Chapter 16

OPTIMAL PORTFOLIO APPLICATION WITH DOUBLE-UNIFORM JUMP MODEL*

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Dedicated to Suresh P. Sethi on his 60th birthday for his fundamental contributions to optimal portfolio theory.

Abstract This paper treats jump-diffusion processes in continuous time, with emphasis on the jump-amplitude distributions, developing more appropriate models using parameter estimation for the market in one phase and then applying the resulting model to a stochastic optimal portfolio application in a second phase. The new developments are the use of double-uniform jump-amplitude distributions and time-varying market parameters, introducing more realism into the application model – a log-normal diffusion, log-double-uniform jump-amplitude model. Although unlimited borrowing and short-selling play an important role in pure diffusion models, it is shown that borrowing and shorting is limited for jump-diffusions, but finite jump-amplitude models can allow very large limits in contrast to infinite range models which severely restrict the instant stock fraction to [0,1]. Among all the time-dependent parameters modeled, it appears that the interest and discount rate have the strongest effects.

Keywords: Optimal portfolio with consumption, portfolio policy, jump-diffusion, double-uniform jump-amplitude

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1. Introduction

The empirical distribution of daily log-returns for actual financial instruments differs in many ways from the ideal pure diffusion process with its log-normal distribution as assumed in the Black-Scholes-Merton option pricing model [4, 27]. The log-returns are the log-differences between two successive trading days, representing the logarithm of the relative size. The most significant difference is that actual log-returns exhibit occasional large jumps in value, whereas the diffusion process in Black-Scholes [4] is continuous. Statistical evidence of jumps in various financial markets is given by Ball and Torous [3], Jarrow and Rosenfeld [18] and Jorion [19]. Hence, some jump-diffusion models were proposed including Merton's pioneering log-normal [28] (also [29, Chap. 9]), Kou and Wang's log-double-exponential [21, 22] and Hanson and Westman's log-uniform [13, 15] jump-diffusion models.

Another difference is that the empirical log-returns are usually negatively skewed, since the negative jumps or crashes are likely to be larger or more numerous than the positive jumps for many instruments, whereas the normal distribution associated with the diffusion process is symmetric. Thus, the coefficient of skew [5] is negative,

$$\eta_3 \equiv M_3/(M_2)^{1.5} < 0, \tag{16.1}$$

where M_2 and M_3 are the 2nd and 3rd central moments of the log-return distribution here. A third difference is that the empirical distribution is usually leptokurtic since the coefficient of kurtosis [5] satisfies

$$\eta_4 \equiv M_4/(M_2)^2 > 3, \tag{16.2}$$

where the value 3 is the normal distribution kurtosis value and M_4 is the fourth central moment. Qualitatively, this means that the tails are fatter than a normal with the same mean and standard deviation, compensated by a distribution that is also more slender about the mode (local maximum). A fourth difference is that the market exhibits time-dependence in the distributions of log-returns, so that the associated parameters are time-dependent.

For option pricing with jump-diffusions, in 1976 Merton [28] (see also [29, Chap. 8]) introduced Poisson jumps with independent identically distributed random jump-amplitudes with fixed mean and variances into the Black-Scholes model, but the ability to hedge the volatilities as with the Black-Sholes options model was not possible. Also for option pricing, Kou [21, 22] used a jump-diffusion model with a double exponential (Laplace) jump-amplitude distribution, having leptokurtic and negative skewness properties. However, it is difficult to see the empirical justification for this or any other jump-amplitude distribution due to the problem of separating the outlying jumps from the diffusion (see Aït-Sahalia [1]), although separating out the diffusion is a reasonable task.

For optimal portfolio with consumption theory Merton in another pioneering paper, prior to the Black-Scholes model, [25, 26] (see also [29, Chapters 4-6]) analyzed the optimal consumption and investment portfolio with geometric Brownian motion and examined an example of hyperbolic absolute risk-aversion (HARA) utility having explicit solutions. Generalizations to jump-diffusions consisting of Brownian motion and compound Poisson processes with general random finite amplitudes are briefly discussed. Earlier in [24] ([29, Chapter 4]), Merton also examined constant relative risk-aversion problems.

In the 1971 Merton paper [25, 26] there are a number of errors, in particular in boundary conditions for bankruptcy (non-positive wealth) and vanishing consumption. Some of these problems are directly due to using a general form of the HARA utility model. These errors are very thoroughly discussed in a seminal collection assembled by Suresh P. Sethi [32] from his papers and those of his coauthors. Sethi in his introduction [32, Chapter 1]) thoroughly summarizes these errors and subsequent generalizations. In particular, basic papers of concern here are the *KLSS* paper with Karatzas, Lehoczhy, Shreve [20] (reprint [32, Chapter 2]) for exact solutions in the infinite horizon case and with Taksar [33] (reprint [32, Chapter 2]) pinpointing the errors in Merton's [25, 26] work.

Hanson and Westman [10, 16] reformulated an important external events model of Rishel [31] solely in terms of stochastic differential equations and applied it to the computation of the optimal portfolio and consumption policies problem for a portfolio of stocks and a bond. The stock prices depend on both scheduled and unscheduled jump external events. The complex computations were illustrated with a simple log-bi-discrete jump-amplitude model, either negative or positive jumps, such that both stochastic and quasi-deterministic jump magnitudes were estimated. In [11], they constructed a jump-diffusion model with marked Poisson jumps that had a log-normally distributed jump-amplitude and rigorously derived the density function for the diffusion and log-normal-jump stock price log-return model. In [12], this financial model is applied to the optimal portfolio and consumption problem for a portfolio of stocks and bonds governed by a jump-diffusion process with log-normal jump amplitudes and emphasizing computational results. In two companion papers, Hanson and Westman [13, 14] introduce the log-uniform jump-amplitude jump-diffusion model, estimate the parameter of the jump-diffusion density with weighted least squares using the S&P500 data and apply it to portfolio and consumption optimization. In [15], they study the time-dependence of the jump-diffusion parameter on the portfolio optimization problem for the log-uniform jump-model. The appeal of the log-uniform jump model is that it is consistent with the stock exchange introduction of *circuit breakers* [2] in 1988 to limit extreme changes, such as in the crash of 1987, in stages. On the contrary, the normal and double-exponential jump models have an infinite domain, which is not a problem for the diffusion part of the jump-diffusion distribution since the contribution in the dynamic programming formulation is local appearing only in partial derivatives. However, the influence of the jump part in dynamic programming is global through integrals with integrands that have shifted arguments. This has important consequences for the choice of jump distribution since the portfolio wealth restrictions will depend on the range of support of the jump density.

In this paper, the log-double-uniform jump-amplitude, jump-diffusion asset model is applied to the portfolio and consumption optimizaition problem. In Section 2, the jump-diffusion density is rigorously derived using a modification of the prior theorem [11]. In Section 3, the time dependent parameters for this log-return process are estimated using this theoretical density and the S&P500 Index daily closing data for 16 years. In Section 4, the optimal portfolio and consumption policy application is presented and then solved computationally. Also, in this section, the big difference in borrowing and short-selling limits is formulated in a lemma. Concluding remarks are given in Section 5.

2. Log-Double-Uniform Amplitude Jump-Diffusion Density for Log-Return

Let $S(t)$ be the price of a single financial asset, such as a stock or mutual fund, governed by a Markov, geometric jump-diffusion stochastic differential equation (SDE) with time-dependent coefficients,

$$dS(t) = S(t)\left(\mu_d(t)dt + \sigma_d(t)dG(t) + \sum_{k=1}^{dP(t)} J(T_k^-, Q_k)\right), \qquad (16.3)$$

with $S(0) = S_0$, $S(t) > 0$, where $\mu_d(t)$ is the mean appreciation return rate at time t, $\sigma_d(t)$ is the diffusive volatility, $dG(t)$ is a continuous Gaussian process with zero mean and dt variance, $dP(t)$ is a discontinuous, standard Poisson process with jump rate $\lambda(t)$, with common mean-variance of $\lambda(t)dt$, and associated jump-amplitude $J(t, Q)$ with log-return mark Q mean $\mu_j(t)$ and variance $\sigma_j^2(t)$. The stochastic processes $G(t)$ and $P(t)$ are assumed to be Markov and pairwise independent. The jump-amplitude $J(t,Q)$, given that a Poisson jump in time occurs, is also independently distributed, at pre-jump time T_k^- and mark Q_k. The stock price SDE (16.3) is similar in prior work [11, 12], except that time-dependent coefficients introduce more realism here. The Q_k are IID random variables with Poisson amplitude mark density, $\phi_Q(q;t)$, on the mark-space $\mathcal{Q}$.

