Log-Double-Uniform Jump-Diffusion Model

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Abstract

In this paper, we propose a jump-diffusion model with the log-double-uniform jump amplitudes to govern the dynamics of the asset price. The truncated probability density function and the bin probability distribution formula for the log-returns of the assert are obtained. We use multinomial maximum likelihood estimation method and 1988-2003 Standard and Poor's 500 (S&P 500) market data to esmitate the model parameters. Then, the estimated parameters are used to simulate the 1988-2003 S&P 500 prices and get the histogram for the simulated log-returns. Some comparisons among different jump models are also presented.

Key words: jump-diffusion model, S&P 500, bin probability distribution, parameter estimation, simulation.

1 Introduction

In the recent three decades, many efforts are used to modify the Black-Scholes model [2] since it is incapable of fully capturing the empircal features of the assert prices or option prices. Some models add the jump part to the black-scholes model (jump-diffusion model) and try to catch the large random fluctuations such as crashes and rallies and the nonnormal features such as negative skewness and leptokurtic (peakedness) behavior in the assert return distribution (see Merton [13], Kou [12], Hanson and Westman [5], Hanson and Zhu [8]). The other models are proposed to incorporate the volatility smile, that is, the volatility is not a constant as in the Black-Scholes model, with or without jumps in returns (see Heston [9], Bates [1] and Duffie, Pan and Singleton [3]), we call them stochastic-volatility models.

In general, the jump-diffusion model is simpler than the stochastic-volatility model, but qualitatively catch the empirical market phenomena. Since crashes and rallies are rare

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events, so the Poisson process is reasonable for the timing of jumps. But, it is not clear how to choose the amplitude of jumps. Merton [13] choose a log-normally distributed process for the jump-amplitudes, recently Kou [12] choose a log-double-exponentially distributed process. However, there are at least two problems for the above two choices: one is that the exponentially small tails of the log-normal and log-double-exponential distributions are contrary to the flat and thick tails of the financial market return data; the other is that an infinite jump domain is unrealistic since the jump amplitudes should be bounded in a real world financial markets, especially after October 1987, when "circuit breakers" are adopted to reduce stock market volatility. Therefore, Hanson and Westman [5] proposed a log-uniform process for the jump-amplitudes. But some problems still exists in the log-uniformly jump-diffusion model: there exist significant lumps in the shoulder in the hysteriagram for the log-uniform model which does not appear in the hysteriagram for the S&P 500 sample data and some parameters estimated still need to be improved such as skewness and kurtosis. For details about the above three models' comparison, please see Hanson and Zongwu [8].

Therefore, we propose a jump-diffusion model with log-double-uniform jump amplitudes and try to improve parameter estimations. In this paper, we are not try to incorporate the stochastic volatility to the jump-diffusion model since we want to focus on the effects of jump amplitudes. If more accurate model is needed, stochastic volatility may be considered.

The paper is organized as follows. In Section 2, the model is proposed and analyzed. In Section 3, we get some basic moments which are important to the parameter estimations. In Section4 and Section 5, density function and bin probability distribution formula of log-returns are achieved. In Section 6, Numerical results and figures are given and discussed. Finally, we draw a conclusion in Section 7.

2 Log-Double-Uniform Jump-Diffusion Model

The following constant rate stochastic differential equation (SDE) is used to model the dynamics of the asset price, $S(t)$:

$$dS(t) = S(t)\left(\mu dt + \sigma dW(t) + J(Q)dP(t)\right), \tag{1}$$

where $S_0 = S(0) > 0$, μ is the drift coefficient, σ is the diffusive volatility, $W(t)$ is the Wiener process, $J(Q)$ is the Poisson jump-amplitude, Q is an underlying Poisson amplitude mark process selected so that

$$Q = \ln(J(Q) + 1)$$

for convenience, $P(t)$ is the standard Poisson jump counting process with joint mean and variance

$$\mathrm{E}[P(t)] = \lambda t = \mathrm{Var}[P(t)].$$

Let the density of the jump amplitude mark Q be double-uniformly distributed:

$$\phi_Q(q) = -\frac{p}{a} I_{\{a \le q < 0\}} + \frac{q}{b} I_{\{0 \le q \le b\}}, \tag{2}$$

where $a < 0 < b$ and $0 < p < 1$ represents the probability of downword jumps and $q = 1-p$ is the probability of upward jumps. The set indicator function is $I_{\{\mathcal{S}\}}$ for set S. The mean of Q

is $\mu_j = \frac{1}{2}(pa+qb)$ and the variance of Q is $\sigma_j^2 = \frac{pq}{4}(b-a)^2 + \frac{pa^2+qb^2}{12}$. The third central moment of Q is $M_j^{(3)} \equiv E[(Q-\mu_j)^3] = \frac{pq}{4}(b-a)^2(aq+bp)$ and the fourth central moment of Q is $M_j^{(4)} \equiv E[(Q-\mu_j)^4] = \mu_j^4 + p/5(a^4 - 5a^3\mu_j + 10a^2\mu_j^2 - 10a\mu_j^3) + q/5(b^4 - 5b^3\mu_j + 10b^2\mu_j^2 - 10b\mu_j^3)$.

According to the Itô stochastic chain rule [7] for jump-diffusions, the instant log-return process $d\ln(S(t))$ satisfies the constant coefficient SDE

$$d\ln(S)(t) = \mu_{ld}dt + \sigma dW(t) + \sum_{i=1}^{dP(t)} Q_i, \tag{3}$$

where $\mu_{ld} \equiv \mu - 0.5\sigma^2$ and the Q_i here are independent identically double-uniformly distributed jump-amplitude marks Q. We call $d\ln(S(t))$ the instant log-return since $d\ln(S(t)) \equiv \ln(S(t+dt)) - \ln(S(t)) = \ln(\frac{S(t+dt)}{S(t)}) = \ln(\frac{S(t)+dS(t)}{S(t)}) \approx \frac{dS(t)}{S(t)}$ which is an instant return, the relative gain over an infinitesimal time dt. In the case that the time step Δt is a daily increment rather than an infinitesimal like dt, then $\Delta\ln(S(t))$ satisfies the following SDE

$$\Delta\ln(S)(t) = \mu_{ld}\Delta t + \sigma\Delta W(t) + \sum_{i=1}^{\Delta P(t)} Q_i. \tag{4}$$

$\Delta\ln(S(t)) \equiv \ln(S(t+\Delta t)) - \ln(S(t))$ is called daily log-return in a similar reason as instant log-return.

3 The Basic Moments of daily log-return $\Delta\ln(S(t))$

For the moments of daily return rate $\Delta\ln(S(t))$, we have the following theorem:

Theorem 3.1 If $\Delta\ln(S(t))$ satisfies SDE (4), the first four moments of $\Delta\ln(S(t))$ are the following:

$$M_1^{(jd)} \equiv \mathrm{E}[\Delta\ln(S(t))] = (\mu_{ld} + \lambda\mu_j)\Delta t;$$

$$\begin{aligned} M_2^{(jd)} &\equiv \mathrm{Var}[\Delta\ln(S(t))] \\ &= (\sigma^2 + \lambda(\sigma_j^2 + \mu_j^2))\Delta t; \end{aligned}$$

$$\begin{aligned} M_3^{(jd)} &\equiv \mathrm{E}\left[(\Delta\ln(S(t)) - M_1^{(jd)})^3\right] \\ &= \frac{pa^3+qb^3}{4}\lambda\Delta t; \end{aligned}$$

$$\begin{aligned} M_4^{(jd)} &\equiv \mathrm{E}\left[(\Delta\ln(S(t)) - M_1^{(jd)})^4\right] \\ &= \frac{pa^4+qb^4}{5}\lambda\Delta t \\ &\quad +3(\sigma^2 + \lambda(pa^2+qb^2)/3)^2(\Delta t)^2. \end{aligned}$$

Proof: From Theorem 5.12 in [7], we know directly that the first two moments $M_1^{(jd)}$ and $M_2^{(jd)}$ are true. Also, from the same Theorem 5.12, $M_3^{(jd)} = (M_j^{(3)} + \mu_j(3\sigma_j^2 + \mu_j^2))\lambda\Delta t$ and $M_4^{(jd)} = \left(M_j^{(4)} + 4\mu_j M_j^{(3)} + 6\mu_j^2\sigma_j^2 + \mu_j^4\right)\lambda\Delta t + 3(\sigma^2 + \lambda(\mu_j^2 + \sigma_j^2))^2(\Delta t)^2$. Then, we put the values of μ_j, σ_j^2, $M_j^{(3)}$ and $M_j^{(4)}$ of the doule-uniform jump amplitude mark Q into the above formulae and do some simplifications, will finally get the formulae as in the theorem for the third and fourth moments. □

4 The density of daily log-return $\Delta\ln(S(t))$

In order to calculate the density of $\mathbf{\Delta ln(S(t))}$, we need the following Lemmas.

Lemma 4.1 Shift Property of the Accumulated Normal Distribution Function:

$$\Phi^{(n)}(a+x, b+x; \mu+x, \sigma^2) = \Phi^{(n)}(a, b; \mu, \sigma^2).$$

Proof:

$$\begin{aligned}
\Phi^{(n)}(a+x, b+x; \mu+x, \sigma^2) &= \int_{a+x}^{b+x} \frac{e^{-\frac{(z-\mu-x)^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}} dz \\
&\overset{\text{set } y = z-x}{=} \int_a^b \frac{e^{-\frac{(y-\mu)^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}} dy \\
&= \Phi^{(n)}(a, b; \mu, \sigma^2).
\end{aligned}$$

□

Lemma 4.2 Distribution Property of the Negative Sign:

$$\Phi^{(n)}(-a, -b; -\mu, \sigma^2) = -\Phi^{(n)}(a, b; \mu, \sigma^2).$$

Proof:

$$\begin{aligned}
\Phi^{(n)}(-a, -b; -\mu, \sigma^2) &= \int_{-a}^{-b} \frac{e^{-\frac{(x+\mu)^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}} dx \\
&\overset{\text{set } z = -x}{=} \int_a^b \frac{e^{-\frac{(z-\mu)^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}} - dz \\
&= \int_b^a \frac{e^{-\frac{(z-\mu)^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}} dz \\
&= \Phi^{(n)}(b, a; \mu, \sigma^2) \\
&= -\Phi^{(n)}(a, b; \mu, \sigma^2).
\end{aligned}$$

□

Lemma 4.3

$$
\begin{aligned}
IB_1(x_1,x_2,A,\sigma) &\equiv \int_{x_1}^{x_2} x\frac{e^{-\frac{(x-A)^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}}dx \\
&= \frac{\sigma}{\sqrt{2\pi}}(e^{-\frac{(x_1-A)^2}{2\sigma^2}} - e^{-\frac{(x_2-A)^2}{2\sigma^2}}) + A\Phi^{(n)}(x_1,x_2;A,\sigma^2).
\end{aligned}
$$

Proof:

$$
\begin{aligned}
IB_1(x_1,x_2,A,\sigma) &= \int_{x_1}^{x_2} (x-A+A)\frac{e^{-\frac{(x-A)^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}}dx \\
&= \int_{x_1}^{x_2} (x-A)\frac{e^{-\frac{(x-A)^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}}dx + A\int_{x_1}^{x_2} \frac{e^{-\frac{(x-A)^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}}dx \\
&= -\sigma^2\int_{x_1}^{x_2} \frac{e^{-\frac{(x-A)^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}}d-\frac{(x-A)^2}{2\sigma^2} + A\Phi^{(n)}(x_1,x_2;A,\sigma^2) \\
&= \frac{-\sigma}{\sqrt{2\pi}}e^{-\frac{(x-A)^2}{2\sigma^2}}|_{x_1}^{x_2} + A\Phi^{(n)}(x_1,x_2;A,\sigma^2) \\
&= \frac{\sigma}{\sqrt{2\pi}}(e^{-\frac{(x_1-A)^2}{2\sigma^2}} - e^{-\frac{(x_2-A)^2}{2\sigma^2}}) + A\Phi^{(n)}(x_1,x_2;A,\sigma^2).
\end{aligned}
$$

□

According to (4) and the convolution theorem [7], we get the following theorem:

Theorem 4.1 The density of $\mathbf{\Delta ln(S(t))}$ is

$$
\begin{aligned}
\phi(x) &= \sum_{k=0}^{\infty} p_k(\lambda\Delta t)\phi_{\Delta G}(*\phi_Q)^k(x) && (5) \\
&\approx \sum_{k=0}^{2} p_k(\lambda\Delta t)\phi^{(k)}(x). && (6)
\end{aligned}
$$

where $\Delta G \equiv \mu_{ld}\Delta t + \sigma\Delta W(t)$, $p_k(\lambda\Delta t) = e^{-\lambda\Delta t}(\lambda\Delta t)^k/k!$ and

$$
\begin{aligned}
\phi^{(0)}(x) &= \phi^{(n)}(x;\bar{\mu},\bar{\sigma}^2),\\
\phi^{(1)}(x) &= \frac{q}{b}\Phi^{(n)}(0,b;x-\bar{\mu},\bar{\sigma}^2) - \frac{p}{a}\Phi^{(n)}(a,0;x-\bar{\mu},\bar{\sigma}^2),\\
\phi^{(2)}(x) &= \frac{\bar{\sigma}}{\sqrt{2\pi}}\Bigg((\frac{p}{a}+\frac{q}{b})^2 e^{-\frac{(x-\bar{\mu})^2}{2\bar{\sigma}^2}} + (\frac{p}{a})^2 e^{-\frac{(x-\bar{\mu}-2a)^2}{2\bar{\sigma}^2}} + (\frac{q}{b})^2 e^{-\frac{(x-\bar{\mu}-2b)^2}{2\bar{\sigma}^2}}\\
&\quad -2\left((\frac{p}{a})^2+\frac{pq}{ab}\right)e^{-\frac{(x-\bar{\mu}-a)^2}{2\bar{\sigma}^2}} - 2\left((\frac{q}{b})^2+\frac{pq}{ab})\right)e^{-\frac{(x-\bar{\mu}-b)^2}{2\bar{\sigma}^2}} + \frac{2pq}{ab}e^{-\frac{(x-\bar{\mu}-a-b)^2}{2\bar{\sigma}^2}}\Bigg)\\
&\quad +(\frac{p}{a})^2\left((x-2a-\bar{\mu})\Phi^{(n)}(2a,a;x-\bar{\mu},\bar{\sigma}^2) - (x-\bar{\mu})\Phi^{(n)}(a,0;x-\bar{\mu},\bar{\sigma}^2)\right)\\
&\quad +\frac{2pq}{ab}\Bigg((x-\bar{\mu})\left(\Phi^{(n)}(0,b;x-\bar{\mu},\bar{\sigma}^2) - \Phi^{(n)}(a,a+b;x-\bar{\mu},\bar{\sigma}^2)\right)\\
&\quad +a\Phi^{(n)}(a,a+b;x-\bar{\mu},\bar{\sigma}^2) - b\Phi^{(n)}(a+b,b;x-\bar{\mu},\bar{\sigma}^2)\Bigg)\\
&\quad +(\frac{q}{b})^2\left((x-\bar{\mu})\Phi^{(n)}(0,b;x-\bar{\mu},\bar{\sigma}^2) - (x-2b-\bar{\mu})\Phi^{(n)}(b,2b;x-\bar{\mu},\bar{\sigma}^2)\right),
\end{aligned}
$$

where $\bar{\mu} \equiv \mu_{ld}\Delta t$ and $\bar{\sigma} \equiv \sigma\sqrt{\Delta t}$.

