our Monte Carlo algorithm, $\rho_{C_i^{(\mathrm{bs})}, C_i^{(\mathrm{abs})}}$ is about -0.83. So, we can get a lot of benefit from the antithetic variate variance reduction technique by equation (5.13). Anyway, if b/a is far away from -1, the correlation coefficient $\rho_{C_i^{(\mathrm{bs})}, C_i^{(\mathrm{abs})}}$ can be positive which will worsen the variance. In this case, we can use the following Monte Carlo estimator using *optimal control variates* technique (OCV) only:

(5.17) $$\overline{Z_n'} = \frac{1}{n}\sum_{i=1}^{n} C_i^{(\mathrm{bs})} - \beta\left(\frac{1}{n}\sum_{i=1}^{n}\exp(\gamma_i) - \exp(\lambda T\bar{J})\right)$$

where $\beta = \mathrm{Cov}[C_i^{(\mathrm{bs})}, e^{\gamma_i}]/\mathrm{Var}[e^{\gamma_i}]$. So,

(5.18) $$\mathrm{Var}[\overline{Z_n'}] = \frac{1}{n}\left(1 - \rho^2_{C_i^{(\mathrm{bs})}, e^{\gamma_i}}\right)\mathrm{Var}[C_i^{(\mathrm{bs})}].$$

Before we compare $\mathrm{Var}[\overline{Z_n'}]$ with $\mathrm{Var}[\overline{Z_n^*}]$, the following Lemma 5.2 and Lemma 5.3 are needed.

Lemma 5.2

$$\rho^2_{X_i, Y_i} = \frac{2\rho^2_{C_i^{(\mathrm{bs})}, Y_i}}{\left(1 + \rho_{C_i^{(\mathrm{bs})}, C_i^{(\mathrm{abs})}}\right)}.$$

Proof: Since the antithetic pair $\left(C_i^{(\mathrm{bs})}, C_i^{(\mathrm{abs})}\right)$ share common statistical properties, e.g.,

$\mathrm{Var}\left[C_i^{(\mathrm{abs})}\right] = \mathrm{Var}[C_i^{(\mathrm{bs})}]$, so

$$\mathrm{Var}\left[C_i^{(\mathrm{bs})} + C_i^{(\mathrm{abs})}\right] = 2\left(1 + \rho_{C_i^{(\mathrm{bs})}, C_i^{(\mathrm{abs})}}\right) \mathrm{Var}\left[C_i^{(\mathrm{bs})}\right]$$

and

$$\begin{aligned}
\rho^2_{X_i,Y_i} &\equiv \frac{\mathrm{Cov}^2[X_i, Y_i]}{\mathrm{Var}[X_i]\mathrm{Var}[Y_i]} = \frac{\mathrm{Cov}^2\left[\frac{C_i^{(\mathrm{bs})}+C_i^{(\mathrm{abs})}}{2}, \frac{e^{\gamma_i}+e^{\gamma_i^{(a)}}}{2}\right]}{\mathrm{Var}\left[\frac{C_i^{(\mathrm{bs})}+C_i^{(\mathrm{abs})}}{2}\right]\mathrm{Var}\left[\frac{e^{\gamma_i}+e^{\gamma_i^{(a)}}}{2}\right]} \\
&= \frac{\mathrm{Cov}^2\left[C_i^{(\mathrm{bs})} + C_i^{(\mathrm{abs})}, e^{\gamma_i} + e^{\gamma_i^{(a)}}\right]}{\mathrm{Var}\left[C_i^{(\mathrm{bs})} + C_i^{(\mathrm{abs})}\right]\mathrm{Var}\left[e^{\gamma_i} + e^{\gamma_i^{(a)}}\right]} \\
&= \frac{\left(\mathrm{Cov}\left[C_i^{(\mathrm{bs})}, e^{\gamma_i} + e^{\gamma_i^{(a)}}\right] + \mathrm{Cov}\left[C_i^{(\mathrm{abs})}, e^{\gamma_i} + e^{\gamma_i^{(a)}}\right]\right)^2}{2\left(1 + \rho_{C_i^{(\mathrm{bs})}, C_i^{(\mathrm{abs})}}\right)\mathrm{Var}\left[C_i^{(\mathrm{bs})}\right]\mathrm{Var}\left[e^{\gamma_i} + e^{\gamma_i^{(a)}}\right]} \\
&= \frac{2\mathrm{Cov}^2\left[C_i^{(\mathrm{bs})}, e^{\gamma_i} + e^{\gamma_i^{(a)}}\right]}{\left(1 + \rho_{C_i^{(\mathrm{bs})}, C_i^{(\mathrm{abs})}}\right)\mathrm{Var}\left[C_i^{(\mathrm{bs})}\right]\mathrm{Var}\left[e^{\gamma_i} + e^{\gamma_i^{(a)}}\right]}, \\
&= \frac{2\rho^2_{C_i^{(\mathrm{bs})}, e^{\gamma_i}+e^{\gamma_i^{(a)}}}}{\left(1 + \rho_{C_i^{(\mathrm{bs})}, C_i^{(\mathrm{abs})}}\right)} = \frac{2\rho^2_{C_i^{(\mathrm{bs})}, Y_i}}{\left(1 + \rho_{C_i^{(\mathrm{bs})}, C_i^{(\mathrm{abs})}}\right)}.
\end{aligned}$$

□

Hence, equation (5.13) can be also written as

(5.19) $$\mathrm{Var}[Z_i^*] = 0.5\left(1 + \rho_{C_i^{(\mathrm{bs})}, C_i^{(\mathrm{abs})}} - 2\rho^2_{C_i^{(\mathrm{bs})}, Y_i}\right)\mathrm{Var}\left[C_i^{(\mathrm{bs})}\right].$$

Remark: From (5.19) and $\rho_{X,\kappa Y} = \rho_{X,Y}$ where κ is some constant, it does not matter whether Y_i or $S_0 Y_i$ is used as the control variate for variance reduction purposes.

Lemma 5.3 *If X, Y and Z are any random variables, then*

$$\rho^2_{X,Y+Z} = \frac{\mathrm{Var}[Y]}{\mathrm{Var}[Y+Z]}\rho^2_{X,Y} + \frac{\mathrm{Var}[Z]}{\mathrm{Var}[Y+Z]}\rho^2_{X,Z}.$$

Proof:

$$\begin{aligned}
\rho^2_{X,Y+Z} &= \frac{\mathrm{Cov}[X,Y+Z]}{\mathrm{Var}[X]\mathrm{Var}[Y+Z]} = \frac{\mathrm{Cov}[X,Y]+\mathrm{Cov}[X,Z]}{\mathrm{Var}[X]\mathrm{Var}[Y+Z]} \\
&= \frac{\mathrm{Cov}[X,Y]}{\mathrm{Var}[X]\mathrm{Var}[Y+Z]} + \frac{\mathrm{Cov}[X,Z]}{\mathrm{Var}[X]\mathrm{Var}[Y+Z]} \\
&= \frac{\mathrm{Cov}[X,Y]}{\mathrm{Var}[X]\mathrm{Var}[Y]}\cdot\frac{\mathrm{Var}[Y]}{\mathrm{Var}[Y+Z]} + \frac{\mathrm{Cov}[X,Z]}{\mathrm{Var}[X]\mathrm{Var}[Z]}\cdot\frac{\mathrm{Var}[Z]}{\mathrm{Var}[Y+Z]} \\
&= \frac{\mathrm{Var}[Y]}{\mathrm{Var}[Y+Z]}\rho^2_{X,Y} + \frac{\mathrm{Var}[Z]}{\mathrm{Var}[Y+Z]}\rho^2_{X,Z}.
\end{aligned}$$

□

Therefore, based on the above two Lemmas, the following Proposition can be obtained.