The infinitesimal moments of the jump process are

$$\mathrm{E}[J(t,Q)dP(t)] = \lambda(t)dt \int_{\mathcal{Q}} J(t,q)\phi_Q(q;t)dq$$

and

$$\mathrm{Var}[J(t,Q)dP(t)] = \lambda(t)dt \int_{\mathcal{Q}} J^2(t,q)\phi_Q(q;t)dq.$$

The differential Poisson process is a counting process with the probability of the jump count given by the usual Poisson distribution,

$$p_k(\lambda(t)dt) = \exp(-\lambda(t)dt)(\lambda(t)dt)^k/k!, \tag{16.4}$$

$k = 0, 1, 2, \ldots$, with parameter $\lambda(t)dt > 0$.

Since the stock price process is geometric, the common multiplicative factor of $S(t)$ can be transformed away yielding the SDE of the stock price log-return using the stochastic chain rule for Markov processes in continuous time,

$$d[\ln(S(t))] = \mu_{ld}(t)dt + \sigma_d(t)dG(t) + \sum_{k=1}^{dP(t)} \ln(1 + J(T_k^-, Q_k)), \tag{16.5}$$

where $\mu_{ld}(t) \equiv \mu_d(t) - \sigma_d^2(t)/2$ is the log-diffusion drift and $\ln(1+J(t,q))$ is the stock log-return jump-amplitude or the logarithm of the relative post-jump-amplitude. This log-return SDE (16.5) is the model that will be used for comparison to the S&P500 log-returns. Since jump-amplitude coefficient $J(t,q) > -1$, it is convenient to select the mark process to be the log-jump-amplitude random variable,

$$Q = \ln\left(1 + J(t,Q)\right), \tag{16.6}$$

on the mark space $\mathcal{Q} = (-\infty, +\infty)$, so $J(t,Q) = e^Q - 1$ in general. Although this is a convenient mark selection, it implies the independence of the jump-amplitude in time, but not of the jump-amplitude distribution.

Since market jumps are rare and limited, while the tails are relatively fat, a reasonable approximation is the log-double-uniform (duq) jump-amplitude distribution with density ϕ_Q on the finite, time-dependent mark interval $[a(t), b(t)]$ as in [15]. However, since the optimistic strategies that play a role in rallies should be different from the pessimistic strategies used for crashes, it would be better to decouple the positive from the negative jumps giving rise to the log-double-uniform jump-amplitude model. The double-uniform density is the juxtaposition of two uniform densities, $\phi_1(q;t) = I_{\{a(t)\le q\le 0\}}/|a|(t)$ on $[a(t), 0]$ and $\phi_2(q;t) = I_{\{0\le q\le b(t)\}}/b(t)$ on $[0, b(t)]$, such that $a(t) < 0 < b(t)$ and $I_{\mathcal{S}}$ is the indicator function for set $\mathcal{S}$. The double-uniform density can be written,

$$\phi_Q(q;t) \equiv \left\{ \begin{array}{ll} 0, & -\infty < q < a(t) \\ p_1(t)/|a|(t), & a(t) \le q < 0 \\ p_2(t)/b(t), & 0 \le q \le b(t) \\ 0, & b(t) < q < +\infty \end{array} \right\}, \tag{16.7}$$

essentially undefined or doubly defined at $q = 0$, except $p_1(t)$ is the probability of a negative jump and $p_2(t)$ is the probability of a non-negative jump, conserving probability by assigning the null jump to the uniform sub-distribution with the positive jumps. Otherwise, $\phi_Q(q;t)$ is undefined as the derivative of the double-uniform distribution for the point of jump discontinuity at 0, but the distribution

$$\begin{aligned}\Phi_Q(q;t) = \; & p_1(t)\tfrac{q-a(t)}{|a|(t)} I_{\{a(t)\le q<0\}} + \left(p_1(t) + p_2(t)\tfrac{q}{b(t)}\right) I_{\{0\le q<b(t)\}} \\ & + I_{\{b\le q<\infty\}}\end{aligned}$$

is well-defined and continuous since points of zero measure do not contribute. The assumption that $a(t) < 0 < b(t)$ is to make sure that both negative jumps (including crashes) and positive jumps (including rallies) are represented. The form of this double-uniform model was motivated by Kou's [21] double-exponential model.

The density $\phi_Q(q;t)$ yields the mean

$$\mathrm{E}_Q[Q] = \mu_j(t) = (p_1(t)a(t) + p_2(t)b(t))/2$$

and variance

$$\mathrm{Var}_Q[Q] = \sigma_j^2(t) = (p_1(t)a^2(t) + p_2(t)b^2(t))/3 - \mu_j^2(t)$$

which define the basic log-return jump-amplitude moment parameters. The third and fourth central moments are, respectively,

$$\begin{aligned} M_3^{(\mathrm{duq})}(t) &\equiv \mathrm{E}_Q\big[(Q - \mu_j(t))^3\big] \\ &= (p_1(t)a^3(t) + p_2(t)b^3(t))/4 - \mu_j(t)(3\sigma_j^2(t) + \mu_j^2(t)) \end{aligned}$$

and

$$\begin{aligned} M_4^{(\mathrm{duq})}(t) &\equiv \mathrm{E}_Q\big[(Q-\mu_j(t))^4\big] \\ &= (p_1(t)a^4(t)+p_2(t)b^4(t))/5 - 4\mu_j(t)M_3^{(\mathrm{duq})}(t) - 6\mu_j^2(t)\sigma_j^2(t) - \mu_j^4(t). \end{aligned}$$

The log-double-uniform distribution is treated as time-dependent in this paper, so $a(t)$, $b(t)$, $\mu_j(t)$ and $\sigma_j^2(t)$ all depend on t.

The difficulty in separating out the small jumps about the mode or maximum of real market distributions is explained by the fact that a diffusion approximation for small marks can be used for the jump process that will be indistinguishable from the continuous Gaussian process anyway.

The first four moments of the difference form stock log-return,

$$\Delta \ln(S(t)) \equiv \ln(S(t+\Delta t)) - \ln(S(t)),$$

assuming that a sufficiently close approximation of the double-uniform jump-diffusion (dujd) by (16.5), i.e.,

$$\begin{aligned} \Delta \ln(S(t)) \simeq\ & \mu_{ld}(t)\Delta t + \sigma_d(t)\Delta G(t) + \textstyle\sum_{k=1}^{\Delta P(t)} Q_k \\ =\ & (\mu_{ld}(t) + \lambda(t)\mu_j(t))\Delta t + \sigma_d(t)\Delta G(t) \\ & +\mu_j(t)(\Delta P(t) - \lambda(t)\Delta t) + \textstyle\sum_{k=1}^{\Delta P(t)}(Q_k - \mu_j(t)) , \end{aligned} \tag{16.8}$$

the latter in a more convenient zero-mean and independent terms form, are

$$M_1^{(\mathrm{dujd})} \equiv \mathrm{E}[\Delta \ln(S(t))] = (\mu_{ld}(t) + \lambda(t)\mu_j(t))\Delta t, \tag{16.9}$$

$$M_2^{(\mathrm{dujd})} \equiv \mathrm{Var}[\Delta \ln(S(t))] = \left(\sigma_d^2(t) + \lambda(t)\left(\mu_j^2(t) + \sigma_j^2(t)\right)\right)\Delta t, \tag{16.10}$$

$$\begin{aligned} M_3^{(\mathrm{dujd})}(t) &\equiv \mathrm{E}\left[\left(\Delta[\ln(S(t))] - M_1^{(\mathrm{dujd})}(t)\right)^3\right] \\ &= (p_1(t)a^3(t) + p_2(t)b^3(t))\lambda(t)\Delta t/4, \end{aligned} \tag{16.11}$$

$$\begin{aligned} M_4^{(\mathrm{dujd})}(t) &\equiv \mathrm{E}\left[\left(\Delta[\ln(S(t))] - M_1^{(\mathrm{dujd})}(t)\right)^4\right] \\ &= (p_1(t)a^4(t) + p_2(t)b^4(t))\lambda(t)\Delta t/5 \\ &\quad +3(\sigma_d^2(t) + \lambda(t)(\mu_j^2(t) + \sigma_j^2(t)))^2(\Delta t)^2. \end{aligned} \tag{16.12}$$