Proof: For the first part (5), please see Chapter 0 in Hanson's book [7]. Now we come to prove the second part (6).
For $k = 0$, $\phi^{(0)}(x) = \phi_{\Delta G}(*\phi_Q)^0(x) = \phi_{\Delta G}(x) = \phi^{(n)}(x;\bar{\mu},\bar{\sigma}^2)$.
For $k = 1$,

$$
\begin{aligned}
\phi^{(1)}(x) &= \phi_{\Delta G}(*\phi_Q)^1(x)\\
&= \phi_{\Delta G} * \phi_Q(x)\\
&= \int_{-\infty}^{\infty}\phi^{(n)}(x-y;\bar{\mu},\bar{\sigma}^2)\phi_Q(y)dy\\
&= \int_{-\infty}^{\infty}\phi^{(n)}(x-y;\bar{\mu},\bar{\sigma}^2)(-\frac{p}{a}I_{\{a\le y<0\}} + \frac{q}{b}I_{\{0\le y\le b\}})dy\\
&= -\frac{p}{a}\int_a^0\phi^{(n)}(x-y;\bar{\mu},\bar{\sigma}^2)dy + \frac{q}{b}\int_0^b\phi^{(n)}(x-y;\bar{\mu},\bar{\sigma}^2)dy\\
&= \frac{q}{b}\Phi^{(n)}(x-b,x;\bar{\mu},\bar{\sigma}^2) - \frac{p}{a}\Phi^{(n)}(x,x-a;\bar{\mu},\bar{\sigma}^2).\\
&\overset{Lemma}{\underset{4.2}{=}} -\frac{q}{b}\Phi^{(n)}(b-x,-x;-\bar{\mu},\bar{\sigma}^2) + \frac{p}{a}\Phi^{(n)}(-x,a-x;-\bar{\mu},\bar{\sigma}^2).\\
&\overset{Lemma}{\underset{4.1}{=}} -\frac{q}{b}\Phi^{(n)}(b,0;x-\bar{\mu},\bar{\sigma}^2) + \frac{p}{a}\Phi^{(n)}(0,a;x-\bar{\mu},\bar{\sigma}^2).\\
&= \frac{q}{b}\Phi^{(n)}(0,b;x-\bar{\mu},\bar{\sigma}^2) - \frac{p}{a}\Phi^{(n)}(a,0;x-\bar{\mu},\bar{\sigma}^2).
\end{aligned}
$$

For $k=2$, first of all, let us calculate $(\phi_Q * \phi_Q)(y)$.

$$\begin{aligned}
(\phi_Q * \phi_Q)(y) &= \int_{-\infty}^{\infty} \phi_Q(y-z)\phi_Q(z)dz \\
&= \int_{-\infty}^{\infty} (-\frac{p}{a} I_{\{a \le y-z<0\}} + \frac{q}{b} I_{\{0 \le y-z \le b\}})(-\frac{p}{a} I_{\{a \le z<0\}} + \frac{q}{b} I_{\{0 \le z \le b\}})dz \\
&= (\frac{p}{a})^2 \int_{-\infty}^{\infty} I_{\{y<z \le y-a\ ,\ a \le z<0\}}dz + (\frac{q}{b})^2 \int_{-\infty}^{\infty} I_{\{y-b \le z \le y\ ,\ 0 \le z \le b\}}dz \\
&\quad -\frac{pq}{ab} \int_{-\infty}^{\infty} I_{\{y<z \le y-a\ ,\ 0 \le z \le b\}}dz - \frac{pq}{ab} \int_{-\infty}^{\infty} I_{\{y-b \le z \le y\ ,\ a \le z<0\}}dz \\
&= (\frac{p}{a})^2 (min(y-a,0) - max(y,a))^+ + (\frac{q}{b})^2 (min(y,b) - max(y-b,0))^+ \\
&\quad -\frac{pq}{ab}(min(y-a,b) - max(y,0))^+ - \frac{pq}{ab}(min(y,0) - max(y-b,a))^+.
\end{aligned}$$

But, $min(y-a,b) - max(y,0) = -max(a-y,-b) + min(-y,0) = (-max(a,y-b)+y) + (min(0,y)-y) = min(0,y) - max(a,y-b) = min(y,0) - max(y-b,a)$ since $min(z_1,z_2) = -max(-z_1,-z_2)$, $min(z_1,z_2)+c = min(z_1+c, z_2+c)$ and $max(z_1,z_2)+c = max(z_1+c, z_2+c)$. Hence, we have,

$$\begin{aligned}
(\phi_Q * \phi_Q)(y) &= (\frac{p}{a})^2 (min(y-a,0) - max(y,a))^+ + (\frac{q}{b})^2 (min(y,b) - max(y-b,0))^+ \\
&\quad -\frac{2pq}{ab}(min(y-a,b) - max(y,0))^+.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\phi^{(2)}(x) &= \phi_{\Delta G}(*\phi_Q)^2(x) \\
&= \int_{-\infty}^{\infty} \phi^{(n)}(x-y;\bar{\mu},\bar{\sigma}^2)(\phi_Q * \phi_Q)(y)dy \\
&= (\frac{p}{a})^2 \int_{-\infty}^{\infty} \frac{e^{-\frac{(x-y-\bar{\mu})^2}{2\bar{\sigma}^2}}}{\sqrt{2\pi\bar{\sigma}^2}} (min(y-a,0) - max(y,a))^+ dy \\
&\quad -\frac{2pq}{ab} \int_{-\infty}^{\infty} \frac{e^{-\frac{(x-y-\bar{\mu})^2}{2\bar{\sigma}^2}}}{\sqrt{2\pi\bar{\sigma}^2}} (min(y-a,b) - max(y,0))^+ dy \\
&\quad +(\frac{q}{b})^2 \int_{-\infty}^{\infty} \frac{e^{-\frac{(x-y-\bar{\mu})^2}{2\bar{\sigma}^2}}}{\sqrt{2\pi\bar{\sigma}^2}} (min(y,b) - max(y-b,0))^+ dy.
\end{aligned}$$

However,

$$
\begin{aligned}
I_1 \quad &\equiv \quad \int_{-\infty}^{\infty} \frac{e^{-\frac{(x-y-\bar{\mu})^2}{2\bar{\sigma}^2}}}{\sqrt{2\pi\bar{\sigma}^2}}(min(y-a,0)-max(y,a))^+dy \\
&= \quad (\int_{-\infty}^{a}+\int_{a}^{\infty})\frac{e^{-\frac{(x-y-\bar{\mu})^2}{2\bar{\sigma}^2}}}{\sqrt{2\pi\bar{\sigma}^2}}(min(y-a,0)-max(y,a))^+dy \\
&= \quad \int_{-\infty}^{a} \frac{e^{-\frac{(x-y-\bar{\mu})^2}{2\bar{\sigma}^2}}}{\sqrt{2\pi\bar{\sigma}^2}}(y-2a)^+dy+\int_{a}^{\infty} \frac{e^{-\frac{(x-y-\bar{\mu})^2}{2\bar{\sigma}^2}}}{\sqrt{2\pi\bar{\sigma}^2}}(-y)^+dy \\
&= \quad \int_{2a}^{a} \frac{e^{-\frac{(x-y-\bar{\mu})^2}{2\bar{\sigma}^2}}}{\sqrt{2\pi\bar{\sigma}^2}}(y-2a)dy+\int_{a}^{0} \frac{e^{-\frac{(x-y-\bar{\mu})^2}{2\bar{\sigma}^2}}}{\sqrt{2\pi\bar{\sigma}^2}}(-y)dy \\
&= \quad \int_{2a}^{a} y\frac{e^{-\frac{(x-y-\bar{\mu})^2}{2\bar{\sigma}^2}}}{\sqrt{2\pi\bar{\sigma}^2}}dy-\int_{a}^{0} y\frac{e^{-\frac{(x-y-\bar{\mu})^2}{2\bar{\sigma}^2}}}{\sqrt{2\pi\bar{\sigma}^2}}dy-2a\Phi^{(n)}(2a,a;x-\bar{\mu},\bar{\sigma}^2) \\
&\overset{\text{Lemma}}{\underset{4.3}{=}} \quad \sqrt{\frac{\bar{\sigma}^2}{2\pi}}(e^{-\frac{(2a-(x-\bar{\mu}))^2}{2\bar{\sigma}^2}}-e^{-\frac{(a-(x-\bar{\mu}))^2}{2\bar{\sigma}^2}})+(x-\bar{\mu})\Phi^{(n)}(2a,a;x-\bar{\mu},\bar{\sigma}^2) \\
&\quad\quad -\sqrt{\frac{\bar{\sigma}^2}{2\pi}}(e^{-\frac{(a-(x-\bar{\mu}))^2}{2\bar{\sigma}^2}}-e^{-\frac{(x-\bar{\mu})^2}{2\bar{\sigma}^2}})-(x-\bar{\mu})\Phi^{(n)}(a,0;x-\bar{\mu},\bar{\sigma}^2) \\
&\quad\quad -2a\Phi^{(n)}(2a,a;x-\bar{\mu},\bar{\sigma}^2). \\
&= \quad \sqrt{\frac{\bar{\sigma}^2}{2\pi}}\left(e^{-\frac{(2a-(x-\bar{\mu}))^2}{2\bar{\sigma}^2}}-2e^{-\frac{(a-(x-\bar{\mu}))^2}{2\bar{\sigma}^2}}+e^{-\frac{(x-\bar{\mu})^2}{2\bar{\sigma}^2}}\right)+ \\
&\quad\quad (x-2a-\bar{\mu})\Phi^{(n)}(2a,a;x-\bar{\mu},\bar{\sigma}^2)-(x-\bar{\mu})\Phi^{(n)}(a,0;x-\bar{\mu},\bar{\sigma}^2).
\end{aligned}
$$

In the case $a+b<0$,

$$
\begin{aligned}
I_2 \quad &\equiv \quad \int_{-\infty}^{\infty} \frac{e^{-\frac{(x-y-\bar{\mu})^2}{2\bar{\sigma}^2}}}{\sqrt{2\pi\bar{\sigma}^2}}(min(y-a,b)-max(y,0))^+dy \\
&\overset{a+b<0}{=} \quad (\int_{-\infty}^{a+b}+\int_{a+b}^{0}+\int_{0}^{\infty})\frac{e^{-\frac{(x-y-\bar{\mu})^2}{2\bar{\sigma}^2}}}{\sqrt{2\pi\bar{\sigma}^2}}(min(y-a,b)-max(y,0))^+dy \\
&= \quad \int_{-\infty}^{a+b} \frac{e^{-\frac{(x-y-\bar{\mu})^2}{2\bar{\sigma}^2}}}{\sqrt{2\pi\bar{\sigma}^2}}(y-a)^+dy+b\int_{a+b}^{0} \frac{e^{-\frac{(x-y-\bar{\mu})^2}{2\bar{\sigma}^2}}}{\sqrt{2\pi\bar{\sigma}^2}}dy+\int_{0}^{\infty} \frac{e^{-\frac{(x-y-\bar{\mu})^2}{2\bar{\sigma}^2}}}{\sqrt{2\pi\bar{\sigma}^2}}(b-y)^+dy \\
&= \quad \int_{a}^{a+b} \frac{e^{-\frac{(x-y-\bar{\mu})^2}{2\bar{\sigma}^2}}}{\sqrt{2\pi\bar{\sigma}^2}}(y-a)dy+b\Phi^{(n)}(a+b,0;x-\bar{\mu},\bar{\sigma}^2)+\int_{0}^{b} \frac{e^{-\frac{(x-y-\bar{\mu})^2}{2\bar{\sigma}^2}}}{\sqrt{2\pi\bar{\sigma}^2}}(b-y)dy \\
&= \quad \int_{a}^{a+b} \frac{ye^{-\frac{(x-y-\bar{\mu})^2}{2\bar{\sigma}^2}}}{\sqrt{2\pi\bar{\sigma}^2}}dy-\int_{0}^{b} \frac{ye^{-\frac{(x-y-\bar{\mu})^2}{2\bar{\sigma}^2}}}{\sqrt{2\pi\bar{\sigma}^2}}dy-a\Phi^{(n)}(a,a+b;x-\bar{\mu},\bar{\sigma}^2) \\
&\quad\quad +b\Phi^{(n)}(a+b,0;x-\bar{\mu},\bar{\sigma}^2)+b\Phi^{(n)}(0,b;x-\bar{\mu},\bar{\sigma}^2)
\end{aligned}
$$

$$
\begin{aligned}
&= \int_a^{a+b} \frac{ye^{-\frac{(x-y-\bar{\mu})^2}{2\bar{\sigma}^2}}}{\sqrt{2\pi\bar{\sigma}^2}}dy - \int_0^b \frac{ye^{-\frac{(x-y-\bar{\mu})^2}{2\bar{\sigma}^2}}}{\sqrt{2\pi\bar{\sigma}^2}}dy \\
&\quad +b\Phi^{(n)}(a+b,b;x-\bar{\mu},\bar{\sigma}^2) - a\Phi^{(n)}(a,a+b;x-\bar{\mu},\bar{\sigma}^2) \\
&\underset{4.3}{\overset{\text{Lemma}}{=}} \sqrt{\frac{\bar{\sigma}^2}{2\pi}}(e^{-\frac{(a-(x-\bar{\mu}))^2}{2\bar{\sigma}^2}} - e^{-\frac{(a+b-(x-\bar{\mu}))^2}{2\bar{\sigma}^2}}) + (x-\bar{\mu})\Phi^{(n)}(a,a+b;x-\bar{\mu},\bar{\sigma}^2) \\
&\quad -\sqrt{\frac{\bar{\sigma}^2}{2\pi}}(e^{-\frac{(x-\bar{\mu})^2}{2\bar{\sigma}^2}} - e^{-\frac{(b-(x-\bar{\mu}))^2}{2\bar{\sigma}^2}}) - (x-\bar{\mu})\Phi^{(n)}(0,b;x-\bar{\mu},\bar{\sigma}^2) \\
&\quad +b\Phi^{(n)}(a+b,b;x-\bar{\mu},\bar{\sigma}^2) - a\Phi^{(n)}(a,a+b;x-\bar{\mu},\bar{\sigma}^2) \\
&= \sqrt{\frac{\bar{\sigma}^2}{2\pi}}\left(e^{-\frac{(a-(x-\bar{\mu}))^2}{2\bar{\sigma}^2}} + e^{-\frac{(b-(x-\bar{\mu}))^2}{2\bar{\sigma}^2}} - e^{-\frac{(a+b-(x-\bar{\mu}))^2}{2\bar{\sigma}^2}} - e^{-\frac{(x-\bar{\mu})^2}{2\bar{\sigma}^2}}\right) \\
&\quad +(x-\bar{\mu})\left(\Phi^{(n)}(a,a+b;x-\bar{\mu},\bar{\sigma}^2) - \Phi^{(n)}(0,b;x-\bar{\mu},\bar{\sigma}^2)\right) \\
&\quad +b\Phi^{(n)}(a+b,b;x-\bar{\mu},\bar{\sigma}^2) - a\Phi^{(n)}(a,a+b;x-\bar{\mu},\bar{\sigma}^2).
\end{aligned}
$$