Proposition 5.2 *If $\rho_{C_i^{(\mathrm{bs})},C_i^{(\mathrm{abs})}} \leq 0$ and $\rho_{e^{\gamma_i},e^{\gamma_i^{(a)}}} \leq 0$, then $\mathrm{Var}[\overline{Z_n^*}] \leq \mathrm{Var}[\overline{Z_n'}]$.*

Proof: From Lemma 5.3,

$$\begin{aligned}
\rho^2_{C_i^{(\mathrm{bs})},e^{\gamma_i}+e^{\gamma_i^{(a)}}} &= \frac{\mathrm{Var}[e^{\gamma_i}]}{\mathrm{Var}\left[e^{\gamma_i}+e^{\gamma_i^{(a)}}\right]}\rho^2_{C_i^{(\mathrm{bs})},e^{\gamma_i}} + \frac{\mathrm{Var}\left[e^{\gamma_i^{(a)}}\right]}{\mathrm{Var}\left[e^{\gamma_i}+e^{\gamma_i^{(a)}}\right]}\rho^2_{C_i^{(\mathrm{bs})},e^{\gamma_i^{(a)}}} \\
&= \frac{1}{2\left(1+\rho_{e^{\gamma_i},e^{\gamma_i^{(a)}}}\right)}\left(\rho^2_{C_i^{(\mathrm{bs})},e^{\gamma_i}} + \rho^2_{C_i^{(\mathrm{bs})},e^{\gamma_i^{(a)}}}\right) \geq 0.5\rho^2_{C_i^{(\mathrm{bs})},e^{\gamma_i}}
\end{aligned}$$

since $\rho_{e^{\gamma_i},e^{\gamma_i^{(a)}}} \leq 0$. Therefore,

$$\rho^2_{C_i^{(\mathrm{bs})},e^{\gamma_i}} - \rho^2_{C_i^{(\mathrm{bs})},e^{\gamma_i}+e^{\gamma_i^{(a)}}} \leq 0.5\rho^2_{C_i^{(\mathrm{bs})},e^{\gamma_i}} \leq 0.5$$

because $\rho^2_{C_i^{(\mathrm{bs})},e^{\gamma_i}} \leq 1$. That is, $2\rho^2_{C_i^{(\mathrm{bs})},e^{\gamma_i}} - 2\rho^2_{C_i^{(\mathrm{bs})},Y_i} - 1 \leq 0$. From Lemma 5.2 and $\rho_{C_i^{(\mathrm{bs})},C_i^{(\mathrm{abs})}} \leq$

0, $\rho^2_{X_i,Y_i} \geq 2\rho^2_{C_i^{(\text{bs})},Y_i}$. Hence, $2\rho^2_{C_i^{(\text{bs})},e^{\gamma_i}} - \rho^2_{X_i,Y_i} - 1 \leq 2\rho^2_{C_i^{(\text{bs})},e^{\gamma_i}} - 2\rho^2_{C_i^{(\text{bs})},Y_i} - 1 \leq 0$. That is, $0.5(1-\rho^2_{X_i,Y_i}) \leq 1 - \rho^2_{C_i^{(\text{bs})},e^{\gamma_i}}$. With the above inequality and $\rho_{C_i^{(\text{bs})},C_i^{(\text{abs})}} \leq 0$, we get

$$0.5\left(1-\rho^2_{X_i,Y_i}\right)\left(1+\rho_{C_i^{(\text{bs})},C_i^{(\text{abs})}}\right) \leq 1 - \rho^2_{C_i^{(\text{bs})},e^{\gamma_i}},$$

and the result follows by (5.16) and (5.18) . □

Remark: From the proof of Lemma 5.4, we know that the condition $\rho_{e^{\gamma_i},e^{\gamma_i^{(a)}}} \leq 0$ is equivalent to $\exp(a+b) - 1 \leq 2\bar{J}$.

5.2 Estimate of Optimal Control Variate Parameter α^*

In general, we do not know the optimal parameter $\alpha^* = \text{Cov}[X_i, Y_i]/\text{Var}[Y_i]$ exactly, so we need some estimation method for it. Before analyzing it, we need the following Lemma.

Lemma 5.4

$$\text{Var}\left[e^{\gamma_i} + e^{\gamma_i^{(a)}}\right] = 2\left(e^{\lambda T\hat{J}} - 2e^{2\lambda T\bar{J}} + e^{\lambda T(e^{a+b}-1)}\right),$$

where $\hat{J} = (\exp(2b) - \exp(2a))/(2(b-a)) - 1$ *and recall* $\bar{J} = (\exp(b) - \exp(a))/(b-a) - 1$ *from (2.3).*

Proof: Using the properties of the antithetic pair $\left(\gamma_i, \gamma_i^{(a)}\right)$,

$$\begin{aligned} \text{Cov}\left[e^{\gamma_i}, e^{\gamma_i^{(a)}}\right] &= \text{E}\left[e^{\gamma_i}e^{\gamma_i^{(a)}}\right] - \text{E}\left[e^{\gamma_i}\right]E\left[e^{\gamma_i^{(a)}}\right] = \text{E}\left[e^{(a+b)N(T)}\right] - \text{E}^2\left[e^{\gamma_i}\right] \\ &= e^{\lambda T(e^{a+b}-1)} - e^{2\lambda T\bar{J}} \end{aligned}$$

and

$$\text{Var}[e^{\gamma_i}] = \text{E}[e^{2\gamma_i}] - \text{E}^2[e^{\gamma_i}] = e^{\lambda T\hat{J}} - e^{2\lambda T\bar{J}} = \text{Var}\left[e^{\gamma_i^{(a)}}\right].$$

Thus,

$$\begin{aligned}
\mathrm{Var}\left[e^{\gamma_i}+e^{\gamma_i^{(a)}}\right] &= \mathrm{Var}[e^{\gamma_i}]+2\mathrm{Cov}\left[e^{\gamma_i},e^{\gamma_i^{(a)}}\right]+\mathrm{Var}\left[e^{\gamma_i^{(a)}}\right]\\
&= 2\mathrm{Var}[e^{\gamma_i}]+2\mathrm{Cov}\left[e^{\gamma_i},e^{\gamma_i^{(a)}}\right]=2\left(e^{\lambda T\hat{J}}-2e^{2\lambda T\bar{J}}+e^{\lambda T(e^{a+b}-1)}\right).
\end{aligned}$$

□

From Lemma 5.4, $\sigma_Y^2\equiv\mathrm{Var}[Y_i]=\mathrm{Var}\left[0.5\left(e^{\gamma_i}+e^{\gamma_i^{(a)}}\right)\right]=0.5\left(e^{\lambda T\hat{J}}-2e^{2\lambda T\bar{J}}+e^{\lambda T(e^{a+b}-1)}\right)$.

Proposition 5.3 *An unbiased estimator for α^* is*

$$\widehat{\alpha}=\left(\frac{1}{n-1}\sum_{i=1}^{n}X_iY_i-\frac{1}{n(n-1)}\sum_{i=1}^{n}\sum_{j=1}^{n}X_iY_j\right)\frac{1}{\sigma_Y^2}\quad=\quad\frac{n}{n-1}\frac{\overline{XY}_n-\overline{X}_n\overline{Y}_n}{\sigma_Y^2}, \tag{5.20}$$

where $\overline{X}_n=\frac{1}{n}\sum_{i=1}^{n}X_i$ is the sample mean, $\overline{XY}_n$ and $\overline{Y}_n$ have the similar meaning.

Proof: It is necessary to show the condition for an unbiased estimate $E[\widehat{\alpha}]=\alpha^*$ is true. Using the standard technique of splitting the diagonal part out of the double sum and the independence as well as the identical distribution property of the random variables at different compound Poisson sample points for $i=1:n$, then

$$\begin{aligned}
\mathrm{E}[\widehat{\alpha}] &= \mathrm{E}\left[\frac{1}{n-1}\sum_{i=1}^{n}\left(X_iY_i-\frac{1}{n}\sum_{j=1}^{n}X_iY_j\right)\frac{1}{\sigma_Y^2}\right]\\
&= \frac{1}{n-1}\sum_{i=1}^{n}\mathrm{E}\left[\left(1-\frac{1}{n}\right)X_iY_i-\frac{1}{n}\sum_{j=1,j\neq i}^{n}X_iY_j\right]\frac{1}{\sigma_Y^2}\\
&= \frac{1}{n-1}\sum_{i=1}^{n}\left(\frac{n-1}{n}\mathrm{E}[X_iY_i]-\frac{1}{n}\sum_{j=1,j\neq i}^{n}\mathrm{E}[X_i]\mathrm{E}[Y_j]\right)\frac{1}{\sigma_Y^2}\\
&= \frac{n}{n-1}\left(\frac{n-1}{n}\mathrm{E}[XY]-\frac{n-1}{n}\mathrm{E}[X]\mathrm{E}[Y]\right)\frac{1}{\sigma_Y^2}\\
&= \left(\mathrm{E}[XY]-\mathrm{E}[X]\mathrm{E}[Y]\right)/\sigma_Y^2=\mathrm{Cov}[X,Y]/\sigma_Y^2=\alpha^*.
\end{aligned}$$

□

Since $\widehat{\alpha}$ depends on Y_i for $i = 1:n$, the estimate $\widehat{\alpha}$ of α^* introduces a bias into the estimate

$$\widehat{Z}_n = \frac{1}{n}\sum_{i=1}^{n} X_i - \widehat{\alpha}\left(\frac{1}{n}\sum_{i=1}^{n} Y_i - e^{\lambda T \bar{J}}\right). \tag{5.21}$$

Fortunately, in this case, we can compute the bias which asympotically goes to zero at the rate $O(1/n)$ as shown in the following Theorem.