The $M_4^{(\mathrm{dujd})}(t)$ moment calculation, in particular, needs a lemma from [9, Chapter 5] for the fourth power of partial sums of zero-mean IID random variables X_i, i.e.,

$$\mathrm{E}\left[\left(\sum_{i=1}^{n} X_i\right)^4\right] = n\mathrm{E}\left[X_i^4\right] + 3n(n-1)\left(\mathrm{E}\left[X_i^2\right]\right)^2.$$

The log-double-uniform jump-diffusion density can be found by basic probabilistic methods following a slight modification to time-dependent coefficients from the constant coefficients assumption used in the theorem of Zhu [36],

Theorem 1 (Probability Density)**.** *The log-double-uniform jump-amplitude jump-diffusion log-return difference, written as*

$$\Delta \ln(S(t)) = \mathcal{G}(t) + \sum_{k=1}^{\Delta P(t)} Q_k$$

specified in the SDE (16.8) with non-standard Gaussian $\mathcal{G}(t) = \mu_{ld}\Delta t + \sigma_d \Delta G(t)$*, has a probability density given by*

$$\begin{aligned} \phi^{(\mathrm{dujd})}_{\Delta \ln(S(t))}(x) \simeq & \quad \textstyle\sum_{k=0}^{\infty} p_k(\lambda(t)\Delta t)\phi^{(\mathrm{dujd})}_{\mathcal{G}(t)+\sum_{i=1}^{k} Q_i}(x) \\ \equiv & \quad \textstyle\sum_{k=0}^{\infty} p_k(\lambda(t)\Delta t)\phi^{(\mathrm{dujd})}_k(x), \end{aligned} \qquad (16.13)$$

for sufficiently small Δt *and* $-\infty < x < +\infty$*, where* $p_k(\lambda(t)\Delta t)$ *is the Poisson distribution (16.4) with parameter* $\lambda(t)\Delta t$ *with multiple-convolution, Poisson coefficients*

$$\phi^{(\mathrm{dujd})}_k(x) = \left(\phi_{\mathcal{G}(t)} \prod_{i=1}^{k} (*\phi_{Q_i})\right)(x). \qquad (16.14)$$

In the case of the corresponding normalized second order approximation,

$$\phi^{(\mathrm{dujd},2)}_{\Delta \ln(S(t))}(x) = \textstyle\sum_{k=0}^{2} p_k(\lambda(t)\Delta t)\phi^{(\mathrm{dujd})}_k(x) / \sum_{k=0}^{2} p_k(\lambda(t)\Delta t), \qquad (16.15)$$

where the density coefficients are given by

$$\phi^{(\mathrm{dujd})}_0(x) = \phi^{(n)}\left(x; \mu, \sigma^2\right), \qquad (16.16)$$

for $k = 0$*, where* $\phi^{(n)}(x; \mu, \sigma^2)$ *is the normal distribution with mean* μ *and variance* σ^2*, while here* $(\mu, \sigma^2) = (\mu_{ld}, \sigma_d^2)\Delta t$*, for* $k = 1$*,*

$$\begin{aligned} \phi^{(\mathrm{dujd})}_1(x) = & \quad + \tfrac{p_1(t)}{|a|(t)} \Phi^{(n)}(a(t), 0; x - \mu, \sigma^2) \\ & \quad + \tfrac{p_2(t)}{b(t)} \Phi^{(n)}(0, b(t); x - \mu, \sigma^2), \end{aligned} \qquad (16.17)$$

where $\Phi^{(n)}(a,b;\mu,\sigma^2)$ *is the normal distribution on* (a,b) *with density* $\phi^{(n)}(x;\mu,\sigma^2)$*, and for* $k=2$*,*

$$\begin{aligned}
\phi_2^{(\mathrm{dujd})}(x) = \sigma^2\Big(&(p_1(t)/a(t)+p_2(t)/b(t))^2\phi^{(n)}(0,*)\\
&+(p_1(t)/a(t))^2\phi^{(n)}(2a(t),*)+(p_2(t)/b(t))^2\phi^{(n)}(2b(t),*)\\
&-2\left((p_1(t)/a(t))^2+p_1(t)p_2(t)/(a(t)b(t))\right)\phi^{(n)}(a(t),*)\\
&-2\left((p_2(t)/b(t))^2+p_1(t)p_2(t)/(a(t)b(t))\right)\phi^{(n)}(b(t),*)\\
&+2p_1(t)p_2(t)/(a(t)b(t))\phi^{(n)}(a(t)+b(t),*)\Big)\\
&+(p_1(t)/a(t))^2\left((x-2a(t)-\mu)\Phi^{(n)}(2a(t),a(t),*)\right.\\
&\left.-(x-\mu)\Phi^{(n)}(a(t),0,*)\right)\\
&+(2p_1(t)p_2(t)/(a(t)b(t)))\left((x-\mu)\Phi^{(n)}(0,b(t),*)\right.\\
&\left.-(x-a(t)-\mu)\Phi^{(n)}(a(t),a(t)+b(t),*)\right)\\
&+(p_2(t)/b(t))^2\left((x-\mu)\Phi^{(n)}(0,b(t),*)\right.\\
&\left.-(x-2b(t)-\mu)\Phi^{(n)}(b(t),2b(t),*)\right)\\
&-2(p_1(t)p_2(t)/a(t))\Phi^{(n)}(a(t)+b(t),b(t),*)
\end{aligned} \tag{16.18}$$

where the symbol $*$ *means that the common parameter argument* $x-\mu,\sigma^2$ *has been suppressed.*

Proof. The sum in (16.13) is merely an expression of the law of total probability [9, Chapters 0 and 5] and the multiple or nested form (16.14) follows from a convolution theorem [9]. When $k=0$ there are no jumps and $\Delta\ln(S(t))=\mathcal{G}(t)$, the purely Gaussian term, so the distribution is normal and is given in (16.16). Note in this case $\sum_{i=1}^{0}Q_i\equiv 0$ by convention.

When $k=1$ jump, consider the double sum of IID random variables $\Delta\ln(S(t))=\mathcal{G}(t)+Q_1$ near the jump for sufficiently small Δt and letting $(\mu,\sigma^2)=(\mu_{ld},\sigma_d^2)\Delta t$,

$$\begin{aligned}
\phi_1^{(\mathrm{dujd})}(x) &= \left(\phi_{\mathcal{G}(t)}*\phi_{Q_1}\right)(x)=\int_{-\infty}^{+\infty}\phi^{(n)}(x-q;\mu,\sigma^2)\phi_{Q_1}(q;t)dq\\
&=\left(\frac{p_1(t)}{|a|(t)}\int_{a(t)}^{0}+\frac{p_2(t)}{b(t)}\int_0^{b(t)}\right)\phi^{(n)}(x-q;\mu,\sigma^2)dq\\
&=\frac{p_1(t)}{|a|(t)}\Phi^{(n)}(a(t),0;x-\mu,\sigma^2)+\frac{p_2(t)}{b(t)}\Phi^{(n)}(0,b(t);x-\mu,\sigma^2),
\end{aligned}$$

verifying (16.17) by the normal argument-mean shift identity [9, Chapter 0], $\phi^{(n)}(x-q;\mu,\sigma^2)=\phi^{(n)}(q;x-\mu,\sigma^2)$. The density $\Phi^{(n)}(\xi,\eta;\mu,\sigma^2)/(\eta-$

ξ) is a **secant-normal density** as the secant approximation to the derivative to the normal distribution [9, Chapter 5].