Otherwese, $a+b \geq 0$,

$$
\begin{aligned}
I_2 &\overset{a+b\geq 0}{=} (\int_{-\infty}^0 + \int_0^{a+b} + \int_{a+b}^{\infty})\frac{e^{-\frac{(x-y-\bar{\mu})^2}{2\bar{\sigma}^2}}}{\sqrt{2\pi\bar{\sigma}^2}}(min(y-a,b)-max(y,0))^+dy \\
&= \int_{-\infty}^0 \frac{e^{-\frac{(x-y-\bar{\mu})^2}{2\bar{\sigma}^2}}}{\sqrt{2\pi\bar{\sigma}^2}}(y-a)^+dy - a\int_{a+b}^0 \frac{e^{-\frac{(x-y-\bar{\mu})^2}{2\bar{\sigma}^2}}}{\sqrt{2\pi\bar{\sigma}^2}}dy + \int_{a+b}^{\infty} \frac{e^{-\frac{(x-y-\bar{\mu})^2}{2\bar{\sigma}^2}}}{\sqrt{2\pi\bar{\sigma}^2}}(b-y)^+dy \\
&= \int_a^0 \frac{e^{-\frac{(x-y-\bar{\mu})^2}{2\bar{\sigma}^2}}}{\sqrt{2\pi\bar{\sigma}^2}}(y-a)dy - a\Phi^{(n)}(0,a+b;x-\bar{\mu},\bar{\sigma}^2) + \int_{a+b}^b \frac{e^{-\frac{(x-y-\bar{\mu})^2}{2\bar{\sigma}^2}}}{\sqrt{2\pi\bar{\sigma}^2}}(b-y)dy \\
&= \int_a^0 \frac{ye^{-\frac{(x-y-\bar{\mu})^2}{2\bar{\sigma}^2}}}{\sqrt{2\pi\bar{\sigma}^2}}dy - \int_{a+b}^b \frac{ye^{-\frac{(x-y-\bar{\mu})^2}{2\bar{\sigma}^2}}}{\sqrt{2\pi\bar{\sigma}^2}}dy \\
&\quad +b\Phi^{(n)}(a+b,b;x-\bar{\mu},\bar{\sigma}^2) - a\Phi^{(n)}(a,a+b;x-\bar{\mu},\bar{\sigma}^2) \\
&\underset{4.3}{\overset{\text{Lemma}}{=}} \sqrt{\frac{\bar{\sigma}^2}{2\pi}}(e^{-\frac{(a-(x-\bar{\mu}))^2}{2\bar{\sigma}^2}} - e^{-\frac{(x-\bar{\mu})^2}{2\bar{\sigma}^2}}) + (x-\bar{\mu})\Phi^{(n)}(a,0;x-\bar{\mu},\bar{\sigma}^2) \\
&\quad -\sqrt{\frac{\bar{\sigma}^2}{2\pi}}(e^{-\frac{(a+b-(x-\bar{\mu}))^2}{2\bar{\sigma}^2}} - e^{-\frac{(b-(x-\bar{\mu}))^2}{2\bar{\sigma}^2}}) - (x-\bar{\mu})\Phi^{(n)}(a+b,b;x-\bar{\mu},\bar{\sigma}^2) \\
&\quad +b\Phi^{(n)}(a+b,b;x-\bar{\mu},\bar{\sigma}^2) - a\Phi^{(n)}(a,a+b;x-\bar{\mu},\bar{\sigma}^2) \\
&= \sqrt{\frac{\bar{\sigma}^2}{2\pi}}\left(e^{-\frac{(a-(x-\bar{\mu}))^2}{2\bar{\sigma}^2}} + e^{-\frac{(b-(x-\bar{\mu}))^2}{2\bar{\sigma}^2}} - e^{-\frac{(a+b-(x-\bar{\mu}))^2}{2\bar{\sigma}^2}} - e^{-\frac{(x-\bar{\mu})^2}{2\bar{\sigma}^2}}\right) \\
&\quad +(x-\bar{\mu})\left(\Phi^{(n)}(a,a+b;x-\bar{\mu},\bar{\sigma}^2) - \Phi^{(n)}(0,b;x-\bar{\mu},\bar{\sigma}^2)\right) \\
&\quad +b\Phi^{(n)}(a+b,b;x-\bar{\mu},\bar{\sigma}^2) - a\Phi^{(n)}(a,a+b;x-\bar{\mu},\bar{\sigma}^2).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
I_2 \;=\;& \sqrt{\frac{\bar{\sigma}^2}{2\pi}}\left(e^{-\frac{(a-(x-\bar{\mu}))^2}{2\bar{\sigma}^2}} + e^{-\frac{(b-(x-\bar{\mu}))^2}{2\bar{\sigma}^2}} - e^{-\frac{(a+b-(x-\bar{\mu}))^2}{2\bar{\sigma}^2}} - e^{-\frac{(x-\bar{\mu})^2}{2\bar{\sigma}^2}}\right) \\
&+(x-\bar{\mu})\left(\Phi^{(n)}(a,a+b;x-\bar{\mu},\bar{\sigma}^2) - \Phi^{(n)}(0,b;x-\bar{\mu},\bar{\sigma}^2)\right) \\
&+b\Phi^{(n)}(a+b,b;x-\bar{\mu},\bar{\sigma}^2) - a\Phi^{(n)}(a,a+b;x-\bar{\mu},\bar{\sigma}^2).
\end{aligned}
$$

$$
\begin{aligned}
I_3 \quad\equiv\quad& \int_{-\infty}^{\infty}\frac{e^{-\frac{(x-y-\bar{\mu})^2}{2\bar{\sigma}^2}}}{\sqrt{2\pi\bar{\sigma}^2}}(min(y,b)-max(y-b,0))^+dy \\
=\quad& \left(\int_{-\infty}^{b}+\int_{b}^{\infty}\right)\frac{e^{-\frac{(x-y-\bar{\mu})^2}{2\bar{\sigma}^2}}}{\sqrt{2\pi\bar{\sigma}^2}}(min(y,b)-max(y-b,0))^+dy \\
=\quad& \int_{-\infty}^{b}\frac{e^{-\frac{(x-y-\bar{\mu})^2}{2\bar{\sigma}^2}}}{\sqrt{2\pi\bar{\sigma}^2}}y^+dy + \int_{b}^{\infty}\frac{e^{-\frac{(x-y-\bar{\mu})^2}{2\bar{\sigma}^2}}}{\sqrt{2\pi\bar{\sigma}^2}}(2b-y)^+dy \\
=\quad& \int_{0}^{b}\frac{e^{-\frac{(x-y-\bar{\mu})^2}{2\bar{\sigma}^2}}}{\sqrt{2\pi\bar{\sigma}^2}}ydy + \int_{b}^{2b}\frac{e^{-\frac{(x-y-\bar{\mu})^2}{2\bar{\sigma}^2}}}{\sqrt{2\pi\bar{\sigma}^2}}(2b-y)dy \\
\underset{4.3}{\overset{\text{Lemma}}{=}}\quad& \sqrt{\frac{\bar{\sigma}^2}{2\pi}}\left(e^{-\frac{(x-\bar{\mu})^2}{2\bar{\sigma}^2}} - e^{-\frac{(b-(x-\bar{\mu}))^2}{2\bar{\sigma}^2}}\right) + (x-\bar{\mu})\Phi^{(n)}(0,b;x-\bar{\mu},\bar{\sigma}^2) \\
&-\sqrt{\frac{\bar{\sigma}^2}{2\pi}}\left(e^{-\frac{(b-(x-\bar{\mu}))^2}{2\bar{\sigma}^2}} - e^{-\frac{(2b-(x-\bar{\mu}))^2}{2\bar{\sigma}^2}}\right) - (x-\bar{\mu})\Phi^{(n)}(b,2b;x-\bar{\mu},\bar{\sigma}^2) \\
&+2b\Phi^{(n)}(b,2b;x-\bar{\mu},\bar{\sigma}^2) \\
=\quad& \sqrt{\frac{\bar{\sigma}^2}{2\pi}}\left(e^{-\frac{(x-\bar{\mu})^2}{2\bar{\sigma}^2}} - 2e^{-\frac{(b-(x-\bar{\mu}))^2}{2\bar{\sigma}^2}} + e^{-\frac{(2b-(x-\bar{\mu}))^2}{2\bar{\sigma}^2}}\right) \\
&+(x-\bar{\mu})\Phi^{(n)}(0,b;x-\bar{\mu},\bar{\sigma}^2) - (x-2b-\bar{\mu})\Phi^{(n)}(b,2b;x-\bar{\mu},\bar{\sigma}^2)
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\phi^{(2)}(x) &= \phi_{\Delta G}(*\phi_Q)^2(x) \\
&= (\frac{p}{a})^2\left(\sqrt{\frac{\bar{\sigma}^2}{2\pi}}\left(e^{-\frac{(2a-(x-\bar{\mu}))^2}{2\bar{\sigma}^2}} - 2e^{-\frac{(a-(x-\bar{\mu}))^2}{2\bar{\sigma}^2}} + e^{-\frac{(x-\bar{\mu})^2}{2\bar{\sigma}^2}}\right) + \right. \\
&\quad \left.(x-2a-\bar{\mu})\Phi^{(n)}(2a,a;x-\bar{\mu},\bar{\sigma}^2) - (x-\bar{\mu})\Phi^{(n)}(a,0;x-\bar{\mu},\bar{\sigma}^2)\right) \\
&\quad -\frac{2pq}{ab}\left(\sqrt{\frac{\bar{\sigma}^2}{2\pi}}\left(e^{-\frac{(a-(x-\bar{\mu}))^2}{2\bar{\sigma}^2}} + e^{-\frac{(b-(x-\bar{\mu}))^2}{2\bar{\sigma}^2}} - e^{-\frac{(a+b-(x-\bar{\mu}))^2}{2\bar{\sigma}^2}} - e^{-\frac{(x-\bar{\mu})^2}{2\bar{\sigma}^2}}\right)\right. \\
&\quad +(x-\bar{\mu})\left(\Phi^{(n)}(a,a+b;x-\bar{\mu},\bar{\sigma}^2) - \Phi^{(n)}(0,b;x-\bar{\mu},\bar{\sigma}^2)\right) \\
&\quad \left.+b\Phi^{(n)}(a+b,b;x-\bar{\mu},\bar{\sigma}^2) - a\Phi^{(n)}(a,a+b;x-\bar{\mu},\bar{\sigma}^2)\right) \\
&\quad +(\frac{q}{b})^2\left(\sqrt{\frac{\bar{\sigma}^2}{2\pi}}(e^{-\frac{(x-\bar{\mu})^2}{2\bar{\sigma}^2}} - 2e^{-\frac{(b-(x-\bar{\mu}))^2}{2\bar{\sigma}^2}} + e^{-\frac{(2b-(x-\bar{\mu}))^2}{2\bar{\sigma}^2}})\right. \\
&\quad \left.+(x-\bar{\mu})\Phi^{(n)}(0,b;x-\bar{\mu},\bar{\sigma}^2) - (x-2b-\bar{\mu})\Phi^{(n)}(b,2b;x-\bar{\mu},\bar{\sigma}^2)\right) \\
&= \frac{\bar{\sigma}}{\sqrt{2\pi}}\left((\frac{p}{a}+\frac{q}{b})^2 e^{-\frac{(x-\bar{\mu})^2}{2\bar{\sigma}^2}} + (\frac{p}{a})^2 e^{-\frac{(x-\bar{\mu}-2a)^2}{2\bar{\sigma}^2}} + (\frac{q}{b})^2 e^{-\frac{(x-\bar{\mu}-2b)^2}{2\bar{\sigma}^2}}\right. \\
&\quad \left.-2\left((\frac{p}{a})^2 + \frac{pq}{ab}\right)e^{-\frac{(x-\bar{\mu}-a)^2}{2\bar{\sigma}^2}} - 2\left((\frac{q}{b})^2 + \frac{pq}{ab})\right)e^{-\frac{(x-\bar{\mu}-b)^2}{2\bar{\sigma}^2}} + \frac{2pq}{ab}e^{-\frac{(x-\bar{\mu}-a-b)^2}{2\bar{\sigma}^2}}\right) \\
&\quad +(\frac{p}{a})^2\left((x-2a-\bar{\mu})\Phi^{(n)}(2a,a;x-\bar{\mu},\bar{\sigma}^2) - (x-\bar{\mu})\Phi^{(n)}(a,0;x-\bar{\mu},\bar{\sigma}^2)\right) \\
&\quad +\frac{2pq}{ab}\left((x-\bar{\mu})\left(\Phi^{(n)}(0,b;x-\bar{\mu},\bar{\sigma}^2) - \Phi^{(n)}(a,a+b;x-\bar{\mu},\bar{\sigma}^2)\right)\right. \\
&\quad \left.+a\Phi^{(n)}(a,a+b;x-\bar{\mu},\bar{\sigma}^2) - b\Phi^{(n)}(a+b,b;x-\bar{\mu},\bar{\sigma}^2)\right) \\
&\quad +(\frac{q}{b})^2\left((x-\bar{\mu})\Phi^{(n)}(0,b;x-\bar{\mu},\bar{\sigma}^2) - (x-2b-\bar{\mu})\Phi^{(n)}(b,2b;x-\bar{\mu},\bar{\sigma}^2)\right).
\end{aligned}
$$

□

5 The Bin Probability Distribution for $\Delta \ln(S(t))$

First of all, let us derive the following lemmas.