Theorem 5.1 *The estimate $\widehat{Z}_n$ of the call price $\mathcal{C}(S_0, T)$ has bias*

$$b \equiv \mathrm{E}[\widehat{Z}_n] - \mathcal{C}(S_0, T) = \frac{1}{n} \cdot \frac{\mathrm{Cov}[X, (2\mu_Y - Y])Y]]}{\sigma_Y^2},$$

where $\mu_Y = \mathrm{E}[Y_i] = E[Y] = \exp(\lambda T \bar{J})$, $\sigma_Y^2 = \mathrm{Var}[Y_i] = \mathrm{Var}[Y]$, Y has the same distribution as Y_i for $i = 1 : n$.

Proof: Set $\eta_k = \sigma_Y^2 \widehat{\alpha}(Y_k - \mu_Y)$. Then,

$$\begin{aligned}
\eta_k &= \left(\frac{\sum_{i=1}^n X_i Y_i}{n-1} - \frac{\sum_{i=1}^n \sum_{j=1}^n X_i Y_j}{n(n-1)}\right)(Y_k - \mu_Y) \\
&= \frac{\sum_{i=1}^n X_i Y_i Y_k}{n-1} - \frac{\sum_{i=1}^n \sum_{j=1}^n X_i Y_j Y_k}{n(n-1)} - \frac{\mu_Y \sum_{i=1}^n X_i Y_i}{n-1} + \frac{\mu_Y \sum_{i=1}^n \sum_{j=1}^n X_i Y_j}{n(n-1)} \\
&= \frac{1}{n}\sum_{i=1}^n X_i Y_i Y_k - \frac{\sum_{i=1}^n \sum_{j\neq i} X_i Y_j Y_k}{n(n-1)} - \frac{\mu_Y \sum_{i=1}^n X_i Y_i}{n} + \frac{\mu_Y \sum_{i=1}^n \sum_{j\neq i} X_i Y_j}{n(n-1)} \\
&= \frac{X_k Y_k^2 + \sum_{i\neq k} X_i Y_i Y_k}{n} - \frac{\sum_{j\neq k} X_k Y_j Y_k + \sum_{i\neq k}\sum_{j\neq i} X_i Y_j Y_k}{n(n-1)} \\
&\quad - \frac{\mu_Y \sum_{i=1}^n X_i Y_i}{n} + \frac{\mu_Y \sum_{i=1}^n \sum_{j\neq i} X_i Y_j}{n(n-1)} \\
&= \frac{X_k Y_k^2 + \sum_{i\neq k} X_i Y_i Y_k}{n} - \frac{\sum_{j\neq k} X_k Y_j Y_k + \sum_{i\neq k} X_i Y_k^2 + \sum_{i\neq k}\sum_{j\neq i,k} X_i Y_j Y_k}{n(n-1)} \\
&\quad - \frac{\mu_Y \sum_{i=1}^n X_i Y_i}{n} + \frac{\mu_Y \sum_{i=1}^n \sum_{j\neq i} X_i Y_j}{n(n-1)}.
\end{aligned}$$

By the independence of $\{X_i, Y_i\}$ and $\{X_j, Y_j\}$ for $j \neq i$ as well as the identical distribution properties,

$$\begin{aligned} E[\eta_k] &= \frac{\overline{XY^2} + (n-1)\overline{XY}\mu_Y}{n} - \frac{(n-1)\overline{XY}\mu_Y + (n-1)\mu_X\overline{Y^2} + (n-1)(n-2)\mu_X{\mu_Y}^2}{n(n-1)} \\ & -n\frac{\mu_Y\overline{XY}}{n} + \frac{n(n-1)\mu_Y^2\mu_X}{n(n-1)} = \frac{\overline{XY^2} - 2\overline{XY}\mu_Y - \mu_X\overline{Y^2} + 2\mu_X\mu_Y^2}{n} \\ &= \frac{\mathrm{Cov}[X, Y^2] - 2\mu_Y\mathrm{Cov}[X, Y]}{n} = \frac{\mathrm{Cov}[X, Y(Y - 2\mu_Y)]}{n}\,, \end{aligned}$$

where $\mu_X = \mathrm{E}[X_i]$, $\mu_Y = \mathrm{E}[Y_i]$, $\overline{XY} = \mathrm{E}[X_iY_i]$, $\overline{Y^2} = \mathrm{E}[Y_i^2]$ and $\overline{XY^2} = \mathrm{E}[X_iY_i^2]$.

Therefore, the call price estimate bias

$$\begin{aligned} b &\equiv \mathrm{E}[\widehat{Z}_n] - \mathcal{C}(S_0, T) = \mathrm{E}[-\widehat{\alpha}(Y_k - \mu_Y)] = -\mathrm{E}[\sigma_Y^2\widehat{\alpha}(Y_k - \mu_Y)]/\sigma_Y^2 \\ &= -\mathrm{E}[\eta_k]/\sigma_Y^2 = \frac{1}{n}\frac{\mathrm{Cov}[X, Y(2\mu_Y - Y)]}{\sigma_Y^2}. \end{aligned}$$

□

Remark: From Theorem 5.1, we can make the following correction to the estimate $\widehat{Z}_n$:

$$\widehat{\theta} = \widehat{Z}_n - \widehat{b}, \tag{5.22}$$

where $\widehat{b}$ is an estimate of b similar to the estimate $\widehat{\alpha}$ of α^* in (5.20) as the following:

$$\widehat{b} = \left(\frac{1}{n(n-1)}\sum_{i=1}^{n} X_iY_i^{'} - \frac{1}{n^2(n-1)}\sum_{i=1}^{n}\sum_{j=1}^{n} X_iY_j^{'}\right)\frac{1}{\sigma_Y^2} = \frac{1}{n-1}\frac{\overline{XY_n'} - \overline{X}_n\overline{Y'}_n}{\sigma_Y^2}\,, \tag{5.23}$$

where $Y_i^{'} = Y_i(2\mu_Y - Y_i)$, for $i = 1 : n$, $\overline{XY'}_n$, $\overline{X}_n$ and $\overline{Y'}_n$ are sample means. Then, the estimate $\widehat{\theta}$ is an unbiased estimate of $\mathcal{C}(S_0, T)$.

5.3 The Monte Carlo Algorithm

Now it is ready for us to give the Monte Carlo algorithm with antithetic variates and optimal control variates techniques as the following:

The Monte Carlo Algorithm with antithetic and optimal control variates techniques

for $i = 1, ..., n$

  Randomly generate N_i by Poisson inverse transform method;

  for $j = 1 : N_i$

    Randomly generate IID uniform random variables $U_{i,j}$;

  end for j

  Set $\gamma_i = aN_i + (b-a)\sum_{j=1}^{N_i} U_{i,j}$;

  Set $\gamma_i^{(a)} = (a+b)N_i - \gamma_i$;

  Set $C_i^{(\mathrm{bs})} = C^{(\mathrm{bs})}\left(S_0 \exp\left(\gamma_i - \lambda T \bar{J}\right), T; K, \sigma^2, r\right)$;

  Set $C_i^{(\mathrm{abs})} = C^{(\mathrm{bs})}\left(S_0 \exp\left(\gamma_i^{(a)} - \lambda T \bar{J}\right), T; K, \sigma^2, r\right)$;

  Set $X_i = 0.5\left(C_i^{(\mathrm{bs})} + C_i^{(\mathrm{abs})}\right)$;

  Set $Y_i = 0.5\left(\exp(\gamma_i) + \exp\left(\gamma_i^{(a)}\right)\right)$;

end for i

Compute $\widehat{\alpha}$ according to (5.20);

Set $\widehat{Z}_n = \frac{1}{n}\sum_{i=1}^{n} X_i - \widehat{\alpha}(\frac{1}{n}\sum_{i=1}^{n} Y_i - e^{\lambda T \bar{J}})$;

Estimate bias $\widehat{b}$ according to (5.23);

Get European call option estimation $\widehat{\theta} = \widehat{Z}_n - \widehat{b}$;

Get European put option estimation $\widehat{\mathcal{P}}$

    by put-call parity (4.7).