For $k = 2$ jumps, we consider the triple IID random variables $\mathcal{G}(t) + Q_1 + Q_2$, first treating the sum of the two double-uniform IID RVs,

$$\begin{aligned}
(\phi_{Q_1} * \phi_{Q_2})(x) &= \int_{-\infty}^{+\infty} \phi_{Q_2}(x-q;t)\phi_{Q_1}(q;t)dq \\
&= \tfrac{p_1^2(t)}{a^2(t)} \int_{a(t)}^{0} I_{\{a(t)\le x-q<0\}}dq + \tfrac{p_2^2(t)}{b^2(t)} \int_0^{b(t)} I_{\{0\le x-q\le b(t)\}}dq \\
&\quad + \tfrac{2p_1(t)p_2(t)}{b(t)|a|(t)} \int_{a(t)}^{0} I_{\{0\le x-q\le b(t)\}}dq \\
&= \tfrac{p_1^2(t)}{a^2(t)} \min(x - 2*a(t), -x) I_{\{a(t)\le x<0\}} \\
&\quad + \tfrac{p_2^2(t)}{b^2(t)} \min(x, 2*b(t) - x) I_{\{0\le x\le b(t)\}} \\
&\quad + \tfrac{2p_1(t)p_2(t)}{b(t)|a|(t)} \min(x-a(t), \min(|a|(t), b(t)), b(t) - x) I_{\{a\le x\le b(t)\}},
\end{aligned}$$

comprising two triangular densities [9, Chapter 5] plus one trapezoidal density. On substituting this density composite and again using the argument-mean normal shift identity again into the double convolution leads to

$$\begin{aligned}
\phi_2^{(\mathrm{dujd})}(x) = {}& \left(\phi_{\mathcal{G}(t)} * (\phi_{Q_1} * \phi_{Q_2})\right)(x) \\
= {}& \tfrac{p_1^2(t)}{a^2(t)} \left(\int_{2a(t)}^{a(t)} (q - 2a(t))\phi^{(n)}(q;*)dq + \int_{a(t)}^{0} (-q)\phi^{(n)}(q;*)dq\right) \\
&+ \tfrac{p_2^2(t)}{b^2(t)} \left(\int_0^{b(t)} q\phi^{(n)}(q;*)dq + \int_{b(t)}^{2b(t)} (2a(t) - q)\phi^{(n)}(q;*)dq\right) \\
&+ \tfrac{2p_1(t)p_2(t)}{b(t)|a|(t)} \left(\int_{a(t)}^{\min(a(t)+b(t),0)} (q - a(t))\phi^{(n)}(q;*)dq\right. \\
&+ \min(|a|(t), b(t)) \int_{\min(a(t)+b(t),0)}^{\max(a(t)+b(t),0)} \phi^{(n)}(q;*)dq \\
&\left. + \int_{\max(a(t)+b(t),0)}^{b(t)} (b(t) - q)\phi^{(n)}(q;*)dq\right),
\end{aligned} \tag{16.19}$$

where again the symbol $*$ denotes $x - \mu, \sigma^2$. The last equation follows from using the following normal integral identity,

$$\pm\int_\alpha^\beta (q-\gamma)\phi^{(n)}(q;*)dq = \pm(x-\mu-\gamma)\Phi^{(n)}(\alpha,\beta;*) \mp \sigma^2\Big(\phi^{(n)}(\beta;*) - \phi^{(n)}(\alpha;*)\Big).$$

Finally after some analysis for two cases: $a(t)+b(t) < 0$ or $a(t)+b(t) >= 0$, the equation (16.19) for $\phi_2^{(\mathrm{dujd})}(x)$ can be recollected and simplified as the form in (16.18). However, there are also practical computational considerations since some naive collections of terms lead to *exponential catastrophic cancellation problems* which are detected by checking a form for $\phi_2^{(\mathrm{dujd})}(x)$ for conservation of probability since $\phi_2^{(\mathrm{dujd})}(x)$ must be a proper density. The problem arises for the double-uniform jump-amplitude model coupled with difficulty of computing normal distributions with very small variances and combining similar exponential as well as distribution terms. Corrections to this problem require a very robust normal density integrator like the MATLAB™ [23] basic *erfc* complementary error function and a proper collection of terms. Note that the two forms (16.18) and (16.19) of $\phi_2^{(\mathrm{dujd})}(x)$ are analytically equivalent in infinite precision, but not computationally in finite precision. □

Using the log-normal jump-diffusion log-return density in (16.13), the third and fourth central moment formulas (16.11,16.12) can be confirmed [36].

3. Jump-Diffusion Parameter Estimation

Given the log-normal-diffusion, log-double-uniform jump density (16.13), it is necessary to fit this theoretical model to realistic empirical data to estimate the parameters of the log-return model (16.5) for $d[\ln(S(t))]$. For realistic empirical data, the daily closings of the S&P500 Index during the years from 1988 to 2003 are used from data available on-line [35]. The data consists of $n^{(\mathrm{sp})} = 4036$ daily closings. The S&P500 (sp) data can be viewed as an example of one large mutual fund rather than a single stock. The data has been transformed into the discrete analog of the continuous log-return, i.e., into changes in the natural logarithm of the index closings, $\Delta[\ln(SP_i)] \equiv \ln(SP_{i+1}) - \ln(SP_i)$ for $i = 1, \ldots, n^{(\mathrm{sp})} - 1$ daily closing pairs. For the period, the mean is $M_1^{(\mathrm{sp})} \simeq 3.640 \times 10^{-4}$ and the variance is $M_2^{(\mathrm{sp})} \simeq 1.075 \times 10^{-4}$, the coefficient of skewness is

$$\eta_3^{(\mathrm{sp})} \equiv M_3^{(\mathrm{sp})}/(M_2^{(\mathrm{sp})})^{1.5} \simeq -0.1952 < 0,$$

demonstrating the typical negative skewness property, and the coefficient of kurtosis is

$$\eta_4^{(\mathrm{sp})} \equiv M_4^{(\mathrm{sp})}/(M_2^{(\mathrm{sp})})^{2} \simeq 6.974 > 3,$$

demonstrating the typical leptokurtic behavior of many real markets.

The S&P500 log-returns, $\Delta[\ln(SP_i)]$ for $i = 1 : n^{(\mathrm{sp})}$ data points, are partitioned into 16 yearly (spy) data sets, $\Delta[\ln(SP_{j_y,k}^{(\mathrm{spy})})]$ for $k = 1 : n_{y,j_y}^{(\mathrm{sp})}$ yearly data points for $j_y = 1 : 16$ years, where $\sum_{j_y=1}^{16} n_{y,j_y}^{(\mathrm{sp})} = n^{(\mathrm{sp})}$. For each of these yearly sets, the six parameters

$$\mathbf{y}_{j_y} = \left(\mu_{ld,j_y}, \sigma_{d,j_y}^2, \mu_{j,j_y}, \sigma_{j,j_y}^2, p_{1,j_y}, \lambda_{j_y}\right),$$

are estimated for each year j_y to specify the jump-diffusion log-return distribution by *maximum likelihood estimation objective*,

$$f(\mathbf{y}_{j_y}) = -\sum_{k=1}^{n_{y,j_y}^{(\mathrm{sp})}} \log\left(\phi_{\Delta\ln(S(t))}^{(\mathrm{dujd},2)}(x_k;\mathbf{y}_{j_y})\right). \tag{16.20}$$

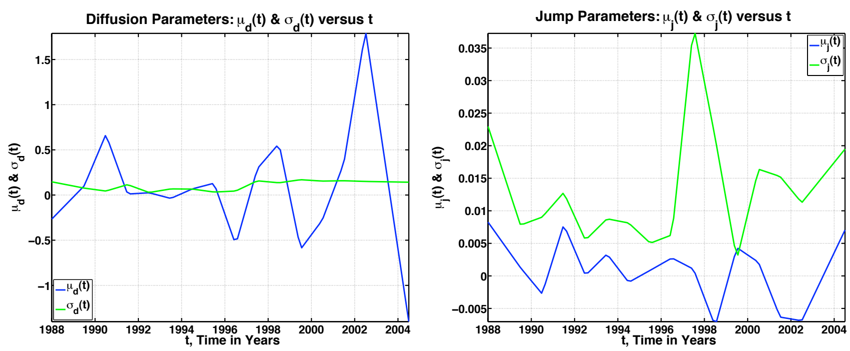
The time step $\Delta t = \Delta T_{j_y}$ is the reciprocal of the number of trading days per year, close to 252 days, but varies a little for $j_y = 1:16$ years used here for parameter estimation. The maximum likelihood estimation is performed for convenience directly on the set

$$\widehat{\mathbf{y}}_{j_y} = \left(\mu_{ld,j_y}, \sigma_{d,j_y}^2, a_{j_y}, b_{j_y}, p_{1,j_y}, \lambda_{j_y}\right),$$

since it is easier to get the pair $\{\mu_{j,j_y}, \sigma_{j,j_y}^2\}$ from $\{a_{j_y}, b_{j_y}\}$, rather then the other way around which would require a quadratic inversion.