Lemma 5.1

$$\begin{aligned} IB_2(x_1,x_2,A,B) &\equiv \int_{x_1}^{x_2} \Phi^{(n)}(A,B;x-\mu,\sigma^2)dx \\ &= x\Phi^{(n)}(A,B;x-\mu,\sigma^2)|_{x_1}^{x_2} \\ &\quad +\frac{\sigma}{\sqrt{2\pi}}(e^{-\frac{(x_1-(\mu+B))^2}{2\sigma^2}} + e^{-\frac{(x_2-(\mu+A))^2}{2\sigma^2}} - e^{-\frac{(x_1-(\mu+A))^2}{2\sigma^2}} - e^{-\frac{(x_2-(\mu+B))^2}{2\sigma^2}}) \\ &\quad +(\mu+B)\Phi^{(n)}(x_1,x_2;\mu+B,\sigma^2) - (\mu+A)\Phi^{(n)}(x_1,x_2;\mu+A,\sigma^2). \end{aligned}$$

Proof:

$$\begin{aligned} IB_2(x_1,x_2,A,B) &\overset{By}{\underset{IBP}{=}} x\Phi^{(n)}(A,B;x-\mu,\sigma^2)|_{x_1}^{x_2} - \int_{x_1}^{x_2} x\frac{d\Phi^{(n)}(A,B;x-\mu,\sigma^2)}{dx}dx \\ &= x\Phi^{(n)}(A,B;x-\mu,\sigma^2)|_{x_1}^{x_2} - \int_{x_1}^{x_2} x(-\frac{e^{-\frac{(x-\mu-y)^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}}|_A^B)dx \\ &= x\Phi^{(n)}(A,B;x-\mu,\sigma^2)|_{x_1}^{x_2} + \int_{x_1}^{x_2} x(\frac{e^{-\frac{(x-\mu-y)^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}}|_A^B)dx \\ &= x\Phi^{(n)}(A,B;x-\mu,\sigma^2)|_{x_1}^{x_2} + \int_{x_1}^{x_2} x(\frac{e^{-\frac{(x-\mu-B)^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}} - \frac{e^{-\frac{(x-\mu-A)^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}})dx \\ &\overset{Lemma}{\underset{4.3}{=}} x\Phi^{(n)}(A,B;x-\mu,\sigma^2)|_{x_1}^{x_2} \\ &\quad +\frac{\sigma}{\sqrt{2\pi}}(e^{-\frac{(x_1-(\mu+B))^2}{2\sigma^2}} - e^{-\frac{(x_2-(\mu+B))^2}{2\sigma^2}}) + (\mu+B)\Phi^{(n)}(x_1,x_2;\mu+B,\sigma^2) \\ &\quad -\frac{\sigma}{\sqrt{2\pi}}(e^{-\frac{(x_1-(\mu+A))^2}{2\sigma^2}} - e^{-\frac{(x_2-(\mu+A))^2}{2\sigma^2}}) - (\mu+A)\Phi^{(n)}(x_1,x_2;\mu+A,\sigma^2) \\ &= x\Phi^{(n)}(A,B;x-\mu,\sigma^2)|_{x_1}^{x_2} \\ &\quad +\frac{\sigma}{\sqrt{2\pi}}(e^{-\frac{(x_1-(\mu+B))^2}{2\sigma^2}} + e^{-\frac{(x_2-(\mu+A))^2}{2\sigma^2}} - e^{-\frac{(x_1-(\mu+A))^2}{2\sigma^2}} - e^{-\frac{(x_2-(\mu+B))^2}{2\sigma^2}}) \\ &\quad +(\mu+B)\Phi^{(n)}(x_1,x_2;\mu+B,\sigma^2) - (\mu+A)\Phi^{(n)}(x_1,x_2;\mu+A,\sigma^2). \end{aligned}$$

□

Lemma 5.2

$$\begin{aligned} IB_3(x_1,x_2,A,\sigma) &\equiv \int_{x_1}^{x_2} x^2\frac{e^{-\frac{(x-A)^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}}dx \\ &= \frac{-\sigma}{\sqrt{2\pi}}(z+2A)e^{-\frac{z^2}{2\sigma^2}}|_{x_1-A}^{x_2-A} + (\sigma^2+A^2)\Phi^{(n)}(x_1,x_2;A,\sigma^2). \end{aligned}$$

Proof:

$$
\begin{aligned}
IB_3(x_1,x_2,A,\sigma) &\underset{x=z+A}{\overset{Set}{=}} \int_{x_1-A}^{x_2-A}(z+A)^2\frac{e^{-\frac{z^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}}dz \\
&= \int_{x_1-A}^{x_2-A}(z^2+2Az+A^2)\frac{e^{-\frac{z^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}}dz \\
&= \int_{x_1-A}^{x_2-A}z^2\frac{e^{-\frac{z^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}}dz+2A\int_{x_1-A}^{x_2-A}z\frac{e^{-\frac{z^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}}dz+A^2\int_{x_1-A}^{x_2-A}\frac{e^{-\frac{z^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}}dz \\
&= \frac{-\sigma}{\sqrt{2\pi}}\int_{x_1-A}^{x_2-A}zde^{-\frac{z^2}{2\sigma^2}}+\frac{-2A\sigma}{\sqrt{2\pi}}\int_{x_1-A}^{x_2-A}de^{-\frac{z^2}{2\sigma^2}}+A^2\int_{x_1-A}^{x_2-A}\frac{e^{-\frac{z^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}}dz \\
&= \frac{-\sigma}{\sqrt{2\pi}}\left(ze^{-\frac{z^2}{2\sigma^2}}|_{x_1-A}^{x_2-A}-\sqrt{2\pi\sigma^2}\int_{x_1-A}^{x_2-A}\frac{e^{-\frac{z^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}}dz\right) \\
&\quad+\frac{-2A\sigma}{\sqrt{2\pi}}e^{-\frac{z^2}{2\sigma^2}}|_{x_1-A}^{x_2-A}+A^2\int_{x_1-A}^{x_2-A}\frac{e^{-\frac{z^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}}dz \\
&= \frac{-\sigma}{\sqrt{2\pi}}\left(ze^{-\frac{z^2}{2\sigma^2}}|_{x_1-A}^{x_2-A}+2Ae^{-\frac{z^2}{2\sigma^2}}|_{x_1-A}^{x_2-A}\right)+(\sigma^2+A^2)\int_{x_1-A}^{x_2-A}\frac{e^{-\frac{z^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}}dz \\
&= \frac{-\sigma}{\sqrt{2\pi}}(z+2A)e^{-\frac{z^2}{2\sigma^2}}|_{x_1-A}^{x_2-A}+(\sigma^2+A^2)\Phi^{(n)}(x_1-A,x_2-A;0,\sigma^2) \\
&\underset{4.1}{\overset{Lemma}{=}} \frac{-\sigma}{\sqrt{2\pi}}(z+2A)e^{-\frac{z^2}{2\sigma^2}}|_{x_1-A}^{x_2-A}+(\sigma^2+A^2)\Phi^{(n)}(x_1,x_2;A,\sigma^2).
\end{aligned}
$$

□

Lemma 5.3

$$
\begin{aligned}
IB_4(x_1,x_2,A,B) &\equiv \int_{x_1}^{x_2}x\Phi^{(n)}(A,B;x-\mu,\sigma^2)dx \\
&= \frac{-\sigma}{2\sqrt{2\pi}}\left((z+2(\mu+B))e^{-\frac{z^2}{2\sigma^2}}|_{x_1-(\mu+B)}^{x_2-(\mu+B)}-(z+2(\mu+A))e^{-\frac{z^2}{2\sigma^2}}|_{x_1-(\mu+A)}^{x_2-(\mu+A)}\right) \\
&\quad+0.5\Big((\sigma^2+(\mu+B)^2)\Phi^{(n)}(x_1,x_2;\mu+B,\sigma^2) \\
&\quad-(\sigma^2+(\mu+A)^2)\Phi^{(n)}(x_1,x_2;\mu+A,\sigma^2)+z^2\Phi^{(n)}(A,B;z-\mu,\sigma^2)|_{x_1}^{x_2}\Big).
\end{aligned}
$$

Proof:

$$
\begin{aligned}
IB_4(x_1,x_2,A,B) \quad &= \quad \frac{1}{2}\int_{x_1}^{x_2}\Phi^{(n)}(A,B;x-\mu,\sigma^2)dx^2 \\
&\overset{By}{\underset{IBP}{=}} \quad \frac{1}{2}\left(x^2\Phi^{(n)}(A,B;x-\mu,\sigma^2)|_{x_1}^{x_2} - \int_{x_1}^{x_2}x^2\frac{d\Phi^{(n)}(A,B;x-\mu,\sigma^2)}{dx}dx\right) \\
&= \quad \frac{1}{2}\left(x^2\Phi^{(n)}(A,B;x-\mu,\sigma^2)|_{x_1}^{x_2} + \int_{x_1}^{x_2}x^2(\frac{e^{-\frac{(x-\mu-y)^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}}|_A^B)dx\right) \\
&= \quad \frac{1}{2}\left(x^2\Phi^{(n)}(A,B;x-\mu,\sigma^2)|_{x_1}^{x_2} + \int_{x_1}^{x_2}x^2(\frac{e^{-\frac{(x-\mu-B)^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}} - \frac{e^{-\frac{(x-\mu-A)^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}})dx\right) \\
&\overset{Lemma}{\underset{5.2}{=}} \quad \frac{1}{2}\Bigg(x^2\Phi^{(n)}(A,B;x-\mu,\sigma^2)|_{x_1}^{x_2} \\
&\quad -\frac{\sigma}{\sqrt{2\pi}}(z+2(\mu+B))e^{-\frac{z^2}{2\sigma^2}}|_{x_1-(\mu+B)}^{x_2-(\mu+B)} + (\sigma^2+(\mu+B)^2)\Phi^{(n)}(x_1,x_2;\mu+B,\sigma^2) \\
&\quad +\frac{\sigma}{\sqrt{2\pi}}(z+2(\mu+A))e^{-\frac{z^2}{2\sigma^2}}|_{x_1-(\mu+A)}^{x_2-(\mu+A)} - (\sigma^2+(\mu+A)^2)\Phi^{(n)}(x_1,x_2;\mu+A,\sigma^2)\Bigg) \\
&= \quad \frac{-\sigma}{2\sqrt{2\pi}}\left((z+2(\mu+B))e^{-\frac{z^2}{2\sigma^2}}|_{x_1-(\mu+B)}^{x_2-(\mu+B)} - (z+2(\mu+A))e^{-\frac{z^2}{2\sigma^2}}|_{x_1-(\mu+A)}^{x_2-(\mu+A)}\right) \\
&\quad +0.5\Bigg((\sigma^2+(\mu+B)^2)\Phi^{(n)}(x_1,x_2;\mu+B,\sigma^2) \\
&\quad -(\sigma^2+(\mu+A)^2)\Phi^{(n)}(x_1,x_2;\mu+A,\sigma^2) + x^2\Phi^{(n)}(A,B;x-\mu,\sigma^2)|_{x_1}^{x_2}\Bigg). \\
&\overset{x\to z}{=} \quad \frac{-\sigma}{2\sqrt{2\pi}}\left((z+2(\mu+B))e^{-\frac{z^2}{2\sigma^2}}|_{x_1-(\mu+B)}^{x_2-(\mu+B)} - (z+2(\mu+A))e^{-\frac{z^2}{2\sigma^2}}|_{x_1-(\mu+A)}^{x_2-(\mu+A)}\right) \\
&\quad +0.5\Bigg((\sigma^2+(\mu+B)^2)\Phi^{(n)}(x_1,x_2;\mu+B,\sigma^2) \\
&\quad -(\sigma^2+(\mu+A)^2)\Phi^{(n)}(x_1,x_2;\mu+A,\sigma^2) + z^2\Phi^{(n)}(A,B;z-\mu,\sigma^2)|_{x_1}^{x_2}\Bigg).
\end{aligned}
$$

$\square$

Based on Lemma 5.3, we get the following corollary.

Corollary 5.1

$$\begin{aligned}
IB_5(x_1,x_2,A,B) &\equiv \int_{x_1}^{x_2}(x-\mu)\Phi^{(n)}(A,B;x-\mu,\sigma^2)dx \\
&= \frac{-\sigma}{2\sqrt{2\pi}}\left((z+2B)e^{-\frac{z^2}{2\sigma^2}}|_{x_1-(\mu+B)}^{x_2-(\mu+B)}-(z+2A)e^{-\frac{z^2}{2\sigma^2}}|_{x_1-(\mu+A)}^{x_2-(\mu+A)}\right) \\
&\quad +0.5\Big((\sigma^2+B^2)\Phi^{(n)}(x_1,x_2;\mu+B,\sigma^2) \\
&\quad -(\sigma^2+A^2)\Phi^{(n)}(x_1,x_2;\mu+A,\sigma^2)+z^2\Phi^{(n)}(A,B;z,\sigma^2)|_{x_1-\mu}^{x_2-\mu}\Big).
\end{aligned}$$

Proof:

$$\begin{aligned}
IB_5(x_1,x_2,A,B) &\underset{x=z+\mu}{\overset{Set}{=}} \int_{x_1-\mu}^{x_2-\mu} z\Phi^{(n)}(A,B;z,\sigma^2)dz \\
&\underset{5.3}{\overset{Lemma}{=}} \frac{-\sigma}{2\sqrt{2\pi}}\left((z+2B)e^{-\frac{z^2}{2\sigma^2}}|_{x_1-(\mu+B)}^{x_2-(\mu+B)}-(z+2A)e^{-\frac{z^2}{2\sigma^2}}|_{x_1-(\mu+A)}^{x_2-(\mu+A)}\right) \\
&\quad +0.5\Big((\sigma^2+B^2)\Phi^{(n)}(x_1,x_2;\mu+B,\sigma^2) \\
&\quad -(\sigma^2+A^2)\Phi^{(n)}(x_1,x_2;\mu+A,\sigma^2)+z^2\Phi^{(n)}(A,B;z,\sigma^2)|_{x_1-\mu}^{x_2-\mu}\Big).
\end{aligned}$$

□

Lemma 5.4

$$\Phi^{(n)}(x_1,x_2;\mu_1,\sigma^2)-\Phi^{(n)}(x_1,x_2;\mu_2,\sigma^2)=\Phi^{(n)}(\mu_1,\mu_2;z,\sigma^2)|_{x_1}^{x_2}.$$

Proof:

$$\begin{aligned}
LHS &\underset{4.1}{\overset{Lemma}{=}} \Phi^{(n)}(x_1-\mu_1,x_2-\mu_1;0,\sigma^2)-\Phi^{(n)}(x_1-\mu_2,x_2-\mu_2;0,\sigma^2) \\
&= \Phi^{(n)}(x_1-\mu_1,x_2-\mu_1;0,\sigma^2)+\Phi^{(n)}(x_2-\mu_1,x_1-\mu_2;0,\sigma^2) \\
&\quad -\Phi^{(n)}(x_2-\mu_1,x_1-\mu_2;0,\sigma^2)-\Phi^{(n)}(x_1-\mu_2,x_2-\mu_2;0,\sigma^2) \\
&= \Phi^{(n)}(x_1-\mu_1,x_1-\mu_2;0,\sigma^2)-\Phi^{(n)}(x_2-\mu_1,x_2-\mu_2;0,\sigma^2) \\
&\underset{4.1}{\overset{Lemma}{=}} \Phi^{(n)}(-\mu_1,-\mu_2;-x_1,\sigma^2)-\Phi^{(n)}(-\mu_1,-\mu_2;-x_2,\sigma^2) \\
&\underset{4.2}{\overset{Lemma}{=}} -\Phi^{(n)}(\mu_1,\mu_2;x_1,\sigma^2)+\Phi^{(n)}(\mu_1,\mu_2;x_2,\sigma^2) \\
&= RHS.
\end{aligned}$$

□

Now, we are ready to calculate the bin probability distribution formula for $\Delta\ln(S(t))$.