6. MONTE CARLO CALL AND PUT PRICE RESULTS

In this section we provide some numerical results and discussions to illustrate the particular algorithm version of the Monte Carlo method used for the jump-diffusion process in this paper. First of all, we compare our Monte Carlo method using antithetic and optimal control variates (AOCV) techniques having variance given in (5.16) to the elementary Monte Carlo estimation (EMCE) method given in (5.3), as well as to other Monte Carlo variations.

The compound Poisson process is simulated by first using the *inverse transform method* in the leading step of the algorithm given by Glasserman (2004) for the jump counting component process N_i for $i = 1\!:\!n$ and then the N_i jump amplitude antithetic pairs $(\gamma_{i,j}, \gamma_{i,j}^{(a)})$ are simulated together by a standard uniform random number generator to get the $U_{i,j}$ for $j = 1 : N_i$. We implement them with MATLAB 6.5 and run them on the PC with a Pentium4 @ 1.6GHz CPU. The numerical test results for elementary Monte Carlo estimation (EMCE) method are listed in Table 1, Monte Carlo with antithetic variates (AV) only in Table 2, Monte Carlo with optimal control variates (OCV) only in Table 3, and for the antithetic variates combined with the optimal control variates in Table 4. The uniform distribution parameters in these first four tables were chosen arbitrarily but satisfying $a/b \simeq -1$ and $\exp(a+b) - 1 \leq 2\bar{J}$, i.e., $a = -0.3$ and $b = 0.2$. Another jump parameter is $\lambda = 100$ per year, while the option parameters are $S0 = \$100$ and $T = 0.2$ years with interest rate $r = 5\%$ per year for testing convenience in the first four tables. Sample volatilities σ are given in the tables.

The results in these first four tables show that the antithetic variates combined with optimal control variates (AOCV) achieves the smallest standard error than the other three Monte Carlo algorithms. This confirms our theoretical results about the comparisons of their variances in the above section, but AOCV needs the longest time for computing. Therefore, we use standard error multiplying square root of computng time $\epsilon\sqrt{t}$ as a benchmark for the trade-off in the estimate variance and computing time. For a detailed explanation of the benchmark $\epsilon\sqrt{t}$, please see Boyle, Broadie and Glasserman (1997) and Glasserman (2004). Seen from these results, the Monte Carlo method with AOCV is overall the best estimate among the four Monte Carlo methods mentioned above. Also, these results show that the European call option price is a decreasing function of strike price K and the European put option is an increasing function of it. Both the call and put option prices increase as the volatility σ of stock price increases.

Table 1: Call and Put Prices for Elementary Monte Carlo Estimation (EMCE) Method

σ	K/S_0	Call, $\mathcal{C}^{(\text{emce})}$	Put. $\mathcal{P}^{(\text{emce})}$	Std. Error, ϵ	t (seconds)	Benchmark, $\epsilon\sqrt{t}$
0.2	0.9	29.980	19.085	0.569	2.671	0.929
	1.0	25.851	24.856	0.540	2.469	0.848
	1.1	22.293	31.199	0.511	2.579	0.821
0.4	0.9	30.588	19.693	0.566	2.546	0.903
	1.0	26.524	25.529	0.538	2.547	0.858
	1.1	23.011	31.916	0.510	2.515	0.808
0.6	0.9	31.574	20.678	0.563	2.531	0.896
	1.0	27.599	26.604	0.535	2.562	0.857
	1.1	24.148	33.054	0.508	2.594	0.817

The financial and jump-diffusion parameters are $S0 = 100$, $T = 0.2$, $r = 0.05$, $\lambda = 100$, $a = -0.3$ and $b = 0.2$. The simulation number is $n = 10,000$. The EMCE standard error is abbreviated by $\epsilon = \sigma_{\widehat{C}_n} = \sigma^{(\text{bs})}/\sqrt{n}$ and $\epsilon\sqrt{t}$ is a combined benchmark index.

Table 2: Call and Put Prices for Monte Carlo with Antithetic Variates (AV) only

σ	K/S_0	Call, $\mathcal{C}^{(\text{av})}$	Put, $\mathcal{P}^{(\text{av})}$	Std. Error, ϵ	t (seconds)	Benchmark, $\epsilon\sqrt{t}$
0.2	0.9	30.372	19.477	0.348	4.453	0.734
	1.0	26.223	25.228	0.340	4.672	0.735
	1.1	22.642	31.547	0.330	4.656	0.711
0.4	0.9	30.981	20.085	0.344	4.594	0.738
	1.0	26.892	25.897	0.336	5.312	0.775
	1.1	23.352	32.258	0.326	5.360	0.755
0.6	0.9	31.965	21.069	0.340	5.203	0.773
	1.0	27.963	26.968	0.331	5.391	0.768
	1.1	24.485	33.391	0.321	5.469	0.751

The financial and jump-diffusion parameters are $S0 = 100$, $T = 0.2$, $r = 0.05$, $\lambda = 100$, $a = -0.3$ and $b = 0.2$. The simulation number is $n = 10,000$. The AC standard error is abbreviated by $\epsilon = \sigma_{\widehat{H}_n}$, where $\widehat{H}_n = \sum_{i=1}^{n} X_i/n$, X_i is given in (5.6) and $\epsilon\sqrt{t}$ is a combined benchmark index.

Table 3: Call and Put Prices for Monte Carlo with Optimal Control Variates (OCV) only

σ	K/S_0	Call, $\mathcal{C}^{(\text{ocv})}$	Put, $\mathcal{P}^{(\text{ocv})}$	Std. Error, ϵ	t (seconds)	Benchmark, $\epsilon\sqrt{t}$
0.2	0.9	30.580	19.684	0.167	5.031	0.375
	1.0	26.412	25.417	0.183	5.016	0.410
	1.1	22.816	31.721	0.196	4.828	0.430
0.4	0.9	31.187	20.291	0.161	4.922	0.358
	1.0	27.085	26.090	0.176	3.093	0.310
	1.1	23.534	32.440	0.188	4.938	0.418
0.6	0.9	32.171	21.276	0.153	4.797	0.335
	1.0	28.160	27.165	0.167	3.375	0.306
	1.1	24.674	33.579	0.178	4.734	0.386

The financial and jump-diffusion parameters are $S0 = 100$, $T = 0.2$, $r = 0.05$, $\lambda = 100$, $a = -0.3$ and $b = 0.2$. The simulation number is $n = 10,000$. The AV standard error is abbreviated by $\epsilon = \sigma_{\overline{Z'_n}}$, where $\overline{Z'_n}$ is given in (5.17) and $\epsilon\sqrt{t}$ is a combined benchmark index.

Table 4: Call and Put Prices for Monte Carlo with combined techniques (AOCV)

σ	K/S_0	Call, $\mathcal{C}^{(aocv)}$	Put, $\mathcal{P}^{(aocv)}$	Std. Error, ϵ	t (seconds)	Benchmark, $\epsilon\sqrt{t}$
0.2	0.9	30.645	19.749	0.106	5.656	0.253
	1.0	26.487	25.492	0.114	5.531	0.267
	1.1	22.895	31.800	0.119	5.781	0.286
0.4	0.9	31.251	20.356	0.102	5.797	0.245
	1.0	27.154	26.159	0.109	6.250	0.272
	1.1	23.604	32.509	0.114	6.672	0.294
0.6	0.9	32.232	21.337	0.096	5.593	0.227
	1.0	28.222	27.227	0.103	4.922	0.227
	1.1	24.735	33.640	0.107	6.109	0.265

The financial and jump-diffusion parameters are $S0 = 100$, $T = 0.2$, $r = 0.05$, $\lambda = 100$, $a = -0.3$ and $b = 0.2$. The simulation number is $n = 10,000$. The AC standard error is abbreviated by $\epsilon = \sigma_{\widehat{Z}_n} = \sigma_Z/\sqrt{n}$, where $\widehat{Z}_n$ is given in (5.21) and $\epsilon\sqrt{t}$ is a combined benchmark index.