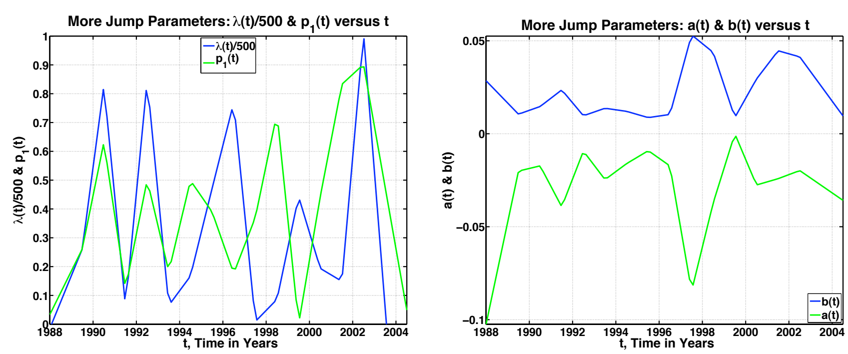
Thus, we have a six dimensional global minimization problem for a highly complex discretized jump-diffusion density function (16.5). Due to the high level of flexibility with six free parameters, *barrier techniques* using large values in excluded regions are adopted to avoid negative variances σ_{d,j_y}^2, non-positive a_{j_y}, negative b_{j_y}, $p_{1,j_y} \notin [0,1)$ and negative λ_{j_y}. The analytical complexity indicates that a general global optimization method that does not require derivatives would be useful. For this purpose, such a method, the *Nelder-Mead downhill simplex* method [30], implemented in MATLAB™ [23] as the function `fminsearch` is used, since simple techniques are desirable in financial engineering. The method is quite efficient since it requires only one new function evaluation for each successive step to test for the best new search direction from the old simplex.

The jump-diffusion estimated yearly parameter results in the present log-double-uniform-jump amplitude case are summarized in the Figures 16.1 and 16.2. The graphs are piecewise linear interpolations of the yearly averages when the averages are assigned to the mid-year. Perhaps cubic splines or moving averages would portray the parameters better, but since a total of eight or more time dependent parameters are needed for the optimal portfolio and consumption application that follows, the piecewise linear interpolation is more convenient due to time constraints. Figure 16.1 displays the time-variation from 1988 to mid 2004 for the diffusion mean $\mu_d(t)$ and variance $\sigma_d(t)$ in Subfigure 16.1(a) and the jump mean $\mu_j(t)$ and variance $\sigma_j(t)$ in Subfigure 16.1(b). The average values of $(\mu_d, \sigma_d, \mu_j, \sigma_j)$ are (0.1654, 0.1043, 3.110e-4, 8.645e-3), respectively. In Figure 16.2, more jump parameters are displayed, but the three double-uniform distribution parameters $p_1(t)$, $a(t)$ and $b(t)$ determine the jump mean $\mu_j(t)$ and variance $\sigma_j(t)$. The biggest interest here is that the jump rate $\lambda(t)$, scaled by 500 to keep on the same graph, with both $\lambda(t)$ and $p_1(t)$ similarly variable in Subfigure 16.2(a). The double-uniform bounds, $a(t)$ and $b(t)$, vary quite a bit as would be expected from the variability of $\mu_j(t)$ and $\sigma_j(t)$.



(a) Diffusion parameters: $\mu_d(t)$ and $\sigma_d(t)$. (b) Jump parameters: $\mu_j(t)$ and $\sigma_j(t)$.

Figure 16.1. Jump-diffusion mean and variance parameters, $(\mu_d(t),\ \sigma_d(t)\ )$ and $(\mu_j(t),\ \sigma_j(t))$ on $t \in [1988, 2004.5]$, represented as piecewise linear interpolation of yearly averages assigned to the mid-year.



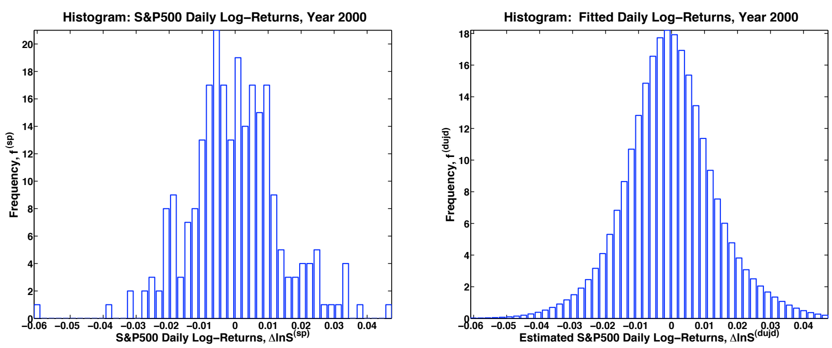
(a) Jump parameters: $\lambda(t)/500$ and $p_1(t)$. (b) Jump parameters: $a(t)$ and $b(t)$.

Figure 16.2. More jump parameters, $(\lambda(t)/500,\ p_1(t)\ )$ and $(a(t),\ b(t))$ on $t \in [1988, 2004.5]$, represented as piecewise linear interpolation of yearly averages assigned to the mid-year.

The *fminsearch* tolerances are *tolx* $= 0.5$e-6 and *toly* $= 0.5$e-6. All yearly iterations are converged in a range from 399 to 750 steps each.

The time needed for the yearly estimations is in a range from 2 to 5 seconds using a Dual 2GHz PowerPC G5 computer processor.

In Figure 16.3 a sample comparison is made for the empirical S&P500 histogram on the left for the year of 2000 with the corresponding theoretical jump-diffusion histogram on the right using the fitted, optimized parameters and the same number of centered bins on the domain. The jump-diffusion histogram is a very idealized version of the empirical distribution, with the asymmetry of the tails clearly illustrated.

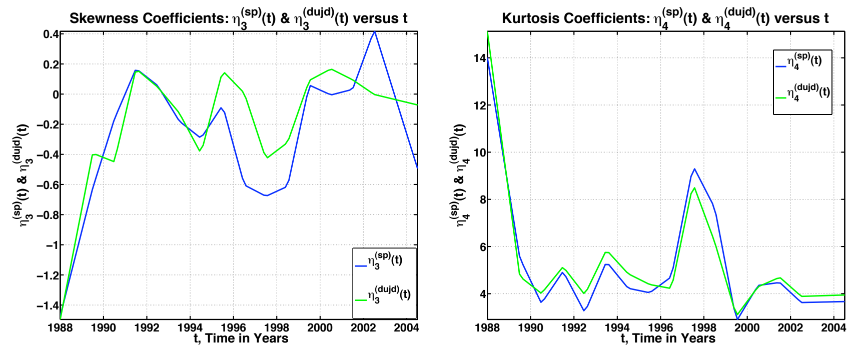


(a) Histogram sample of S&P500 log returns for year 2000.

(b) Histogram sample of S&P500 log returns for fitted jump-diffusion in the same year.

Figure 16.3. Sample comparison for year 2000 of the empirical S&P500 histogram on the left with the corresponding fitted theoretical log-double-uniform jump-diffusion histogram on the right, using 50 bins.

For reference, the summaries of the coefficients of skewness and kurtosis are given in Figure 16.4 for both the estimated theoretical jump-diffusion model and the empirical S&P500 data to facilitate comparison. The jump-diffusion skewness values $\eta_3^{(\mathrm{dujd})}$ in Subfigure 16.4(a) are in the range of -308% to +204% of the empirical S&P500 values, except in the case of the year 2000 when the empirical value is near zero and the relative error is ill-defined. Note that contrary to the legendary long-term negative market skewness, the skewness for some years is positive and the change in sign of the skewness is reflected in the large differences from the empirical results. The jump-diffusion kurtosis values $\eta_4^{(\mathrm{dujd})}$ in Subfigure 16.4(b) are in the range of -19% to +24% with a mean of 3.2% of the empirical values, which is very good considering the difficulty of accurately estimated fourth moments and the results are very much better than that the skewness results. Any discrepancy between the estimated theoretical and observed data for kurtosis is likely due to the relative smallness of the yearly sample as well as the bin size and the fixed yearly double-uniform domain. The concept that the market data is usually leptokurtic refers to long term data and not to shorter term data.