Theorem 5.1 The second order truncated approximation to $[x_1, x_2]$ bin probability distribution for the linear jump-diffusion daily log-return $\Delta\ln(S(t))$ with log-double-uniformly distributed jump-amplitude is given by

$$\Phi(x_1, x_2) \approx \sum_{k=0}^{2} p_k(\lambda\Delta t)\Phi^{(k)}(x_1, x_2), \tag{7}$$

where

$$\Phi^{(0)}(x_1, x_2) \equiv \Phi^{(n)}(x_1, x_2; \bar{\mu}, \bar{\sigma}^2),$$

$$\begin{aligned}
\Phi^{(1)}(x_1, x_2) \;=\;& \frac{q}{b}x\Phi^{(n)}(0, b; x-\bar{\mu}, \bar{\sigma}^2)|_{x_1}^{x_2} - \frac{p}{a}x\Phi^{(n)}(a, 0; x-\bar{\mu}, \bar{\sigma}^2)|_{x_1}^{x_2} \\
&+\frac{\bar{\sigma}}{\sqrt{2\pi}}\Bigg(\frac{q}{b}(e^{-\frac{(x_1-(\bar{\mu}+b))^2}{2\bar{\sigma}^2}} - e^{-\frac{(x_2-(\bar{\mu}+b))^2}{2\bar{\sigma}^2}}) + (\frac{q}{b}+\frac{p}{a})(e^{-\frac{(x_2-\bar{\mu})^2}{2\bar{\sigma}^2}} - e^{-\frac{(x_1-\bar{\mu})^2}{2\bar{\sigma}^2}}) \\
&+\frac{p}{a}(e^{-\frac{(x_1-(\bar{\mu}+a))^2}{2\bar{\sigma}^2}} - e^{-\frac{(x_2-(\bar{\mu}+a))^2}{2\bar{\sigma}^2}})\Bigg) \\
&+\frac{p}{a}(\bar{\mu}+a)\Phi^{(n)}(x_1, x_2; \bar{\mu}+a, \bar{\sigma}^2) + \frac{q}{b}(\bar{\mu}+b)\Phi^{(n)}(x_1, x_2; \bar{\mu}+b, \bar{\sigma}^2) \\
&-(\frac{q}{b}+\frac{p}{a})\bar{\mu}\Phi^{(n)}(x_1, x_2; \bar{\mu}, \bar{\sigma}^2),
\end{aligned}$$

$$\begin{aligned}
\Phi^{(2)}(x_1, x_2) \;=\;& 0.5\bar{\sigma}^2\Bigg((\frac{p}{a}+\frac{q}{b})^2\Phi(x_1, x_2; \bar{\mu}, \bar{\sigma}^2) + (\frac{p}{a})^2\Phi(x_1, x_2; \bar{\mu}+2a, \bar{\sigma}^2) \\
&+(\frac{q}{b})^2\Phi(x_1, x_2; \bar{\mu}+2b, \bar{\sigma}^2) - 2\left((\frac{p}{a})^2 + \frac{pq}{ab}\right)\Phi(x_1, x_2; \bar{\mu}+a, \bar{\sigma}^2) \\
&-2\left((\frac{q}{b})^2 + \frac{pq}{ab})\right)\Phi(x_1, x_2; \bar{\mu}+b, \bar{\sigma}^2) + \frac{2pq}{ab}\Phi(x_1, x_2; \bar{\mu}+a+b, \bar{\sigma}^2)\Bigg) \\
&+0.5\frac{\bar{\sigma}}{\sqrt{2\pi}}\Bigg((\frac{p}{a})^2\left(ze^{-\frac{z^2}{2\bar{\sigma}^2}}|_{x_1-(\bar{\mu}+2a)}^{x_2-(\bar{\mu}+2a)} - 2ze^{-\frac{z^2}{2\bar{\sigma}^2}}|_{x_1-(\bar{\mu}+a)}^{x_2-(\bar{\mu}+a)} + ze^{-\frac{z^2}{2\bar{\sigma}^2}}|_{x_1-\bar{\mu}}^{x_2-\bar{\mu}}\right) \\
&+(\frac{2pq}{ab})\left(ze^{-\frac{z^2}{2\bar{\sigma}^2}}|_{x_1-(\bar{\mu}+a+b)}^{x_2-(\bar{\mu}+a+b)} - ze^{-\frac{z^2}{2\bar{\sigma}^2}}|_{x_1-a-\bar{\mu}}^{x_2-a-\bar{\mu}} + ze^{-\frac{z^2}{2\bar{\sigma}^2}}|_{x_1-\bar{\mu}}^{x_2-\bar{\mu}} - ze^{-\frac{z^2}{2\bar{\sigma}^2}}|_{x_1-b-\bar{\mu}}^{x_2-b-\bar{\mu}}\right) \\
&+(\frac{q}{b})^2\left(ze^{-\frac{z^2}{2\bar{\sigma}^2}}|_{x_1-(\bar{\mu}+2b)}^{x_2-(\bar{\mu}+2b)} - 2ze^{-\frac{z^2}{2\bar{\sigma}^2}}|_{x_1-(\bar{\mu}+b)}^{x_2-(\bar{\mu}+b)} + ze^{-\frac{z^2}{2\bar{\sigma}^2}}|_{x_1-\bar{\mu}}^{x_2-\bar{\mu}}\right)\Bigg) \\
&+0.5\Bigg((\frac{p}{a})^2(z^2\Phi^{(n)}(0, -a; z, \bar{\sigma}^2)|_{x_1-(\bar{\mu}+2a)}^{x_2-(\bar{\mu}+2a)} - z^2\Phi^{(n)}(a, 0; z, \bar{\sigma}^2)|_{x_1-\bar{\mu}}^{x_2-\bar{\mu}}) \\
&+\frac{2pq}{ab}z^2\left(\Phi^{(n)}(0, b; z, \bar{\sigma}^2) - \Phi^{(n)}(a, a+b; z, \bar{\sigma}^2)\right)|_{x_1-\bar{\mu}}^{x_2-\bar{\mu}} \\
&+(\frac{q}{b})^2(z^2\Phi^{(n)}(0, b; z, \bar{\sigma}^2)|_{x_1-\bar{\mu}}^{x_2-\bar{\mu}} - z^2\Phi^{(n)}(-b, 0; z, \bar{\sigma}^2)|_{x_1-2b-\bar{\mu}}^{x_2-2b-\bar{\mu}})\Bigg) \\
&+p^2\Phi^{(n)}(x_1, x_2; \bar{\mu}+a, \bar{\sigma}^2) + q^2\Phi^{(n)}(x_1, x_2; \bar{\mu}+b, \bar{\sigma}^2) + 2pq\Phi^{(n)}(x_1, x_2; \bar{\mu}+a+b, \bar{\sigma}^2) \\
&+\frac{pq}{ab}\Bigg((2a(z-\bar{\mu}) - a^2)\Phi^{(n)}(a, a+b; z-\bar{\mu}, \bar{\sigma}^2) \\
&-(2b(z-\bar{\mu}) - b^2)\Phi^{(n)}(a+b, b; z-\bar{\mu}, \bar{\sigma}^2)\Bigg)|_{x_1}^{x_2}.
\end{aligned}$$

Remark: It is important that the formula be properly arranged to avoid the catastrophe cancellation in the computation.

Proof:

$$
\begin{aligned}
\Phi^{(1)}(x_1,x_2) &\equiv \int_{x_1}^{x_2} \phi^{(1)}(x)dx \\
&\overset{Theorem}{\underset{4.1}{\equiv}} \int_{x_1}^{x_2} (\frac{q}{b}\Phi^{(n)}(0,b;x-\bar{\mu},\bar{\sigma}^2) - \frac{p}{a}\Phi^{(n)}(a,0;x-\bar{\mu},\bar{\sigma}^2))dx \\
&= \frac{q}{b}\int_{x_1}^{x_2} \Phi^{(n)}(0,b;x-\bar{\mu},\bar{\sigma}^2)dx - \frac{p}{a}\int_{x_1}^{x_2} \Phi^{(n)}(a,0;x-\bar{\mu},\bar{\sigma}^2)dx \\
&\overset{Lemma}{\underset{5.1}{\equiv}} \frac{q}{b}\Bigg(x\Phi^{(n)}(0,b;x-\bar{\mu},\bar{\sigma}^2)|_{x_1}^{x_2} \\
&\quad + \frac{\bar{\sigma}}{\sqrt{2\pi}}(e^{-\frac{(x_1-(\bar{\mu}+b))^2}{2\bar{\sigma}^2}} + e^{-\frac{(x_2-\bar{\mu})^2}{2\bar{\sigma}^2}} - e^{-\frac{(x_1-\bar{\mu})^2}{2\bar{\sigma}^2}} - e^{-\frac{(x_2-(\bar{\mu}+b))^2}{2\bar{\sigma}^2}}) \\
&\quad +(\bar{\mu}+b)\Phi^{(n)}(x_1,x_2;\bar{\mu}+b,\bar{\sigma}^2) - \bar{\mu}\Phi^{(n)}(x_1,x_2;\bar{\mu},\bar{\sigma}^2)\Bigg) \\
&\quad -\frac{p}{a}\Bigg(x\Phi^{(n)}(a,0;x-\bar{\mu},\bar{\sigma}^2)|_{x_1}^{x_2} \\
&\quad + \frac{\bar{\sigma}}{\sqrt{2\pi}}(e^{-\frac{(x_1-\bar{\mu})^2}{2\bar{\sigma}^2}} + e^{-\frac{(x_2-(\bar{\mu}+a))^2}{2\bar{\sigma}^2}} - e^{-\frac{(x_1-(\bar{\mu}+a))^2}{2\bar{\sigma}^2}} - e^{-\frac{(x_2-\bar{\mu})^2}{2\bar{\sigma}^2}}) \\
&\quad +\bar{\mu}\Phi^{(n)}(x_1,x_2;\bar{\mu},\bar{\sigma}^2) - (\bar{\mu}+a)\Phi^{(n)}(x_1,x_2;\bar{\mu}+a,\bar{\sigma}^2)\Bigg) \\
&= \frac{q}{b}x\Phi^{(n)}(0,b;x-\bar{\mu},\bar{\sigma}^2)|_{x_1}^{x_2} - \frac{p}{a}x\Phi^{(n)}(a,0;x-\bar{\mu},\bar{\sigma}^2)|_{x_1}^{x_2} \\
&\quad + \frac{\bar{\sigma}}{\sqrt{2\pi}}\Bigg(\frac{q}{b}(e^{-\frac{(x_1-(\bar{\mu}+b))^2}{2\bar{\sigma}^2}} - e^{-\frac{(x_2-(\bar{\mu}+b))^2}{2\bar{\sigma}^2}}) + (\frac{q}{b}+\frac{p}{a})(e^{-\frac{(x_2-\bar{\mu})^2}{2\bar{\sigma}^2}} - e^{-\frac{(x_1-\bar{\mu})^2}{2\bar{\sigma}^2}}) \\
&\quad + \frac{p}{a}(e^{-\frac{(x_1-(\bar{\mu}+a))^2}{2\bar{\sigma}^2}} - e^{-\frac{(x_2-(\bar{\mu}+a))^2}{2\bar{\sigma}^2}})\Bigg) \\
&\quad + \frac{p}{a}(\bar{\mu}+a)\Phi^{(n)}(x_1,x_2;\bar{\mu}+a,\bar{\sigma}^2) + \frac{q}{b}(\bar{\mu}+b)\Phi^{(n)}(x_1,x_2;\bar{\mu}+b,\bar{\sigma}^2) \\
&\quad -(\frac{q}{b}+\frac{p}{a})\bar{\mu}\Phi^{(n)}(x_1,x_2;\bar{\mu},\bar{\sigma}^2).
\end{aligned}
$$

$$
\begin{aligned}
\Phi^{(2)}(x_1,x_2) &\equiv \int_{x_1}^{x_2}\phi^{(2)}(x)dx \\
&\overset{Theorem}{\underset{4.1}{=}} \int_{x_1}^{x_2}\Bigg(\frac{\bar\sigma}{\sqrt{2\pi}}\Bigg((\frac{p}{a}+\frac{q}{b})^2 e^{-\frac{(x-\bar\mu)^2}{2\bar\sigma^2}} + (\frac{p}{a})^2 e^{-\frac{(x-\bar\mu-2a)^2}{2\bar\sigma^2}} + (\frac{q}{b})^2 e^{-\frac{(x-\bar\mu-2b)^2}{2\bar\sigma^2}} \\
&\quad -2\left((\frac{p}{a})^2+\frac{pq}{ab}\right)e^{-\frac{(x-\bar\mu-a)^2}{2\bar\sigma^2}} - 2\left((\frac{q}{b})^2+\frac{pq}{ab})\right)e^{-\frac{(x-\bar\mu-b)^2}{2\bar\sigma^2}} + \frac{2pq}{ab}e^{-\frac{(x-\bar\mu-a-b)^2}{2\bar\sigma^2}}\Bigg) \\
&\quad +(\frac{p}{a})^2\left((x-2a-\bar\mu)\Phi^{(n)}(2a,a;x-\bar\mu,\bar\sigma^2)-(x-\bar\mu)\Phi^{(n)}(a,0;x-\bar\mu,\bar\sigma^2)\right) \\
&\quad +\frac{2pq}{ab}\Bigg((x-\bar\mu)\left(\Phi^{(n)}(0,b;x-\bar\mu,\bar\sigma^2)-\Phi^{(n)}(a,a+b;x-\bar\mu,\bar\sigma^2)\right) \\
&\quad +a\Phi^{(n)}(a,a+b;x-\bar\mu,\bar\sigma^2)-b\Phi^{(n)}(a+b,b;x-\bar\mu,\bar\sigma^2)\Bigg) \\
&\quad +(\frac{q}{b})^2\left((x-\bar\mu)\Phi^{(n)}(0,b;x-\bar\mu,\bar\sigma^2)-(x-2b-\bar\mu)\Phi^{(n)}(b,2b;x-\bar\mu,\bar\sigma^2)\right)\Bigg)dx \\
&= \bar\sigma^2\Bigg((\frac{p}{a}+\frac{q}{b})^2\Phi(x_1,x_2;\bar\mu,\bar\sigma^2)+(\frac{p}{a})^2\Phi(x_1,x_2;\bar\mu+2a,\bar\sigma^2) \\
&\quad +(\frac{q}{b})^2\Phi(x_1,x_2;\bar\mu+2b,\bar\sigma^2)-2\left((\frac{p}{a})^2+\frac{pq}{ab}\right)\Phi(x_1,x_2;\bar\mu+a,\bar\sigma^2) \\
&\quad -2\left((\frac{q}{b})^2+\frac{pq}{ab})\right)\Phi(x_1,x_2;\bar\mu+b,\bar\sigma^2)+\frac{2pq}{ab}\Phi(x_1,x_2;\bar\mu+a+b,\bar\sigma^2)\Bigg) \\
&\quad +(\frac{p}{a})^2\int_{x_1}^{x_2}\left((x-2a-\bar\mu)\Phi^{(n)}(2a,a;x-\bar\mu,\bar\sigma^2)-(x-\bar\mu)\Phi^{(n)}(a,0;x-\bar\mu,\bar\sigma^2)\right)dx \\
&\quad +\frac{2pq}{ab}\int_{x_1}^{x_2}\Bigg((x-\bar\mu)\left(\Phi^{(n)}(0,b;x-\bar\mu,\bar\sigma^2)-\Phi^{(n)}(a,a+b;x-\bar\mu,\bar\sigma^2)\right) \\
&\quad +a\Phi^{(n)}(a,a+b;x-\bar\mu,\bar\sigma^2)-b\Phi^{(n)}(a+b,b;x-\bar\mu,\bar\sigma^2)\Bigg)dx \\
&\quad +(\frac{q}{b})^2\int_{x_1}^{x_2}\left((x-\bar\mu)\Phi^{(n)}(0,b;x-\bar\mu,\bar\sigma^2)-(x-2b-\bar\mu)\Phi^{(n)}(b,2b;x-\bar\mu,\bar\sigma^2)\right)dx
\end{aligned}
$$