Also, the Black-Scholes call price $C^{(\text{bs})}(S_0, T; K, \sigma^2, r)$ are computed directly from the formula (4.5) and from it the put price $P^{(\text{bs})}(S_0, T; K, \sigma^2, r)$ is computed from the put-call parity relation,

$$P^{(\text{bs})}(S_0, T; K, \sigma^2, r) = C^{(\text{bs})}(S_0, T; K, \sigma^2, r) + K\exp(-rT) - S_0. \tag{6.1}$$

The numerical results for call and prices for these Black-Scholes values and the AOCV values are compared in Table 5. For Table 5, the estimated jump-diffusion model parameters $\sigma = 0.1074$, $\lambda = 64.16$, $a = -0.028$ and $b = 0.026$ used come from the double-unform distribution paper of Zhu and Hanson (2005). The option parameters are $S0 = 1000$, $T = 0.25$ year, and $r = 3.45\%$ which is the 3-month U.S. Treasury bill yeild rate on August 2, 2005. These parameters are also used to compute the Standard & Poor 500 index option prices. (Note that these parameters are different from those for Tables 1-4.)

Table 5: Comparison of Call and Put Option Prices

K/S_0	$\mathcal{C}^{(aocv)}$	$\mathcal{P}^{(aocv)}$	ϵ	t (sec.)	$C^{(\text{bs})}$	$P^{(\text{bs})}$	$\mathcal{C}^{(\text{true})}$	$\mathcal{P}^{(\text{true})}$
0.8	206.927	0.057	0.003	6.219	206.870	8.e-5	206.937	0.067
0.9	110.787	3.058	0.043	6.516	108.040	0.3087	110.792	3.063
1.0	37.358	28.770	0.118	6.109	25.897	17.309	37.248	28.660
1.1	6.506	97.059	0.069	6.500	1.260	91.813	6.479	97.033
1.2	0.553	190.248	0.015	6.203	0.010	189.700	0.575	190.269

The option parameters are $S0 = 1000$, $r = 0.0345$, $T = 0.25$, $\sigma = 0.1074$, $\lambda = 64.16$, $a = -0.028$ and $b = 0.026$. The simulation number is $n = 10,000$ for AOCV values, but a much larger number, $n = 400,000$ sample points, are used for the approximation to the true values. The Black-Scholes values come from (4.5) and put-call parity (6.1). The standard error is abbreviated by $\epsilon = \sigma_{\widehat{Z}_n} = \sigma_Z/\sqrt{n}$, where $\widehat{Z}_n$ is given in (5.21) for AOCV.

The numerical results in Table 5 show that the estimated call and put values by the Monte Carlo method with AOCV are within the 95% confidence interval of the true call $\mathcal{C}^{(\text{true})}$ i.e.,

$$\mathcal{C}^{(aocv)} \in [\mathcal{C}^{(\text{true})} - 1.96\epsilon, \mathcal{C}^{(\text{true})} + 1.96\epsilon]$$

and put values $\mathcal{P}^{(\text{true})}$, i.e.,

$$\mathcal{P}^{(aocv)} \in [\mathcal{P}^{(\text{true})} - 1.96\epsilon, \mathcal{P}^{(\text{true})} + 1.96\epsilon]$$

by the central limit theorem except the case when $K/S0 = 0.8$. In Table 5, the true call and put prices are approximated with a much larger number of simulations, $n = 400,000$ compared to $n = 10,000$ in Tables 1-4. Also, the estimated European call and put option prices are observed to be bigger than the Black-Scholes call and put option prices, respectively. This is not just a numerical fact, but can be stated and proven with the following theorem:

Theorem 6.1 *The European call and put option prices based on the jump-diffusion model in (2.1) are bigger than the Black-Scholes call and put option prices respectively, i.e.,*

$$\mathcal{C}(S_0, T; K, \sigma^2, r) > C^{(\text{bs})}(S_0, T; K, \sigma^2, r),$$

and

$$\mathcal{P}(S_0, T; K, \sigma^2, r) > P^{(\text{bs})}(S_0, T; K, \sigma^2, r).$$

Proof: Since the Black-Scholes call option pricing formula (4.5) for $C^{(\text{bs})}(S, T; K, \sigma^2, r)$ is a strictly convex function about S and by Jensen's inequality (see Hanson (2005, Chapter 0) for instance), we have

$$\begin{aligned} \mathcal{C}(S_0, T; K, \sigma^2, r) &\overset{(5.1)}{=} E_{\gamma(T)}\left[C^{(\text{bs})}\left(S_0 e^{\gamma(T)-\lambda\bar{J}T}, T; K, \sigma^2, r\right)\right] \\ &> C^{(\text{bs})}\left(E_{\gamma(T)}[S_0 e^{\gamma(T)-\lambda\bar{J}T}], T; K, \sigma^2, r\right) = C^{(\text{bs})}\left(S_0, T; K, \sigma^2, r\right). \end{aligned}$$

By put-call parity and the above proven inequality,

$$\begin{aligned} \mathcal{P}(S_0, T; K, \sigma^2, r) &= \mathcal{C}(S_0, T; K, \sigma^2, r) + Ke^{-rT} - S_0 \\ &> C^{(\text{bs})}(S_0, T; K, \sigma^2, r) + Ke^{-rT} - S_0 = P^{(\text{bs})}(S_0, T; K, \sigma^2, r). \end{aligned}$$

□

Remark: In the proof of the Theorem 6.1, no special jump mark distribution of Q in the jump-diffusion model (2.1) is needed. Hence, this is a general result also suitable for the jump-diffusion jump-amplitude models such as the log-normal of Merton (1976), the log-double-exponential of Kou (2002) and Kou and Wang (2004) and the log-double-uniform of Zhu and Hanson (2005).

7. CONCLUSIONS

The original SDE is transformed to a risk-neutral SDE by setting the stock price increases at the risk-free interest rate. Based on this risk-neutral SDE, a reduced European call option pricing formula is derived and then by the put-call parity the European put option price can be easily computed. Also, some useful binomial lemmas and a partial sum density theorem for the uniformly distributed IID random variables are established. Unfortunately, the analysis of the log-uniform jump-amplitude jump-diffusion density is too complicated to get a closed-form option pricing formula like that of Black-Scholes, excluding infinite sums. However, that is true for many complex problems where computational methods are important. Hence, a Monte Carlo algorithm with both antithetic variate and control variate techniques for variance reduction for jump-diffusions is applied. This algorithm is easy to implement and the simulation results show that it is also efficient within seven seconds to get the practical accuracy. Finally, we show that the European call and put option prices based on general jump-diffusion models satisfying linear constant coefficient SDEs are bigger than the Black-Scholes call and put option prices, respectively.

Appendix A.

SUMS OF UNIFORMLY DISTRIBUTED VARIABLES

The main purpose of this Appendix is to derive the partial sum density function for the uniformly distributed IID variables, but first we need the following lemmas.