(a) Skewness coefficients: $\eta_3^{(\mathrm{sp})}$ and $\eta_3^{(\mathrm{dujd})}$ (b) Kurtosis coefficients : $\eta_4^{(\mathrm{sp})}$ and $\eta_4^{(\mathrm{dujd})}$.

Figure 16.4. Comparison of skewness and kurtosis coefficients for both the S&P500 data and the estimated double-uniform jump diffusion values on $t \in [1988, 2004.5]$, represented as piecewise linear interpolation of yearly averages assigned to the mid-year.

The main purpose of this parameter estimation has been to have an estimate of the many time-dependence parameters . Hence, we use the simple piecewise linear interpolation to fit the jump-diffusion parameters in time assigning the estimate yearly averages to the mid-year as interpolation points.

4. Application to Optimal Portfolio and Consumption Policies

Consider a portfolio consisting of a riskless asset, called a bond, with price $B(t)$ dollars at time t years, and a risky asset, called a stock, with price $S(t)$ at time t. Let the instantaneous portfolio change fractions be $U_0(t)$ for the bond and $U_1(t)$ for the stock, so that the total satisfies $U_0(t) + U_1(t) = 1$. This does not necessarily imply bounds for $U_0(t)$ and $U_1(t)$, as will be seen later that their bounds depend on the jump-amplitude distribution in the presence of a non-negative of wealth (no bankruptcy) condition.

The bond price process is deterministic exponential,

$$dB(t) = r(t)B(t)dt \ , \quad B(0) = B_0 \ . \tag{16.21}$$

where $r(t)$ is the bond rate of interest at time t. The stock price $S(t)$ has been given in (16.3). The portfolio wealth process changes due to changes in the portfolio fractions less the instantaneous consumption of wealth $C(t)dt$,

$$\begin{aligned} dW(t) = W(t)\Big(r(t)dt + U_1(t)\Big((\mu_d(t) - r(t))dt \\ +\sigma_d(t)dG(t) + \textstyle\sum_{k=1}^{dP(t)}\left(e^{Q_k} - 1\right)\Big)\Big) - C(t)dt\,, \end{aligned} \tag{16.22}$$

such that, consistent with non-negative constraints Sethi and Taksar [33] show are needed, $W(t) \geq 0$ and that the consumption rate is constrained relative to wealth $0 \leq C(t) \leq C^{(0)}_{\max}W(t)$. In addition, the stock fraction is bounded by fixed constants. $U^{(0)}_{\min} \leq U_1(t) \leq U^{(0)}_{\max}$, so borrowing and short-selling is permissible, and $U_0(t) = 1 - U_1(t)$ has been eliminated [12].

The investor's portfolio objective is to maximize the conditional, expected current value of the discounted utility $\mathcal{U}_f(w)$ of terminal wealth at the end of the investment terminal time t_f and the discounted utility of instantaneous consumption $\mathcal{U}(c)$, i.e.,

$$\begin{aligned} v^*(t, w) = \max_{\{u,c\}}\Big[\mathrm{E}\Big[e^{-\beta(t,t_f)}\mathcal{U}_f(W(t_f)) \\ + \textstyle\int_t^{t_f} e^{-\beta(t,s)}\mathcal{U}(C(s))\,ds\Big|\,\mathcal{C}\Big]\Big]\,, \end{aligned} \tag{16.23}$$

conditioned on the state-control set $\mathcal{C} = \{W(t) = w, U_1(t) = u, C(t) = c\}$, where the time horizon is assumed to be finite, $0 \leq t < t_f$, and $\beta(t, s)$ is the cumulative time discount over time in (t, s) with $\beta(t, t) = 0$ and discount rate $\widehat{\beta}(t) = \partial\beta/\partial s(t, t)$ at time t. In order to avoid Merton's [25] problems with utility functions, $\mathcal{U}'(C) \to +\infty$ as $C \to 0^+$ will be assumed for the utility of consumption, while a similar form will be used for the final bequest $\mathcal{U}_f(W)$. Thus, the instantaneous consumption $c = C(t)$ and stock portfolio fraction $u = U_1(t)$ serve as control variables, while the wealth $w = W(t)$ is the single state variable.

Absorbing Boundary Condition at Zero Wealth: Eq. (16.23) is subject to zero wealth absorbing natural boundary condition (avoids arbitrage as pointed out by Karatzas, Lehoczky, Sethi, Shreve and Taksar ([20] or [32, Chapter 2] and [33] or [32, Chapter 3]) that it is necessary to enforce non-negativity feasibility conditions on both wealth and consumption. They formally derived explicit solutions for consumption-investment dynamic programming models with a time-to-bankruptcy horizon that qualitatively corrects the results of Merton [25, 26] ([29, Chapter 6]). See also Sethi and Taksar [33] and much more in the *Sethi* *volume* [32], which includes Sethi's very broad and excellent summary [32, Chapter 1].

Here the Merton correction [29, Chap. 6]) is used,

$$v^*(t,0^+) \;\;=\;\; \mathcal{U}_f(0)e^{-\beta(t,t_f)} + \mathcal{U}(0)\int_t^{t_f} e^{-\beta(t,s)}ds, \tag{16.24}$$

where the terminal wealth condition, $v^*(t_f,w) = \mathcal{U}_f(w)$, has been applied, following from the fact that the consumption must be zero when the wealth is zero.

Portfolio Stochastic Dynamic Programming: Assuming the optimal value $v^*(t,w)$ is continuously differentiable in t and twice continuously differentiable in w, then the stochastic dynamic programming equation (see [12]) follows from an application of the (Itô) stochastic chain rule to the principle of optimality,

$$\begin{aligned}0 = {} & v_t^*(t,w) - \widehat{\beta}(t)v^*(t,w) + \mathcal{U}(c^*(t,w)) \\ & + \left[(r(t) + (\mu_d(t) - r(t))u^*(t,w))w - c^*(t,w)\right] v_w^*(t,w) \\ & + \tfrac{1}{2}\sigma_d^2(t)(u^*)^2(t,w)w^2 v_{ww}^*(t,w) + \lambda(t)\left(\tfrac{p_1(t)}{|a|(t)}\int_{a(t)}^0 + \tfrac{p_2(t)}{b(t)}\int_0^{b(t)}\right) \\ & \cdot \left(v^*(t,(1+(e^q-1)u^*(t,w))w) - v^*(t,w)\right)dq,\end{aligned} \tag{16.25}$$

where $u^* = u^*(t,w) \in [U_{\min}^{(0)}, U_{\max}^{(0)}]$ and $c^* = c^*(t,w) \in [0, C_{\max}^{(0)}w]$ are the optimal controls if they exist, while $v_w^*(t,w)$ and $v_{ww}^*(t,w)$ are the partial derivatives with respect to wealth w when $0 \le t < t_f$.

Non-Negativity of Wealth and Jump Distribution: The non-negativity of wealth implies an additional consistency condition for the control since the jump in wealth argument $(1 + (e^q - 1)u^*)w$ in the stochastic dynamic programming equation (16.25) requires $\kappa(u,q) \equiv 1 + (e^q - 1)u \ge 0$ on the support interval of the jump-amplitude mark density $\phi_Q(q;t)$. Hence, it will make a difference in the optimal portfolio stock fraction u^* bounds if the support interval $[a(t), b(t)]$ is finite or if the support interval is $(-\infty, +\infty)$, i.e., had infinite range. Our results will be restricted to the usual case when $a(t) < 0 < b(t)$, i.e., when both crashes and rallies are modeled.

Lemma 1. ***Bounds on Optimal Stock Fraction due to Non-Negativity of Wealth Jump Argument***
If the support of $\phi_Q(q;t)$ is the finite interval $q \in [a(t), b(t)]$ with $a(t) < 0 < b(t)$, then $u^*(t,w)$ is restricted by (16.25) to

$$\frac{-1}{\left(e^{b(t)} - 1\right)} \le u^*(t,w) \le \frac{1}{\left(1 - e^{a(t)}\right)}, \tag{16.26}$$

but if the support of $\phi_Q(q)$ is fully infinite, i.e., $(-\infty, +\infty)$, then $u^*(t, w)$ is restricted by (16.25) to

$$0 \leq u^*(t, w) \leq 1. \tag{16.27}$$

Proof. Since $\kappa(u, q) = 1 + (e^q - 1)u$ and it is necessary that $\kappa(u, q) \geq 0$ so that $\kappa(u, q)w \geq 0$ when the wealth and its jump argument need to be non-negative. The most basic instantaneous stock fraction case is when $u = 0$, so $\kappa(0, q) = 1 > 0$.