But,

$$
\begin{aligned}
IA &\equiv \int_{x_1}^{x_2}\left((x-2a-\bar\mu)\Phi^{(n)}(2a,a;x-\bar\mu,\bar\sigma^2)-(x-\bar\mu)\Phi^{(n)}(a,0;x-\bar\mu,\bar\sigma^2)\right)dx \\
&\overset{Lemma}{\underset{4.1}{=}} \int_{x_1}^{x_2}\left((x-2a-\bar\mu)\Phi^{(n)}(0,-a;x-\bar\mu-2a,\bar\sigma^2)-(x-\bar\mu)\Phi^{(n)}(a,0;x-\bar\mu,\bar\sigma^2)\right)dx \\
&= \int_{x_1}^{x_2}(x-2a-\bar\mu)\Phi^{(n)}(0,-a;x-\bar\mu-2a,\bar\sigma^2)dx-\int_{x_1}^{x_2}(x-\bar\mu)\Phi^{(n)}(a,0;x-\bar\mu,\bar\sigma^2)dx
\end{aligned}
$$

$$
\begin{aligned}
\overset{Corollary}{\underset{5.1}{=}} \quad & \frac{-\bar{\sigma}}{2\sqrt{2\pi}}\left((z-2a)e^{-\frac{z^2}{2\bar{\sigma}^2}}\Big|_{x_1-(\bar{\mu}+a)}^{x_2-(\bar{\mu}+a)} - ze^{-\frac{z^2}{2\bar{\sigma}^2}}\Big|_{x_1-(\bar{\mu}+2a)}^{x_2-(\bar{\mu}+2a)}\right)\\
& +0.5\Bigg((\bar{\sigma}^2+a^2)\Phi^{(n)}(x_1,x_2;\bar{\mu}+a,\bar{\sigma}^2)\\
& -\bar{\sigma}^2\Phi^{(n)}(x_1,x_2;\bar{\mu}+2a,\bar{\sigma}^2) + z^2\Phi^{(n)}(0,-a;z,\bar{\sigma}^2)\Big|_{x_1-(\bar{\mu}+2a)}^{x_2-(\bar{\mu}+2a)}\Bigg)\\
& -\frac{-\bar{\sigma}}{2\sqrt{2\pi}}\left(ze^{-\frac{z^2}{2\bar{\sigma}^2}}\Big|_{x_1-\bar{\mu}}^{x_2-\bar{\mu}} - (z+2a)e^{-\frac{z^2}{2\bar{\sigma}^2}}\Big|_{x_1-(\bar{\mu}+a)}^{x_2-(\bar{\mu}+a)}\right)\\
& -0.5\Bigg(\bar{\sigma}^2\Phi^{(n)}(x_1,x_2;\bar{\mu},\bar{\sigma}^2)\\
& -(\bar{\sigma}^2+a^2)\Phi^{(n)}(x_1,x_2;\bar{\mu}+a,\bar{\sigma}^2) + z^2\Phi^{(n)}(a,0;z,\bar{\sigma}^2)\Big|_{x_1-\bar{\mu}}^{x_2-\bar{\mu}}\Bigg)\\
= \quad & \frac{\bar{\sigma}}{2\sqrt{2\pi}}\left(ze^{-\frac{z^2}{2\bar{\sigma}^2}}\Big|_{x_1-\bar{\mu}}^{x_2-\bar{\mu}} - 2ze^{-\frac{z^2}{2\bar{\sigma}^2}}\Big|_{x_1-(\bar{\mu}+a)}^{x_2-(\bar{\mu}+a)} + ze^{-\frac{z^2}{2\bar{\sigma}^2}}\Big|_{x_1-(\bar{\mu}+2a)}^{x_2-(\bar{\mu}+2a)}\right)\\
& +0.5\left(z^2\Phi^{(n)}(0,-a;z,\bar{\sigma}^2)\Big|_{x_1-(\bar{\mu}+2a)}^{x_2-(\bar{\mu}+2a)} - z^2\Phi^{(n)}(a,0;z,\bar{\sigma}^2)\Big|_{x_1-\bar{\mu}}^{x_2-\bar{\mu}}\right)\\
& -0.5\bar{\sigma}^2\left(\Phi^{(n)}(x_1,x_2;\bar{\mu},\bar{\sigma}^2) + \Phi^{(n)}(x_1,x_2;\bar{\mu}+2a,\bar{\sigma}^2)\right)\\
& +(\bar{\sigma}^2+a^2)\Phi^{(n)}(x_1,x_2;\bar{\mu}+a,\bar{\sigma}^2).
\end{aligned}
$$

$$
\begin{aligned}
IB \quad &\equiv \quad \int_{x_1}^{x_2}\Bigg((x-\bar{\mu})\left(\Phi^{(n)}(0,b;x-\bar{\mu},\bar{\sigma}^2)-\Phi^{(n)}(a,a+b;x-\bar{\mu},\bar{\sigma}^2)\right)\\
&\quad +a\Phi^{(n)}(a,a+b;x-\bar{\mu},\bar{\sigma}^2)-b\Phi^{(n)}(a+b,b;x-\bar{\mu},\bar{\sigma}^2)\Bigg)dx\\
&= \int_{x_1}^{x_2}(x-\bar{\mu})\Phi^{(n)}(0,b;x-\bar{\mu},\bar{\sigma}^2)dx-\int_{x_1}^{x_2}(x-\bar{\mu})\Phi^{(n)}(a,a+b;x-\bar{\mu},\bar{\sigma}^2)dx\\
&\quad +a\int_{x_1}^{x_2}\Phi^{(n)}(a,a+b;x-\bar{\mu},\bar{\sigma}^2)dx-b\int_{x_1}^{x_2}\Phi^{(n)}(a+b,b;x-\bar{\mu},\bar{\sigma}^2)dx\\
&\overset{Set}{\underset{z=x-\bar{\mu}}{=}} \int_{x_1-\bar{\mu}}^{x_2-\bar{\mu}}z\Phi^{(n)}(0,b;z,\bar{\sigma}^2)dz-\int_{x_1-\bar{\mu}}^{x_2-\bar{\mu}}z\Phi^{(n)}(a,a+b;z,\bar{\sigma}^2)dz\\
&\quad +a\int_{x_1}^{x_2}\Phi^{(n)}(a,a+b;x-\bar{\mu},\bar{\sigma}^2)dx-b\int_{x_1}^{x_2}\Phi^{(n)}(a+b,b;x-\bar{\mu},\bar{\sigma}^2)dx\\
&\overset{Lemma}{\underset{5.3,\ 5.1}{=}} -\frac{\bar{\sigma}}{2\sqrt{2\pi}}\left((z+2b)e^{-\frac{z^2}{2\bar{\sigma}^2}}\Big|_{x_1-(\bar{\mu}+b)}^{x_2-(\bar{\mu}+b)}-ze^{-\frac{z^2}{2\bar{\sigma}^2}}\Big|_{x_1-\bar{\mu}}^{x_2-\bar{\mu}}\right)\\
&\quad +0.5\Bigg((\bar{\sigma}^2+b^2)\Phi^{(n)}(x_1,x_2;\bar{\mu}+b,\bar{\sigma}^2)\\
&\quad -\bar{\sigma}^2\Phi^{(n)}(x_1,x_2;\bar{\mu},\bar{\sigma}^2)+z^2\Phi^{(n)}(0,b;z,\bar{\sigma}^2)\Big|_{x_1-\bar{\mu}}^{x_2-\bar{\mu}}\Bigg)\\
&\quad +\frac{\bar{\sigma}}{2\sqrt{2\pi}}\left((z+2(a+b))e^{-\frac{z^2}{2\bar{\sigma}^2}}\Big|_{x_1-(\bar{\mu}+a+b)}^{x_2-(\bar{\mu}+a+b)}-(z+2a)e^{-\frac{z^2}{2\bar{\sigma}^2}}\Big|_{x_1-(\bar{\mu}+a)}^{x_2-(\bar{\mu}+a)}\right)\\
&\quad -0.5\Bigg((\bar{\sigma}^2+(a+b)^2)\Phi^{(n)}(x_1,x_2;\bar{\mu}+a+b,\bar{\sigma}^2)\\
&\quad -(\bar{\sigma}^2+a^2)\Phi^{(n)}(x_1,x_2;\bar{\mu}+a,\bar{\sigma}^2)+z^2\Phi^{(n)}(a,a+b;z,\bar{\sigma}^2)\Big|_{x_1-\bar{\mu}}^{x_2-\bar{\mu}}\Bigg)\\
&\quad +a\Bigg(z\Phi^{(n)}(a,a+b;z-\bar{\mu},\bar{\sigma}^2)\Big|_{x_1}^{x_2}\\
&\quad +\frac{\bar{\sigma}}{\sqrt{2\pi}}(e^{-\frac{(x_1-(\bar{\mu}+a+b))^2}{2\bar{\sigma}^2}}+e^{-\frac{(x_2-(\bar{\mu}+a))^2}{2\bar{\sigma}^2}}-e^{-\frac{(x_1-(\bar{\mu}+a))^2}{2\bar{\sigma}^2}}-e^{-\frac{(x_2-(\bar{\mu}+a+b))^2}{2\bar{\sigma}^2}})\\
&\quad +(\bar{\mu}+a+b)\Phi^{(n)}(x_1,x_2;\bar{\mu}+a+b,\bar{\sigma}^2)-(\bar{\mu}+a)\Phi^{(n)}(x_1,x_2;\bar{\mu}+a,\bar{\sigma}^2)\Bigg)\\
&\quad -b\Bigg(z\Phi^{(n)}(a+b,b;x-\bar{\mu},\bar{\sigma}^2)\Big|_{x_1}^{x_2}\\
&\quad +\frac{\bar{\sigma}}{\sqrt{2\pi}}(e^{-\frac{(x_1-(\bar{\mu}+b))^2}{2\bar{\sigma}^2}}+e^{-\frac{(x_2-(\bar{\mu}+a+b))^2}{2\bar{\sigma}^2}}-e^{-\frac{(x_1-(\bar{\mu}+a+b))^2}{2\bar{\sigma}^2}}-e^{-\frac{(x_2-(\bar{\mu}+b))^2}{2\bar{\sigma}^2}})\\
&\quad +(\bar{\mu}+b)\Phi^{(n)}(x_1,x_2;\bar{\mu}+b,\bar{\sigma}^2)-(\bar{\mu}+a+b)\Phi^{(n)}(x_1,x_2;\bar{\mu}+a+b,\bar{\sigma}^2)\Bigg)
\end{aligned}
$$

That is,

$$
\begin{aligned}
IB &= \frac{\bar{\sigma}}{2\sqrt{2\pi}}\left(ze^{-\frac{z^2}{2\bar{\sigma}^2}}|_{x_1-(\bar{\mu}+a+b)}^{x_2-(\bar{\mu}+a+b)} - ze^{-\frac{z^2}{2\bar{\sigma}^2}}|_{x_1-(\bar{\mu}+a)}^{x_2-(\bar{\mu}+a)} - ze^{-\frac{z^2}{2\bar{\sigma}^2}}|_{x_1-(\bar{\mu}+b)}^{x_2-(\bar{\mu}+b)} + ze^{-\frac{z^2}{2\bar{\sigma}^2}}|_{x_1-\bar{\mu}}^{x_2-\bar{\mu}}\right) \\
&\quad +0.5z^2\left(\Phi^{(n)}(0,b;z,\bar{\sigma}^2) - \Phi^{(n)}(a,a+b;z,\bar{\sigma}^2)\right)|_{x_1-\bar{\mu}}^{x_2-\bar{\mu}} \\
&\quad +z\left(a\Phi^{(n)}(a,a+b;z-\bar{\mu},\bar{\sigma}^2) - b\Phi^{(n)}(a+b,b;z-\bar{\mu},\bar{\sigma}^2)\right)|_{x_1}^{x_2} \\
&\quad +\bar{\mu}\left((a+b)\Phi^{(n)}(x_1,x_2;\bar{\mu}+a+b,\bar{\sigma}^2) - a\Phi^{(n)}(x_1,x_2;\bar{\mu}+a,\bar{\sigma}^2) - b\Phi^{(n)}(x_1,x_2;\bar{\mu}+b,\bar{\sigma}^2)\right) \\
&\quad -0.5\left(a^2\Phi^{(n)}(x_1,x_2;\bar{\mu}+a,\bar{\sigma}^2) - (a+b)^2\Phi^{(n)}(x_1,x_2;\bar{\mu}+a+b,\bar{\sigma}^2) + b^2\Phi^{(n)}(x_1,x_2;\bar{\mu}+b,\bar{\sigma}^2)\right) \\
&\quad +0.5\bar{\sigma}^2\Bigg(\Phi^{(n)}(x_1,x_2;\bar{\mu}+b,\bar{\sigma}^2) - \Phi^{(n)}(x_1,x_2;\bar{\mu},\bar{\sigma}^2) \\
&\quad +\Phi^{(n)}(x_1,x_2;\bar{\mu}+a,\bar{\sigma}^2) - \Phi^{(n)}(x_1,x_2;\bar{\mu}+a+b,\bar{\sigma}^2)\Bigg).
\end{aligned}
$$