Lemma A.1 ***Partial Sum Density Recursion:***

Let X_i *for* $i = 1 : n$ *be a sequence of independent identically distributed (IID) random variables with uniform distribution over* $[0, 1]$. *Let*

$$\mathcal{S}_n = \sum_{i=1}^{n} X_n$$

be the partial sum for $n \geq 1$ *with distribution*

$$\Phi_{\mathcal{S}_n}(s) = \mathrm{Prob}[\mathcal{S}_n \leq s]$$

and assume the density

$$\phi_{\mathcal{S}_n}(s) = \Phi'_{\mathcal{S}_n}(s)$$

exists. Then for any real number s, *such that* $0 \leq s \leq n + 1$,

$$\phi_{\mathcal{S}_{n+1}}(s) = \int_{s-1}^{s} \phi_{\mathcal{S}_n}(y)dy. \tag{A.1}$$

Proof: Application convolution theorem (see Hanson (2005, Chapter 0)) to the recursion $\mathcal{S}_{n+1} = \mathcal{S}_n + X_{n+1}$ and the uniform IID of the X_i on $[0, 1]$ yield the density of $\mathcal{S}_{n+1}$,

$$\begin{aligned}
\phi_{\mathcal{S}_{n+1}}(s) &= \left(\phi_{\mathcal{S}_n} * \phi_{X_{n+1}}\right)(s) = \int_{-\infty}^{+\infty} \phi_{\mathcal{S}_n}(s-x)\phi_{X_{n+1}}(x)dx \\
&= \int_{0}^{1} \phi_{\mathcal{S}_n}(s-x)dx = \int_{s-1}^{s} \phi_{\mathcal{S}_n}(x)dx.
\end{aligned}$$

That is, $\phi_{\mathcal{S}_{n+1}}(s) = \int_{s-1}^{s} \phi_{\mathcal{S}_n}(x)dx$. □

A.1 Preliminary Binomial Formulas

Lemma A.2 ***Binomial Formula Derivative Identity:***

$$\text{(A.2)} \qquad \sum_{j=0}^{n} \binom{n}{j} (-1)^j j^i = \left\{ \begin{array}{ll} 0, & n = 0 \text{ or } n \geq i+1 \\ (-1)^n n! c_{i,n}, & 1 \leq n \leq i \end{array} \right\},$$

for some set of constants $c_{i,n}$.

Proof: Consider the basic binomial formula:

$$\text{(A.3)} \qquad B_0(x;n) = (1-x)^n = \sum_{j=0}^{n} \binom{n}{j} (-1)^j x^j$$

whose derivatives are easy to calculate by induction giving

$$B_0^{(i)}(x;n) = \left\{ \begin{array}{ll} \frac{n!}{(n-i)!}(1-x)^{n-i}, & n \geq i+1 \\ (-1)^i i!, & n = i \\ 0, & 0 \leq n \leq i-1 \end{array} \right\},$$

so

$$B_0^{(i)}(1^{\pm};n) = \lim_{x \to 1^{\pm}} B_0^{(i)}(x;n) = (-1)^i i! \delta_{n,i} \; ,$$

where $\delta_{n,i}$ is the Kronecker delta. Consider the derivative form

$$x B_0'(x;n) = \sum_{j=0}^{n} \binom{n}{j} (-1)^j j x^j$$

and define $B_1(x;n) \equiv x B_0'(x;n)$, then

$$B_1(1^{\pm};n) = \sum_{j=0}^{n} \binom{n}{j} (-1)^j j = -\delta_{n,1} \; ,$$

which proves (A.2) for the case $i = 1$ with $c_{n,1} = 1$.

For $i > 1$, note that each $x \cdot d/dx$ operation on the basic binomial formula (A.3) introduces another factor of j in the binomial formula summand, leading to the inductive definition of higher order binomial formulas,

$$B_{i+1}(x;n) \equiv x \cdot B_i'(x;n)$$

for $i \geq 0$. Straight-forward induction, shows that for $i \geq 0$,

$$B_i(x;n) \equiv \sum_{j=0}^{n} \binom{n}{j} (-1)^j j^i x^j$$

and that

$$B_i(1^{\pm};n) \equiv \sum_{j=0}^{n} \binom{n}{j} (-1)^j j^i \ ,$$

which is the target binomial formula in (A.2). To evaluate this formula, a further application of induction on the inductive or recursive form definition for $B_{i+1}(x;n)$, leads to the induction hypothesis,

$$B_i(x;n) = \sum_{j=1}^{i} c_{i,j} x^j B_0^{(j)}(x;n) \tag{A.4}$$

where the constants $c_{i,j}$ are determined recursively by equating this induction hypothesis for $i+1$ and the recursive form for $B_{i+1}(x;n)$. Thus for arbitrary x, equating the coefficients of the x terms give $c_{i+1,1} = c_{i,1}$ for $i \geq 1$, so $c_{i,1} = c_{1,1} = 1$ already found from the $j = 1$ case. Similarly, for $j = i+1$, i.e., order x^{i+1}, $c_{i+1,i+1} = c_{i,i}$ for $i \geq 1$, so $c_{i,i} = c_{1,1} = 1$, completing the two boundary cases. In general, comparing coefficients of x^j for $2 \leq j \leq i$ yields the recursion,

$$c_{i+1,j} = c_{i,j-1} + j \cdot c_{i,j} \ ,$$

which can be used to get all constants needed, for example $c_{i,i-1} = i(i-1)/2$.

Finally, using (A.4) with the result $B_0^{(i)}(1^{\pm};n) = (-1)^i i! \delta_{n,i}$ implies the final result (A.2), proving the lemma. □

Lemma A.3 ***Shifted Binomial Formula Identity:***

(A.5) $$\sum_{j=0}^{n} \binom{n}{j} (-1)^j (n-j)^i = \left\{ \begin{array}{ll} 0, & n = 0 \text{ or } n \geq i+1 \\ n! c_{i,n}, & 1 \leq n \leq i \end{array} \right\},$$

where the constants $c_{i,n}$ are given recursively in the above lemma.

Proof: This result follows quite easily from the derivative identity (A.2) by change of variable $j' = n - j$ and a binomial coefficient identity,

$$\binom{n}{n-j} = \frac{n!}{(n-j)!j!} = \binom{n}{j},$$

so

$$\begin{aligned} \sum_{j=0}^{n} \binom{n}{j} (-1)^j j^i &= \sum_{j=0}^{n} \binom{n}{n-j} (-1)^{n-j} (n-j)^i = (-1)^n \sum_{j=0}^{n} \binom{n}{j} (-1)^j (n-j)^i \\ &= \left\{ \begin{array}{ll} 0, & n = 0 \text{ or } n \geq i+1 \\ (-1)^n n! c_{i,n}, & 1 \leq n \leq i \end{array} \right\} \end{aligned}$$

and taking into account the extra factor of $(-1)^n$ proves the result (A.5) as well as the lemma. □

Lemma A.4 ***Key Binomial Formula Application:***

(A.6) $$\sum_{j=0}^{n} \binom{n}{j} (-1)^j ((n-j)^n - (n-j+\xi)^n) = 0,$$

where $n \geq 0$ is an integer and ξ is any value.

Proof: Primarily expanding the factor $(n-j+\xi)$ about $(n-j)$ by the binomial theorem leads to

$$\begin{aligned}\sum_{j=0}^{n}\binom{n}{j}(-1)^j\left((n-j)^n-(n-j+\xi)^n\right) &= -\sum_{j=0}^{n}\binom{n}{j}(-1)^j\sum_{i=0}^{n-1}\binom{n}{i}\xi^{n-i}(n-j)^i \\ &= -\sum_{i=0}^{n-1}\binom{n}{i}\xi^{n-i}\sum_{j=0}^{n}\binom{n}{j}(-1)^j(n-j)^i \\ &\overset{Lemma}{\underset{A.3}{=}} -\sum_{i=0}^{n-1}\binom{n}{i}\xi^{n-i}\cdot 0 = 0\,,\end{aligned}$$

noting that $n > i$ in the application of Lemma A.3. □

A.2 Density of Partial Sums of Uniformly Distributed IID Random Variables

Now, returning to the calculation of the probability density of the partial sum random variable $\mathcal{S}_n = \sum_{i=1}^{n} X_i$, where the X_i for $i = 1 : n$ are an IID sequence of uniform random variables.

Theorem A.1 ***Partial Sum Density:***

Let X_i for $i = 1 : n$ be a sequence of independent identity distribution (IID) random variables each uniformly distributed on $[0, 1]$. Let $\mathcal{S}_n = \sum_{i=1}^{n} X_i$ with $n \geq 1$. Then, the probability density function of $\mathcal{S}_n$ is

$$\text{(A.7)}\qquad \phi_{\mathcal{S}_n}(s) = \left\{\begin{array}{ll} 1, & 0 \leq s \leq 1,\ n = 1 \\ \frac{1}{(n-1)!}\sum_{j=0}^{\lfloor s \rfloor}\binom{n}{j}(-1)^j(s-j)^{n-1}, & 0 \leq s \leq n,\ n > 1 \\ 0, & \text{else} \end{array}\right\},$$

where $\lfloor s \rfloor$ denotes the integer floor function.