First consider the case when the support is the finite $a(t) \leq q \leq b(t)$. When $u > 0$, then

$$0 \leq 1 - \left(1 - e^{a(t)}\right) u \leq \kappa(u, q) \leq 1 + \left(e^{b(t)} - 1\right) u.$$

Since $e^{a(t)} < 1 < e^{b(t)}$, the worse case for enforcing $\kappa(u, q) \geq 0$ is on the left, so

$$u \leq \frac{+1}{\left(1 - e^{a(t)}\right)}.$$

When $u < 0$, then

$$0 \leq 1 - \left(e^{b(t)} - 1\right)(-u) \leq \kappa(u, q) \leq 1 + \left(1 - e^{a(t)}\right)(-u).$$

The worse case for enforcing $\kappa(u, q) \geq 0$ is again on the left so upon reversing signs,

$$u \geq \frac{-1}{e^{b(t)} - 1},$$

completing both sides of the finite case (16.26).

In the infinite range jump model case when $-\infty < q < +\infty$, then $0 < e^q < \infty$. Thus, when $u > 0$,

$$0 \leq 1 - u < \kappa(u, q) < \infty,$$

so $u \leq 1$. However, when $u < 0$, then

$$-\infty < \kappa(u, q) < 1 - u,$$

so $u < 0$ leads to a contradiction since $\kappa(u, q)$ is unbounded on the left and $u \geq 0$, proving (16.27), which is just the limiting case of (16.26). □

Remark 1. *This lemma gives the constraints on the instantaneous stock fraction $u^*(t, w)$ that limits the jumps to the jumps that at most just wipe out the investor's wealth. Unlike the case of pure diffusion where the* *functional terms have local dependence on the wealth mainly through partial derivatives, the case of jump-diffusion has global dependence through jump integrals over finite differences with jump modified wealth arguments, leading to additional constraints under non-negative wealth conditions that do not appear for pure diffusions. The additional constraint comes not from the current wealth or nearby wealth but from the new wealth created by a jump. The more severe restrictions on the optimal stock fraction in the non-finite support case for the jump-amplitude models compared to the compact support case such as the double uniform model gives further justification for the uniform type models.*

Note that the compact support bounds can be rewritten in terms of the original jump-amplitude coefficient

$$-1/J(t,b(t)) \le u^*(t,w) \le -1/J(t,a(t)).$$

In the case of the fitted log-double-uniform jump-amplitude model, the range of the jump-amplitude marks $[a(t), b(t)]$ *is covered by the estimated interval*

$$[a_{\min}, b_{\max}] = \left[\min_t(a(t)), \max_t(b(t))\right] \simeq [-8.470e\text{-}2, 5.320e\text{-}2]$$

over the whole period from 1988-2003. The corresponding overall estimated range of the optimal instantaneous stock fraction $u^*(t,w)$ *is then*

$$[u_{\min}, u_{\max}] = \left[\frac{-1}{(e^{b_{\max}}-1)}, \frac{+1}{(1-e^{a_{\min}})}\right] \simeq [-18.30, +12.31] \quad (16.28)$$

in large contrast to the highly restricted infinite range models where $[\min(u^*(t,w)), \max(u^*(t,w))] = [0,1]$ *is fixed for any* t.

Regular Optimal Control Policies: In absence of control constraints, then the maximum controls are the regular optimal controls $u_{\text{reg}}(t,w)$ and $c_{\text{reg}}(t,w)$, which are given implicitly, provided they are attainable and there is sufficient differentiability in c and u, by the dual critical conditions,

$$\mathcal{U}'(c_{\text{reg}}(t,w)) = v_w^*(t,w)\ , \quad (16.29)$$

$$\begin{aligned} \sigma_d^2(t) w^2 v_{ww}^*(t,w) u_{\text{reg}}(t,w) &= -(\mu_d(t) - r(t)) w v_w^*(t,w) \\ -\lambda(t) w \left(\tfrac{p_1(t)}{|a|(t)} \int_{a(t)}^{0} + \tfrac{p_2(t)}{b(t)} \int_0^{b(t)} \right) & (e^q - 1) v_w^*(t, \kappa(u_{\text{reg}}(t,w), q) w)\, dq\ , \end{aligned} \quad (16.30)$$

for the optimal consumption and portfolio policies with respect to the terminal wealth and instantaneous consumption utilities (16.23). Note that (16.29-16.30) define the set of regular controls implicitly.

CRRA Utility and Canonical Solution Reduction: Assuming the investor is risk adverse, the utilities will be the constant relative risk-aversion (CRRA) power utilities [29, 10], with the same power for both wealth and consumption,

$$\mathcal{U}(x) = \mathcal{U}_f(x) = x^\gamma/\gamma\ , \quad x \geq 0\ , \quad 0 < \gamma < 1\ . \tag{16.31}$$

The CRRA utility designation arises since the relative risk aversion is the negative of the local change in the marginal utility ($\mathcal{U}''(x)$) relative to the average change in the marginal utility ($\mathcal{U}'(x)/x$), or here

$$R(x) \equiv -\mathcal{U}''(x)/(\mathcal{U}'(x)/x) = (1-\gamma) > 0,$$

i.e., a constant, and is a special case of the more general HARA utilities.

The CRRA power utilities for the optimal consumption and portfolio problem lead to a *canonical reduction* of the stochastic dynamic programming PDE problem to a simpler ODE problem in time, by the separation of wealth and time dependence,

$$v^*(t,w) = \mathcal{U}(w)v_0(t), \tag{16.32}$$

where only the time function $v_0(t)$ is to be determined. The regular consumption control is a linear function of the wealth,

$$c_{\rm reg}(t,w) \equiv w \cdot c^{(0)}_{\rm reg}(t) = w/v_0^{1/(1-\gamma)}(t), \tag{16.33}$$

using (16.29) and $\mathcal{U}'(x) = x^{\gamma-1}$ in (16.31). The regular stock fraction u in (16.30) is a wealth independent control, but is given in implicit form:

$$\begin{aligned} u_{\rm reg}(t,w) &= u^{(0)}_{\rm reg}(t) \\ &= \tfrac{1}{(1-\gamma)\sigma_d^2(t)}\left[\mu_d(t) - r(t) + \lambda(t) I_1\left(u^{(0)}_{\rm reg}(t)\right)\right], \end{aligned} \tag{16.34}$$

$$I_1(u) = \left(\tfrac{p_1(t)}{|a|(t)}\int_{a(t)}^0 + \tfrac{p_2(t)}{b(t)}\int_0^{b(t)}\right)(e^q-1)\kappa^{\gamma-1}(u,q)dq, \tag{16.35}$$

The wealth independent property of the regular stock fraction is essential for the separability of the optimal value function (16.32). Since (16.34) only defines $u^{(0)}_{\rm reg}(t)$ implicitly in fixed point form, $u^{(0)}_{\rm reg}(t)$ must be found by an iteration such as Newton's method, while the Gauss-Statistics quadrature [34] can be used for jump integrals (see [12]). The optimal controls, when there are constraints, are given in piecewise form as $c^*(t,w)/w = c^*_0(t) = \max[\min[c^{(0)}_{\rm reg}(t), C^{(0)}_{\max}], 0]$, provided $w > 0$, and $u^*(t,w) = u^*_0(t) = \max[\min[u^{(0)}_{\rm reg}(t), U^{(0)}_{\max}], U^{(0)}_{\min}]$, is independent of w

along with $u_{\rm reg}^{(0)}(t)$. Substitution of the separable power solution (16.32) and the regular controls (16.33-16.34) into the stochastic dynamic programming equation (16.25), leads to an apparent Bernoulli type ODE,

$$0 = v_0'(t) + (1-\gamma)\left(g_1(t, u_0^*(t))v_0(t) + g_2(t)v_0^{\frac{\gamma}{\gamma-1}}(t)\right), \qquad (16.36)$$

$$\begin{aligned} g_1(t,u) \equiv & \tfrac{1}{1-\gamma}\Big[-\widehat{\beta}(t) + \gamma\left(r(t) + u(\mu_d(t) - r(t))\right) \\ & - \tfrac{\gamma(1-\gamma)}{2}\sigma_d^2(t)u^2 + \lambda(t)(I_2(t,u) - 1)\Big]\ , \end{aligned} \qquad (16.37)$$