$$
\begin{aligned}
IC &\equiv \int_{x_1}^{x_2}\left((x-\bar{\mu})\Phi^{(n)}(0,b;x-\bar{\mu},\bar{\sigma}^2) - (x-2b-\bar{\mu})\Phi^{(n)}(b,2b;x-\bar{\mu},\bar{\sigma}^2)\right)dx \\
&\overset{Lemma}{\underset{4.1}{=}} \int_{x_1}^{x_2}\left((x-\bar{\mu})\Phi^{(n)}(0,b;x-\bar{\mu},\bar{\sigma}^2) - (x-2b-\bar{\mu})\Phi^{(n)}(-b,0;x-2b-\bar{\mu},\bar{\sigma}^2)\right)dx \\
&\overset{Corollary}{\underset{5.1}{=}} \frac{-\bar{\sigma}}{2\sqrt{2\pi}}\left((z+2b)e^{-\frac{z^2}{2\bar{\sigma}^2}}|_{x_1-(\bar{\mu}+b)}^{x_2-(\bar{\mu}+b)} - ze^{-\frac{z^2}{2\bar{\sigma}^2}}|_{x_1-\bar{\mu}}^{x_2-\bar{\mu}}\right) \\
&\quad +0.5\Bigg((\bar{\sigma}^2+b^2)\Phi^{(n)}(x_1,x_2;\bar{\mu}+b,\bar{\sigma}^2) \\
&\quad -\bar{\sigma}^2\Phi^{(n)}(x_1,x_2;\bar{\mu},\bar{\sigma}^2) + z^2\Phi^{(n)}(0,b;z,\bar{\sigma}^2)|_{x_1-\bar{\mu}}^{x_2-\bar{\mu}}\Bigg) \\
&\quad -\frac{-\bar{\sigma}}{2\sqrt{2\pi}}\left(ze^{-\frac{z^2}{2\bar{\sigma}^2}}|_{x_1-(\bar{\mu}+2b)}^{x_2-(\bar{\mu}+2b)} - (z-2b)e^{-\frac{z^2}{2\bar{\sigma}^2}}|_{x_1-(\bar{\mu}+b)}^{x_2-(\bar{\mu}+b)}\right) \\
&\quad -0.5\Bigg(\bar{\sigma}^2\Phi^{(n)}(x_1,x_2;\bar{\mu}+2b,\bar{\sigma}^2) \\
&\quad -(\bar{\sigma}^2+b^2)\Phi^{(n)}(x_1,x_2;\bar{\mu}+b,\bar{\sigma}^2) + z^2\Phi^{(n)}(-b,0;z,\bar{\sigma}^2)|_{x_1-2b-\bar{\mu}}^{x_2-2b-\bar{\mu}}\Bigg). \\
&= \frac{\bar{\sigma}}{2\sqrt{2\pi}}\left(ze^{-\frac{z^2}{2\bar{\sigma}^2}}|_{x_1-(\bar{\mu}+2b)}^{x_2-(\bar{\mu}+2b)} - 2ze^{-\frac{z^2}{2\bar{\sigma}^2}}|_{x_1-(\bar{\mu}+b)}^{x_2-(\bar{\mu}+b)} + ze^{-\frac{z^2}{2\bar{\sigma}^2}}|_{x_1-\bar{\mu}}^{x_2-\bar{\mu}}\right) \\
&\quad +0.5\left(z^2\Phi^{(n)}(0,b;z,\bar{\sigma}^2)|_{x_1-\bar{\mu}}^{x_2-\bar{\mu}} - z^2\Phi^{(n)}(-b,0;z,\bar{\sigma}^2)|_{x_1-2b-\bar{\mu}}^{x_2-2b-\bar{\mu}}\right) \\
&\quad -0.5\bar{\sigma}^2\left(\Phi^{(n)}(x_1,x_2;\bar{\mu},\bar{\sigma}^2) + \Phi^{(n)}(x_1,x_2;\bar{\mu}+2b,\bar{\sigma}^2)\right) \\
&\quad +(\bar{\sigma}^2+b^2)\Phi^{(n)}(x_1,x_2;\bar{\mu}+b,\bar{\sigma}^2).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\Phi^{(2)}(x_1,x_2) &= \bar\sigma^2\Bigg((\frac{p}{a}+\frac{q}{b})^2\Phi(x_1,x_2;\bar\mu,\bar\sigma^2)+(\frac{p}{a})^2\Phi(x_1,x_2;\bar\mu+2a,\bar\sigma^2)\\
&\quad+(\frac{q}{b})^2\Phi(x_1,x_2;\bar\mu+2b,\bar\sigma^2)-2\left((\frac{p}{a})^2+\frac{pq}{ab}\right)\Phi(x_1,x_2;\bar\mu+a,\bar\sigma^2)\\
&\quad-2\left((\frac{q}{b})^2+\frac{pq}{ab})\right)\Phi(x_1,x_2;\bar\mu+b,\bar\sigma^2)+\frac{2pq}{ab}\Phi(x_1,x_2;\bar\mu+a+b,\bar\sigma^2)\Bigg)\\
&\quad+(\frac{p}{a})^2IA+\frac{2pq}{ab}IB+(\frac{q}{b})^2IC\\
&= 0.5\bar\sigma^2\Bigg((\frac{p}{a}+\frac{q}{b})^2\Phi(x_1,x_2;\bar\mu,\bar\sigma^2)+(\frac{p}{a})^2\Phi(x_1,x_2;\bar\mu+2a,\bar\sigma^2)\\
&\quad+(\frac{q}{b})^2\Phi(x_1,x_2;\bar\mu+2b,\bar\sigma^2)-2\left((\frac{p}{a})^2+\frac{pq}{ab}\right)\Phi(x_1,x_2;\bar\mu+a,\bar\sigma^2)\\
&\quad-2\left((\frac{q}{b})^2+\frac{pq}{ab})\right)\Phi(x_1,x_2;\bar\mu+b,\bar\sigma^2)+\frac{2pq}{ab}\Phi(x_1,x_2;\bar\mu+a+b,\bar\sigma^2)\Bigg)\\
&\quad+0.5\frac{\bar\sigma}{\sqrt{2\pi}}\Bigg((\frac{p}{a})^2\left(ze^{-\frac{z^2}{2\bar\sigma^2}}|_{x_1-(\bar\mu+2a)}^{x_2-(\bar\mu+2a)}-2ze^{-\frac{z^2}{2\bar\sigma^2}}|_{x_1-(\bar\mu+a)}^{x_2-(\bar\mu+a)}+ze^{-\frac{z^2}{2\bar\sigma^2}}|_{x_1-\bar\mu}^{x_2-\bar\mu}\right)\\
&\quad+(\frac{2pq}{ab})\left(ze^{-\frac{z^2}{2\bar\sigma^2}}|_{x_1-(\bar\mu+a+b)}^{x_2-(\bar\mu+a+b)}-ze^{-\frac{z^2}{2\bar\sigma^2}}|_{x_1-a-\bar\mu}^{x_2-a-\bar\mu}+ze^{-\frac{z^2}{2\bar\sigma^2}}|_{x_1-\bar\mu}^{x_2-\bar\mu}-ze^{-\frac{z^2}{2\bar\sigma^2}}|_{x_1-b-\bar\mu}^{x_2-b-\bar\mu}\right)\\
&\quad+(\frac{q}{b})^2\left(ze^{-\frac{z^2}{2\bar\sigma^2}}|_{x_1-(\bar\mu+2b)}^{x_2-(\bar\mu+2b)}-2ze^{-\frac{z^2}{2\bar\sigma^2}}|_{x_1-(\bar\mu+b)}^{x_2-(\bar\mu+b)}+ze^{-\frac{z^2}{2\bar\sigma^2}}|_{x_1-\bar\mu}^{x_2-\bar\mu}\right)\Bigg)\\
&\quad+0.5\Bigg((\frac{p}{a})^2(z^2\Phi^{(n)}(0,-a;z,\bar\sigma^2)|_{x_1-(\bar\mu+2a)}^{x_2-(\bar\mu+2a)}-z^2\Phi^{(n)}(a,0;z,\bar\sigma^2)|_{x_1-\bar\mu}^{x_2-\bar\mu})\\
&\quad+\frac{2pq}{ab}z^2\left(\Phi^{(n)}(0,b;z,\bar\sigma^2)-\Phi^{(n)}(a,a+b;z,\bar\sigma^2)\right)|_{x_1-\bar\mu}^{x_2-\bar\mu}\\
&\quad+(\frac{q}{b})^2(z^2\Phi^{(n)}(0,b;z,\bar\sigma^2)|_{x_1-\bar\mu}^{x_2-\bar\mu}-z^2\Phi^{(n)}(-b,0;z,\bar\sigma^2)|_{x_1-2b-\bar\mu}^{x_2-2b-\bar\mu})\Bigg)\\
&\quad+p^2\Phi^{(n)}(x_1,x_2;\bar\mu+a,\bar\sigma^2)+q^2\Phi^{(n)}(x_1,x_2;\bar\mu+b,\bar\sigma^2)+2pq\Phi^{(n)}(x_1,x_2;\bar\mu+a+b,\bar\sigma^2)\\
&\quad+\frac{2pq}{ab}\Bigg(0.5\Bigg((a^2+2a\bar\mu)\left(\Phi^{(n)}(x_1,x_2;\bar\mu+a+b,\bar\sigma^2)-\Phi^{(n)}(x_1,x_2;\bar\mu+a,\bar\sigma^2)\right)\\
&\quad+(b^2+2b\bar\mu)\left(\Phi^{(n)}(x_1,x_2;\bar\mu+a+b,\bar\sigma^2)-\Phi^{(n)}(x_1,x_2;\bar\mu+b,\bar\sigma^2)\right)\Bigg)\\
&\quad+z\left(a\Phi^{(n)}(a,a+b;z-\bar\mu,\bar\sigma^2)-b\Phi^{(n)}(a+b,b;z-\bar\mu,\bar\sigma^2)\right)|_{x_1}^{x_2}\Bigg).
\end{aligned}
$$

$$
\begin{aligned}
\overset{Lemma}{\underset{5.4}{=}} \quad & 0.5\bar\sigma^2\Bigg((\frac{p}{a}+\frac{q}{b})^2\Phi(x_1,x_2;\bar\mu,\bar\sigma^2)+(\frac{p}{a})^2\Phi(x_1,x_2;\bar\mu+2a,\bar\sigma^2)\\
&+(\frac{q}{b})^2\Phi(x_1,x_2;\bar\mu+2b,\bar\sigma^2)-2\left((\frac{p}{a})^2+\frac{pq}{ab}\right)\Phi(x_1,x_2;\bar\mu+a,\bar\sigma^2)\\
&-2\left((\frac{q}{b})^2+\frac{pq}{ab})\right)\Phi(x_1,x_2;\bar\mu+b,\bar\sigma^2)+\frac{2pq}{ab}\Phi(x_1,x_2;\bar\mu+a+b,\bar\sigma^2)\Bigg)\\
&+0.5\frac{\bar\sigma}{\sqrt{2\pi}}\Bigg((\frac{p}{a})^2\left(ze^{-\frac{z^2}{2\bar\sigma^2}}|_{x_1-(\bar\mu+2a)}^{x_2-(\bar\mu+2a)}-2ze^{-\frac{z^2}{2\bar\sigma^2}}|_{x_1-(\bar\mu+a)}^{x_2-(\bar\mu+a)}+ze^{-\frac{z^2}{2\bar\sigma^2}}|_{x_1-\bar\mu}^{x_2-\bar\mu}\right)\\
&+(\frac{2pq}{ab})\left(ze^{-\frac{z^2}{2\bar\sigma^2}}|_{x_1-(\bar\mu+a+b)}^{x_2-(\bar\mu+a+b)}-ze^{-\frac{z^2}{2\bar\sigma^2}}|_{x_1-a-\bar\mu}^{x_2-a-\bar\mu}+ze^{-\frac{z^2}{2\bar\sigma^2}}|_{x_1-\bar\mu}^{x_2-\bar\mu}-ze^{-\frac{z^2}{2\bar\sigma^2}}|_{x_1-b-\bar\mu}^{x_2-b-\bar\mu}\right)\\
&+(\frac{q}{b})^2\left(ze^{-\frac{z^2}{2\bar\sigma^2}}|_{x_1-(\bar\mu+2b)}^{x_2-(\bar\mu+2b)}-2ze^{-\frac{z^2}{2\bar\sigma^2}}|_{x_1-(\bar\mu+b)}^{x_2-(\bar\mu+b)}+ze^{-\frac{z^2}{2\bar\sigma^2}}|_{x_1-\bar\mu}^{x_2-\bar\mu}\right)\Bigg)\\
&+0.5\Bigg((\frac{p}{a})^2(z^2\Phi^{(n)}(0,-a;z,\bar\sigma^2)|_{x_1-(\bar\mu+2a)}^{x_2-(\bar\mu+2a)}-z^2\Phi^{(n)}(a,0;z,\bar\sigma^2)|_{x_1-\bar\mu}^{x_2-\bar\mu})\\
&+\frac{2pq}{ab}z^2\left(\Phi^{(n)}(0,b;z,\bar\sigma^2)-\Phi^{(n)}(a,a+b;z,\bar\sigma^2)\right)|_{x_1-\bar\mu}^{x_2-\bar\mu}\\
&+(\frac{q}{b})^2(z^2\Phi^{(n)}(0,b;z,\bar\sigma^2)|_{x_1-\bar\mu}^{x_2-\bar\mu}-z^2\Phi^{(n)}(-b,0;z,\bar\sigma^2)|_{x_1-2b-\bar\mu}^{x_2-2b-\bar\mu})\Bigg)\\
&+p^2\Phi^{(n)}(x_1,x_2;\bar\mu+a,\bar\sigma^2)+q^2\Phi^{(n)}(x_1,x_2;\bar\mu+b,\bar\sigma^2)+2pq\Phi^{(n)}(x_1,x_2;\bar\mu+a+b,\bar\sigma^2)\\
&+\frac{pq}{ab}\left((2a(z-\bar\mu)-a^2)\Phi^{(n)}(a,a+b;z-\bar\mu,\bar\sigma^2)-(2b(z-\bar\mu)-b^2)\Phi^{(n)}(a+b,b;z-\bar\mu,\bar\sigma^2)\right)|_{x_1}^{x_2}.
\end{aligned}
$$

□

6 Numerical Results, Figures and Simulation

We want to compare the log-double-uniform jump-diffusion model with the other three models: log-Normal, log-uniform and log-double-exponential model.

We use 1988-2003 S&P 500 stock index [14] as the sample of the financial market which is considered to be a moderate size simulation of one of these four jump-diffusion processes. These post-87 data have relatively stable diffusive volatility so that the volatility σ in the model thought as a constant parameter is reasonable. The following are some statistic results about the 1988-2003 S&P 500 sample data.

Let $n^{(sp)} = 4036$ be the number of daily closings $S_s^{(sp)}$ for $s = 1{:}n^{(sp)}$, such that there are $ns = 4035$ daily log-returns,

$$\Delta\ln\left(S_s^{(sp)}\right)\equiv\ln\left(S_{s+1}^{(sp)}\right)-\ln\left(S_s^{(sp)}\right) \tag{8}$$

with empirical

- Mean: $M_1^{(sp)} = \frac{\sum_{s=1}^{ns}\Delta\ln\left(S_s^{(sp)}\right)}{ns} \simeq$3.6404e-4.

- Variance: $M_2^{(sp)} = \frac{\sum_{s=1}^{ns}\left(\Delta\ln\left(S_s^{(sp)}\right)-M_1^{(sp)}\right)^2}{ns-1} \simeq 1.0752\text{e-}4$.
- Skewness coefficient: $\beta_3^{(sp)} \equiv M_3^{(sp)}/\left(M_2^{(sp)}\right)^{1.5} \simeq -0.1952 < 0$,where $\beta_3^{(n)} = 0$ is the normal distribution value and $M_3^{(sp)}$ is the 3rd central log-return moment of the data.
- Kurtosis coefficient: $\beta_4^{(sp)} \equiv M_4^{(sp)}/\left(M_2^{(sp)}\right)^2 \simeq 6.9745 > 3$,where $\beta_4^{(n)} = 3$ is the normal distribution value and $M_4^{(sp)}$ is the 4th central log-return moment of the data.