Proof: Again, we apply mathematical induction. When $n = 1$, the conclusion is true from the uniform density given in (2.2) when $a = 0$ and $b = 1$. When $n > 1$, assume the induction hypothesis and $0 \le s \le n$, so

$$\phi_{\mathcal{S}_n}(s) = \frac{1}{(n-1)!} \sum_{j=0}^{\lfloor s \rfloor} \binom{n}{j} (-1)^j (s-j)^{n-1}$$

is true, but otherwise $\phi_{\mathcal{S}_n}(s) = 0$. The objective is to use the hypothesis to show the result

$$\phi_{\mathcal{S}_{n+1}}(s) = \frac{1}{n!} \sum_{j=0}^{\lfloor s \rfloor} \binom{n+1}{j} (-1)^j (s-j)^{n},$$

if $0 \le s \le n+1$, but is 0 otherwise.

- **Case 1:** Let $s < 0$ or $s > n+1$, then $\phi_{\mathcal{S}_{n+1}}(s) = 0$ since the value of $\mathcal{S}_{n+1} = \sum_{i=1}^{n+1} X_i$ and $0 \le X_i \le 1$ for $i = 1 : n+1$, so $\mathcal{S}_{n+1} \in [0, n+1]$.

- **Case 2:** Let $0 \le s < 1$, then $-1 \le s - 1 < 0$. Therefore, starting with Lemma A.1 and using the fact that $\phi_{\mathcal{S}_n}(x) = 0$ when $x < 0$,

$$\begin{aligned}
\phi_{\mathcal{S}_{n+1}}(s) &= \int_{s-1}^{s} \phi_{\mathcal{S}_n}(x)dx = \left(\int_{s-1}^{0} + \int_{0}^{s} \right) \phi_{\mathcal{S}_n}(x)dx = \int_{0}^{s} \phi_{\mathcal{S}_n}(x)dx \\
&= \int_{0}^{s} \frac{1}{(n-1)!} \sum_{j=0}^{\lfloor x \rfloor} \binom{n}{j} (-1)^j (x-j)^{n-1} dx \\
&= \int_{0}^{s} \frac{1}{(n-1)!} \sum_{j=0}^{0} \binom{n}{j} (-1)^j (x-j)^{n-1} dx \\
&= \int_{0}^{s} \frac{1}{(n-1)!} x^{n-1} dx = \frac{s^n}{n!}.
\end{aligned}$$

However, for $0 \le s < 1$,

$$\frac{1}{n!} \sum_{j=0}^{\lfloor s \rfloor} \binom{n+1}{j} (-1)^j (s-j)^n = \frac{1}{n!} \sum_{j=0}^{0} \binom{n+1}{j} (-1)^j (s-j)^n = \frac{s^n}{n!}.$$

Hence, on $0 \leq s < 1$,

$$\phi_{\mathcal{S}_{n+1}}(s) = \frac{1}{n!}\sum_{j=0}^{\lfloor s \rfloor}\binom{n+1}{j}(-1)^j(s-j)^n.$$

- **Case 3:** $n < s \leq n+1$. Then, $n-1 < s-1 \leq n$ and $\lfloor s \rfloor = n$ or $n+1$. Therefore, by Lemma A.1,

$$\begin{aligned}
\phi_{\mathcal{S}_{n+1}}(s) &= \int_{s-1}^{s}\phi_{\mathcal{S}_n}(x)dx = \left(\int_{s-1}^{n} + \int_{n}^{s}\right)\phi_{\mathcal{S}_n}(x)dx = \int_{s-1}^{n}\phi_{\mathcal{S}_n}(x)dx \\
&= \int_{s-1}^{n}\frac{1}{(n-1)!}\sum_{j=0}^{\lfloor x \rfloor}\binom{n}{j}(-1)^j(x-j)^{n-1}dx \\
&= \int_{s-1}^{n}\frac{1}{(n-1)!}\sum_{j=0}^{n-1}\binom{n}{j}(-1)^j(x-j)^{n-1}dx \\
&= \frac{1}{(n-1)!}\sum_{j=0}^{n-1}\binom{n}{j}(-1)^j\int_{s-1}^{n}(x-j)^{n-1}dx \\
&= \frac{1}{n!}\sum_{j=0}^{n-1}\binom{n}{j}(-1)^j[(n-j)^n-(s-1-j)^n] \\
&= \frac{1}{n!}\sum_{j=0}^{n-1}\binom{n}{j}(-1)^{j+1}(s-1-j)^n + \frac{1}{n!}\sum_{j=0}^{n-1}\binom{n}{j}(-1)^j(n-j)^n \\
&= \frac{1}{n!}\sum_{j=1}^{n}\binom{n}{j-1}(-1)^j(s-j)^n + \frac{1}{n!}\sum_{j=0}^{n-1}\binom{n}{j}(-1)^j(n-j)^n \\
&= \frac{1}{n!}\sum_{j=1}^{n}\left(\binom{n+1}{j}-\binom{n}{j}\right)(-1)^j(s-j)^n + \frac{1}{n!}\sum_{j=0}^{n-1}\binom{n}{j}(-1)^j(n-j)^n \\
&= \frac{1}{n!}\sum_{j=0}^{n}\left(\binom{n+1}{j}-\binom{n}{j}\right)(-1)^j(s-j)^n + \frac{1}{n!}\sum_{j=0}^{n}\binom{n}{j}(-1)^j(n-j)^n \\
&= \frac{1}{n!}\sum_{j=0}^{n}\binom{n+1}{j}(-1)^j(s-j)^n + \frac{1}{n!}\sum_{j=0}^{n}\binom{n}{j}(-1)^j\left((n-j)^n-(s-j)^n\right). \\
&\overset{Lemma}{\underset{A.4}{=}} \frac{1}{n!}\sum_{j=0}^{\lfloor s \rfloor}\binom{n+1}{j}(-1)^j(s-j)^n.
\end{aligned}$$

Therefore, the right boundary case for $n < s \leq n+1$ is proven.

- **Case 4:** Let $1 \leq s \leq n$ then by Lemma A.1, the fact that $s-1 \leq \lfloor s \rfloor \leq s$ and the induction hypothesis,

$$
\begin{aligned}
\phi_{\mathcal{S}_{n+1}}(s) &= \int_{s-1}^{s} \phi_{\mathcal{S}_n}(x)dx = \int_{s-1}^{s} \frac{1}{(n-1)!} \sum_{j=0}^{\lfloor x \rfloor} \binom{n}{j} (-1)^j (x-j)^{n-1} dx \\
&= \left(\int_{s-1}^{\lfloor s \rfloor} + \int_{\lfloor s \rfloor}^{s} \right) \frac{1}{(n-1)!} \sum_{j=0}^{\lfloor x \rfloor} \binom{n}{j} (-1)^j (x-j)^{n-1} dx \\
&= \int_{s-1}^{\lfloor s \rfloor} \frac{1}{(n-1)!} \sum_{j=0}^{\lfloor s-1 \rfloor} \binom{n}{j} (-1)^j (x-j)^{n-1} dx \\
&\quad + \int_{\lfloor s \rfloor}^{s} \frac{1}{(n-1)!} \sum_{j=0}^{\lfloor s \rfloor} \binom{n}{j} (-1)^j (x-j)^{n-1} dx \\
&= \frac{1}{(n-1)!} \sum_{j=0}^{\lfloor s-1 \rfloor} \binom{n}{j} (-1)^j \int_{s-1}^{\lfloor s \rfloor} (x-j)^{n-1} dx \\
&\quad + \frac{1}{(n-1)!} \sum_{j=0}^{\lfloor s \rfloor} \binom{n}{j} (-1)^j \int_{\lfloor s \rfloor}^{s} (x-j)^{n-1} dx \\
&= \frac{1}{n!} \sum_{j=0}^{\lfloor s-1 \rfloor} \binom{n}{j} (-1)^j [(\lfloor s \rfloor - j)^n - (s-1-j)^n] \\
&\quad + \frac{1}{n!} \sum_{j=0}^{\lfloor s \rfloor} \binom{n}{j} (-1)^j [(s-j)^n - (\lfloor s \rfloor - j)^n] \\
&= \frac{1}{n!} \sum_{j=0}^{\lfloor s-1 \rfloor} \binom{n}{j} (-1)^{(j+1)} (s-1-j)^n + \frac{1}{n!} \sum_{j=0}^{\lfloor s \rfloor} \binom{n}{j} (-1)^j (s-j)^n \\
&= \frac{1}{n!} \sum_{j=1}^{\lfloor s \rfloor} \binom{n}{j-1} (-1)^j (s-j)^n + \frac{1}{n!} \sum_{j=0}^{\lfloor s \rfloor} \binom{n}{j} (-1)^j (s-j)^n \\
&= \frac{1}{n!} \sum_{j=1}^{\lfloor s \rfloor} \left(\binom{n}{j-1} + \binom{n}{j} \right) (-1)^j (s-j)^n + \frac{s^n}{n!} \\
&= \frac{1}{n!} \sum_{j=1}^{\lfloor s \rfloor} \binom{n+1}{j} (-1)^j (s-j)^n + \frac{s^n}{n!} = \frac{1}{n!} \sum_{j=0}^{\lfloor s \rfloor} \binom{n+1}{j} (-1)^j (s-j)^n.
\end{aligned}
$$