The histogram and hysteriagram of the log-returns of the 1988-2003 S&P 500 are the Figure 1 and Figure 2 respectively.

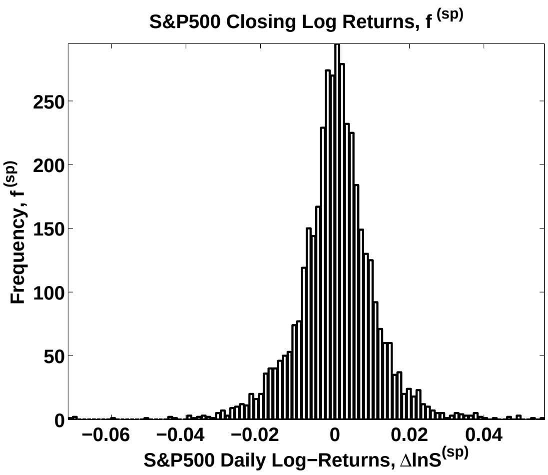

Figure 1: Histogram of S&P500 log return frequencies for 1988-2003, using 100 centered evenly spaced bins.

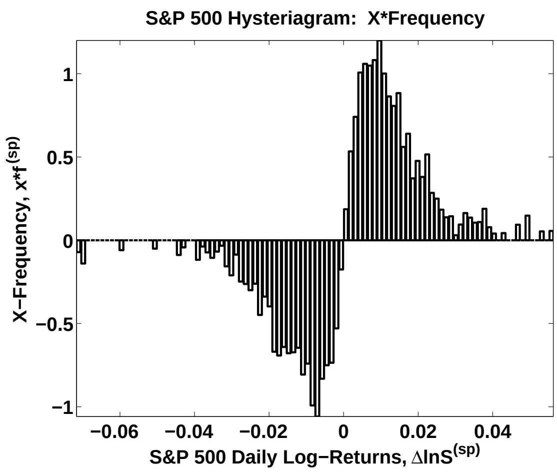

Figure 2: Hysteriagram of S&P500 log returns for 1988-2003, using 100 centered evenly spaced bins.

First of all, we use multinomial maximum likelihood method to estimate the parameters $\{\mu_{ld}, \sigma, p, a, b, \lambda\}$ in (3) using 100 centered evenly spaced bins to sort the daily log-returns of the sample data. Then, from $\mu_{ld} = \mu - 0.5\sigma^2$, μ is obtained. For details about how the multinomial maximum likelihood method works, please see [6] and [8]. The difference is that we do not use the first and the second moment constrains as what the above two papers do since the computational time does not increase too much and the mean and variance differences are more meaningful to readers than the skewness and kurtosis differences and also some biases can be avoid.

The estimated model parameters set $\{\mu, \sigma, p, a, b, \lambda\}$ in (1) are $\{$0.1901, 0.1133, 0.5653 , -0.02918, 0.0293, 48.82 $\}$. In order to compare with the other models, we need to compute μ_j and σ_j according to the formulae $\mu_j = \frac{1}{2}(pa + qb)$ and $\sigma_j^2 = \frac{pq}{4}(b - a)^2 + \frac{pa^2+qb^2}{12}$. The summary of the results for all the four models are listed in the Table 1 and Table 2.

From the Table 1, we see that the overall jump-diffusion parameters $\{\mu, \sigma, \mu_j, \sigma_j, \lambda\}$ have somewhat different distributions among these four jump models but there is no much difference, especially for μ, σ and σ_j. For Normal model , Uniform model and Double-Exponential model, the yearly jump rate λ is 64.01, 64.16 and 67.74 respectively they are

Table 1: Comparison summary of derived distribution parameters for the log-normal, log-uniform, log-double-exponential and log-double-uniform jump-diffusion models, respectively.

Model	μ	σ	μ_j	σ_j	λ
Normal	0.1720	0.1114	-1.146e-3	1.505e-2	64.01
Uniform	0.1626	0.1074	-9.780e-4	1.561e-2	64.16
Dbl-Exp	0.1987	0.1173	-1.467e-3	1.446e-2	67.74
Dbl-Uniform	0.1901	0.1133	-1.876e-3	1.677e-2	48.82

Table 2: The mean, variance, skewness and kurtosis coefficients for the four models are compared to S&P500 values, respectively.

Model	Mean	%	Variance	%	β_3	%	β_4	%
Normal	3.665e-4	0.6823	1.113e-4	-0.4166	-0.1789	-8.358	6.450	-7.524
Uniform	3.732e-4	2.505	1.080e-4	0.4159	-0.1624	-16.79	5.366	-23.07
Dbl-Exp	3.665e-4	0.6816	1.113e-4	3.482	-0.1196	-38.75	8.830	26.86
Dbl-Uniform	3.652e-4	0.3298	1.060e-4	-1.378	-0.1374	-29.63	5.514	-20.94
S&P500	3.640e-4	0.0	1.075e-4	0.0	-0.1952	0.0	6.974	0.0

almost the same. But for the Double-Uniform model, the jump rate is 48.82, about two thirds of the other three models, which may be more reasonable since the trading days per year is about 250 days and the market should be kept stable.

From the Table 2, the Double-Uniform model has the best mean estimation since it has the smallest mean difference percentage, only about 0.3%. The Uniform model gets the best variance estimation but has the worst mean estimation. The Double-Exponential model has the worst varance, skewness and kurtosis estimations. The Normal model has the best skewness and kurtosis estimations although all the four models get qualitative estimate about these nonnormal features. Based on the valuation standards of mean, variance, skewness and kurtosis coefficients, Double-Uniform model and Normal model outperform double-exponential model.

The hysteriagrams for the four models are the figures from Figure 3 to Figure 6. From these figures, we see that Figure 5 resembles Figure 3, especially both have exponentially decreasing thin tails. Also, Figure 6 has about the same shape as Figure 4, especially both have fat tails which fit Figure 2 relatively better than Figure 3 and Figure 5. However, the distinct lumps in the shoulders exists not only in Figure 4 but also in Figure 6.

We use the estimated parameters to simulate the 1988-2003 S&P 500 index prices and get the simulated histograms for the four models. These are the figures from Figure 7 to Figure 10. By comparying these figures to the Figure 1, all these jump models simulate the S&P 500 market data qualitatively well.

7 Conclusion

This paper proposed and analyzed a Double-Uniform Jump-Diffusion model. Some theoretical and numerical results are given. By comparision with the other models, this model

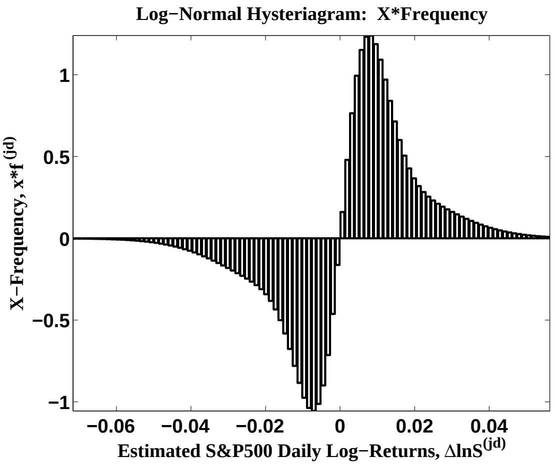

Figure 3: Hysteriagram of the estimated S&P500 log-return frequencies for 1988-2003 by the log-normal jump-diffusion model, using 100 centered evenly spaced bins.

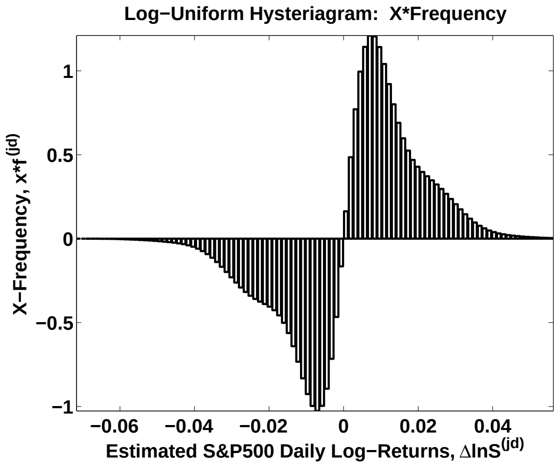

Figure 4: Hysteriagram of the estimated S&P500 log-return frequencies for 1988-2003 by the log-uniform jump-diffusion model, using 100 centered evenly spaced bins.

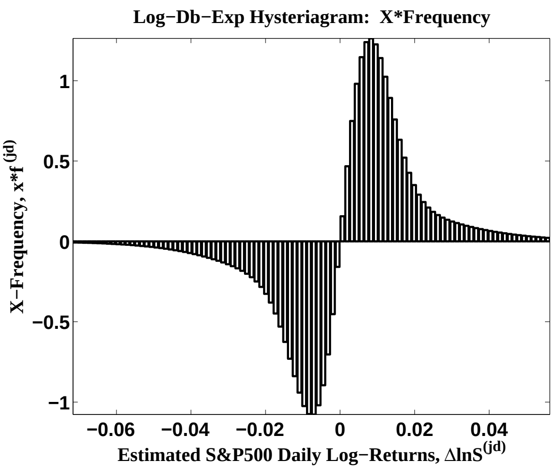

Figure 5: Hysteriagram of the estimated S&P500 log-return frequencies for 1988-2003 by the log-double-exponential jump-diffusion model, using 100 centered evenly spaced bins.

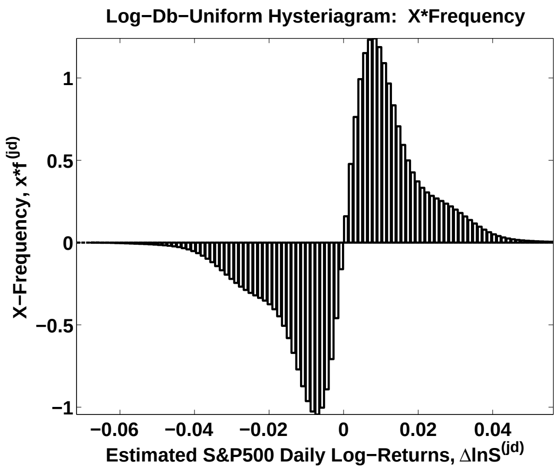

Figure 6: Hysteriagram of the estimated S&P500 log-return frequencies for 1988-2003 by the log-double-uniform jump-diffusion model, using 100 centered evenly spaced bins.

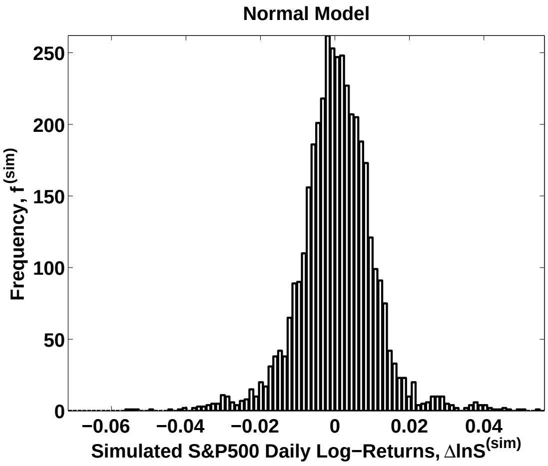

Figure 7: Histogram of the simulated S&P500 log-return frequencies for 1988-2003 by the log-normal jump-diffusion model, using 100 centered evenly spaced bins.

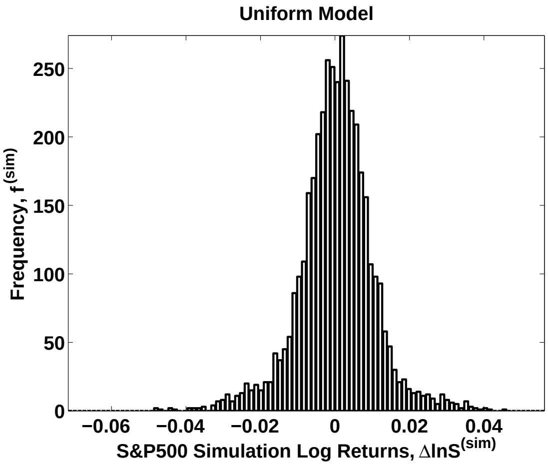

Figure 8: Histogram of the simulated S&P500 log-return frequencies for 1988-2003 by the log-uniform jump-diffusion model, using 100 centered evenly spaced bins.

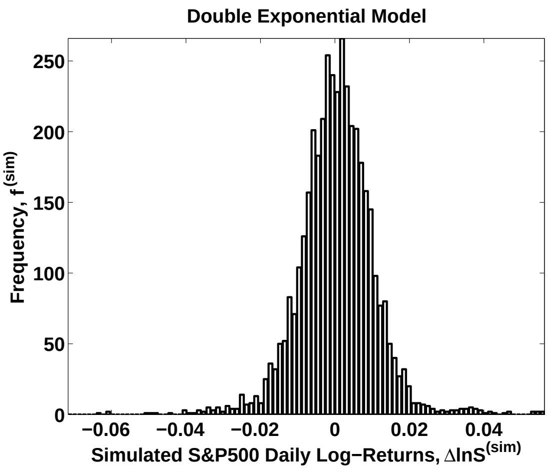

Figure 9: Histogram of the simulated S&P500 log-return frequencies for 1988-2003 by the log-double-exponential jump-diffusion model, using 100 centered evenly spaced bins.

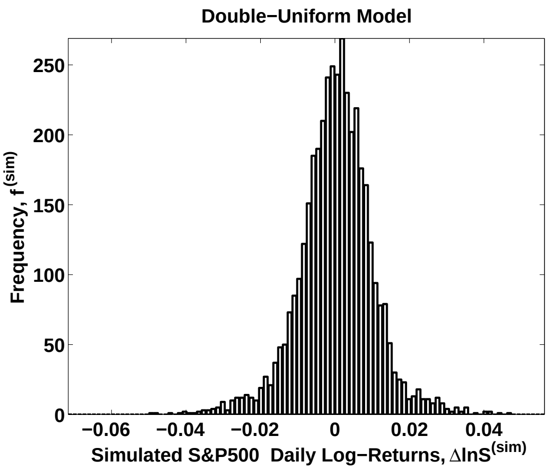

Figure 10: Histogram of the simulated S&P500 log-return frequencies for 1988-2003 by the log-double-uniform jump-diffusion model, using 100 centered evenly spaced bins.

has its own advantages. It is more realic for the Double-Uniform Jump-Diffusion model to gorvern assert prices than the Normal and Double-Exponential Jump-Diffusion models since Double-Uniform model has thick tails and bounded jump amplitudes, however the other two models have exponentially small tails and unbounded jump amplitudes which are against the real world financial market although the infinite jump domain may provide the models some chance to catch some extreme large crashes and rallies. Also, the Double-Uniform Jump-Diffusion model outperforms Double-Exponential Jump-Diffusion model based on the valuations of the differences of mean, varance, skewness and kurtosis between the estimated and the $S\&P500$ market data. The results for the Uniform and Double-Uniform models are mixed up. Finally, we use the estimated parameters to simulate the sample $S\&P500$ index prices and generate the qualitative histograms for the four models.

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