Therefore, from Case 1 to Case 4, the conclusion is true for $n+1$, so by the mathematical induction the theorem is proved. □

Corollary A.1 ***Partial Sum Density on*** $[\mathrm{a}, \mathrm{b}]$***:***

Let Y_i for $i = 1 : n$ be a sequence of independent identically distributed random variables uniformly distributed over $[a, b]$. Let $S_n = \sum_{i=1}^{n} Y_i$, where $n \geq 1$, then, the probability density function of S_n is

$$\text{(A.8)} \quad \phi_{S_n}(s) = \left\{ \begin{array}{ll} 1, & a \leq s \leq b,\ n = 1 \\ \frac{1}{(n-1)!(b-a)} \sum_{j=0}^{\lfloor \frac{s-na}{b-a} \rfloor} \binom{n}{j} (-1)^j (\frac{s-na}{b-a} - j)^{n-1}, & na \leq s \leq nb,\ n > 1 \\ 0, & \textit{else} \end{array} \right\}.$$

Proof: For $s < na$ or $s > nb$, it is obvious that $\phi_{\mathcal{S}_n}(s) = 0$.

Now, we consider $na \leq s \leq nb$. Set

$$X_i = \frac{Y_i - a}{b - a},$$

where Y_i is uniformly distributed random variable from a to b, then it is easy to prove that the distribution of X_i is the transformable to the uniform distributed on $[0, 1]$ (the proof is omitted) . Set

$$\mathcal{S}_n = \sum_{i=1}^{n} X_i.$$

So,

$$\mathcal{S}_n = \sum_{i=1}^{n} \frac{Y_i - a}{b - a} = \frac{\sum_{i=1}^{n} Y_i - na}{b - a} = \frac{S_n - na}{b - a}.$$

The probability distribution function of S_n is

$$\begin{aligned} \Phi_{S_n}(s) &= \text{Prob}[S_n \leq s] = \text{Prob}\left[(b-a)\mathcal{S}_n + na \leq s\right] \\ &= \text{Prob}\left[\mathcal{S}_n \leq \frac{s - na}{b - a}\right] = \int_0^{\frac{s-na}{b-a}} \phi_{S_n}(x)dx. \end{aligned}$$

So, for $na \leq s \leq nb$,

$$\phi_{S_n}(s) = \frac{\phi_{\mathcal{S}_n}(\frac{s-na}{b-a})}{b - a} = \frac{\sum_{j=0}^{\lfloor \frac{s-na}{b-a} \rfloor} \binom{n}{j} (-1)^j (\frac{s-na}{b-a} - j)^{n-1}}{(n-1)!(b-a)}.$$

The Corollary is proved. □

REFERENCES

ANDERSEN, T. G., L. BENZONI, and,J. LUND (2002): An Empirical Investigation of Continuous-Time Equity Return Models, *J. Finance*, 57 (3), 1239-1284.

AOURIR, C. A., D. OKUYAMA, C. LOTT, and C. EGLINTON (2002): *Exchanges - Circuit Breakers, Curbs, and Other Trading Restrictions*, `http://invest-faq.com/articles/exch-circuit-brkr.html`.

BALL, C. A., and W. N. TORUS (1985): On Jumps in Common Stock Prices and Their Impact on Call Option Prices, *J. Finance*, 40, 155-173.

BAXTER, M., and A. RENNIE (1996): *Financial Calculus: An Introduction to Derivative Pricing*. Cambridge, UK: Cambridge University Press.

BLACK, F., and M. SCHOLES (1973): The Pricing of Options and Corporate Liabilities, *J. Polit. Econ.*, 81, 637-659.

BOYLE, P. (1977): Options: A Monte Carlo Approach, *J. Financial Econ.*, 4, 323-338.

BOYLE, P., M. BROADIE and P. GLASSERMAN (1997): Monte Carlo methods for Security Pricing, *J. Econ. Dyn. and Control*, 21, 1267-1321.

CONT, R., and P. TANKOV (1964): *Financial Modelling with Jump Processes*. London: Chapman & Hall/CRC.

GLASSERMAN, P. (2004): *Monte Carlo methods in Financial Engineering*. New York: Springer.

HAMMERSLEY, J. M., and D. C. HANDSCOMB (1964): *Monte Carlo Methods*. London: Chapman & Hall.

HANSON, F. B. (2005): *Applied Stochastic Processes and Control for Jump-Diffusions: Modeling, Analysis and Computation*. Philadelphia, PA: SIAM Books, to appear, `http://www2.math.uic.edu/~hanson/math574/#Text`

HANSON, F. B., and J. J. WESTMAN (2002a): Optimal Consumption and Portfolio Control for Jump-Diffusion Stock Process with Log-Normal Jumps, *Proc. 2002 Amer. Control Conf.*, 4256-4261.

HANSON, F. B., and J. J. WESTMAN (2002b): Jump-Diffusion Stock Return Models in Finance: Stochastic Process Density with Uniform-Jump Amplitude, *Proc. 15th Int. Symp. Math. Theory of Networks and Systems*, 7 pages (CD).

HANSON, F. B., J. J. WESTMAN and Z. ZHU (2004): Multinomial Maximum Likelihood Estimation of Market Parameters for Stock Jump-Diffusion Models, *AMS Contemporary Math.*, 351, 155-169.

HANSON, F. B., and Z. ZHU (2004): Comparison of Market Parameters for Jump-Diffusion Distributions Using Multinomial Maximum Likelihood Estimation, *Proc. 43rd IEEE Conf. Decision and Control*, 3919-3924.

HIGHAM, D. J. (2004): *An Introduction to Financial Option Valuation: Mathematics, Stochastics and Computation*. Cambridge, UK: Cambridge University Press.

HULL, J. C. (2000): *Options, Futures, & Other Derivatives*, 4th Edition. Englewood Cliffs, NJ: Prentice-Hall.

JARROW, R. A., and E. R. ROSENFELD (1984): Jump Risks and the Intertemporal Capital Asset Pricing Model, *J. Business*, 57 (3), 337-351.

JORION. P. (1989): On Jump Processes in the Foreign Exchange and Stock Markets, *Rev. Financial Stud.*, 88 (4), 427-445.

KOU, S. G. (2002): A Jump Diffusion Model for Option Pricing, *Mgmt. Sci.*, 48 (9), 1086-1101.

KOU, S. G., and H. WANG, (2004): Option Pricing Under a Double Exponential Jump Diffusion Model, *Mgmt. Sci.*. 50 (9), 1178-1192.

MERTON, R. C. (1973): Theory of Rational Option Pricing, *Bell J. Econ. Mgmt. Sci.*, 4, 141-183.

MERTON, R. C. (1976): Option Pricing when Underlying Stock Returns are Discontinuous, *J. Fin. Econ.*, 3, 125-144.

MERTON, R. C., and M. SCHOLES (1995): Fischer Black, *J. Finance*, 50 (5), 1359-1370.

ØKSENDAL, B., and A. SULEM (2005): *Applied Stochastic Control of Jump Diffusions*. New York: Springer.

ZHU, Z., and F. B. HANSON (2005): Log-Double-Uniform Jump-Diffusion Model, working paper, Dept. Math., Stat., and Comp. Sci., U. Illinois at Chicago, to be submitted, pp. 1-29. `http://www.math.uic.edu/~hanson/pub/ZZhu/dbuniformfree2.pdf`.