$$g_2(t) \equiv \frac{1}{1-\gamma}\left[\left(\frac{c_0^*(t)}{c_{\rm reg}^{(0)}(t)}\right)^{\gamma} - \gamma\left(\frac{c_0^*(t)}{c_{\rm reg}^{(0)}(t)}\right)\right], \qquad (16.38)$$

$$I_2(t,u) \equiv \left(\frac{p_1(t)}{|a|(t)}\int_{a(t)}^{0} + \frac{p_2(t)}{b(t)}\int_0^{b(t)}\right)\kappa^{\gamma}(u,q)\ dq\ , \qquad (16.39)$$

for $0 \le t < t_f$. The coupling of $v_0(t)$ to the time dependent part of the consumption term $c_{\rm reg}^{(0)}(t)$ in $g_2(t)$ and the relationship of $c_{\rm reg}^{(0)}(t)$ to $v_0(t)$ in (16.33) means that the differential equation (16.36) is implicitly highly nonlinear and thus (16.36) is only of Bernoulli type formally. The apparent Bernoulli equation (16.36) can be transformed to an apparent linear differential equation by using $\theta(t) = v_0^{1/(1-\gamma)}(t)$, to obtain,

$$0 = \theta'(t) + g_1(t, u_0^*)\theta(t) + g_2(t),$$

whose general solution can be inverse transformed to the general solution for the separated time function,

$$\begin{aligned} v_0(t) &= \theta^{1-\gamma}(t) \\ &= \left[e^{-\overline{g}_1(t,t_f)(t_f-t)}\left(1 + \int_t^{t_f} g_2(\tau)e^{\overline{g}_1(\tau,t_f)(t_f-\tau)}d\tau\right)\right]^{1-\gamma}, \end{aligned} \qquad (16.40)$$

given implicitly, where $\overline{g}_1(t,t_f)(t_f - t) \equiv -\int_t^{t_f} g_1(x, u_0^*(x))dx$.

In order to illustrate this stochastic application, a computational approximation of the solution is presented. The main computational changes from the procedure used in [12] are that the jump-amplitude distribution is now double-uniform and the portfolio parameters as well as the jump-amplitude distribution are time-dependent. Parameter time-dependence is approximated by piecewise linear interpolation over the years from 1988-2003. The terminal time is taken to be $t_f = 2004.5$, one half year beyond this range.

For this numerical study, the economic rates are taken to be federal funds historical rates [6] from the U.S. Federal Reserve Bank, because they are readily available. For feasibility of the computation, the daily rates, $r(t)$ for interest and $\widehat{\beta}(t)$ for discounting, are transformed into approximate piecewise linear interpolation representations of the yearly averages of daily rates over the period 1988-2003. As for other time-dependent parameters the yearly averages are assigned to the mid-years as interpolation points. The federal funds rates are shown in Figure 16.5. Note that the economic rates are much more variable that the stock mar-

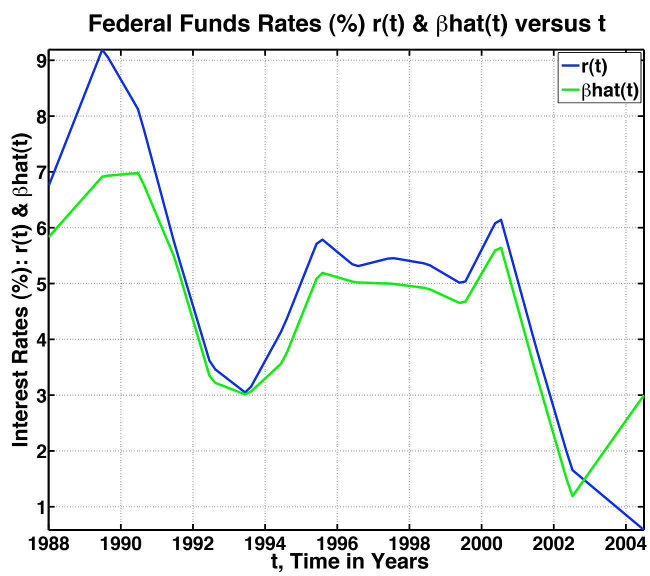


Figure 16.5. Federal funds rate [6] for interest $r(t)$ and discounting $\widehat{\beta}(t)$ on a daily bases, represented by piecewise linear interpolation with yearly averages assigned to the midpoint of each year for $t = 1988.5\!:\!2003.5$.

ket parameters displayed early. Also, the typical approximate ordering of interest and discount rates, $\widehat{\beta}(t) \le r(t)$, is not valid in the recent anomalous low interest period, 2002-present.

The portfolio stock fraction constraints are chosen so that there is at least one active constraint within the time horizon,

$$[U^{(0)}_{\min}, U^{(0)}_{\max}] = [-18, +12],$$

since in a realistic trading environment there would be some bounds on the extremes of borrowing and short-selling, but not as severe as constraining the control to [0,1] as in (16.27). Also, the bound on consumption relative to wealth are assumed to be

$$C_{\max}^{(0)} = 0.75,$$

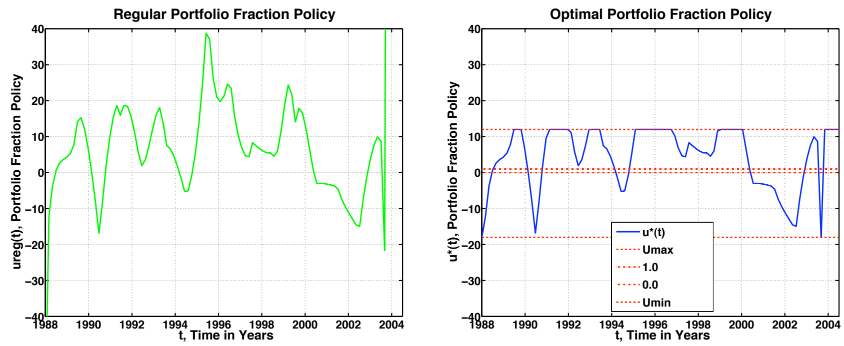
meaning that the investor cannot consume more that 75% of the wealth in the portfolio and $0 \leq c(t,w) \leq C_{\max}^{(0)} w$.

Subfigure 16.6(a) shows the regular or unconstrained optimal instantaneous portfolio stock fraction. Although the $u_{\text{reg}}(t)$ results appear to be out of the conservative range of $[u_{\min}, u_{\max}]$ in (16.28) using $[a_{\min}, b_{max}]$, the results are consistent with the worst case scenario range

$$[\widetilde{u}_{\min}, \widetilde{u}_{\max}] \simeq \text{[-1.150e+02, 1.940e+12]}$$

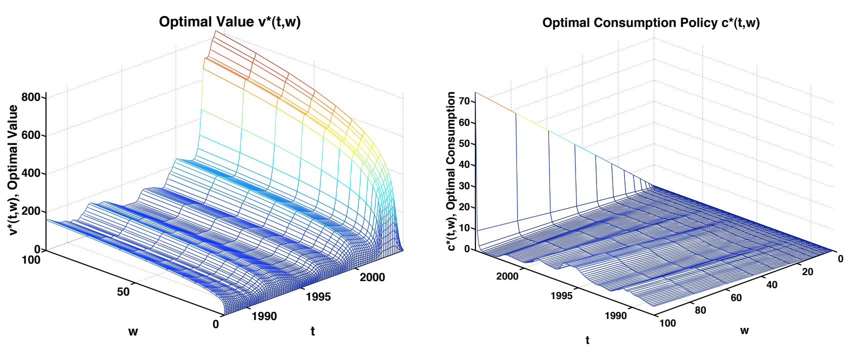
using the tighter distribution range $[a_{\max}, b_{min}]$ of [-5.156e-13, 8.658e-03] in (16.26). In Subfigure 16.6(b), the optimal portfolio stock fraction $u^*(t)$ is displayed. The portfolio policy is not monotonic in time and the maximum control constraint at $U_{\max}^{(0)}$ is active during the interval just prior to the end of the time horizon $t \in [0, t_f]$, while the minimum constraint $U_{\min}^{(0)}$ remains unused since the stock fraction remains mostly in the borrowing range with the corresponding bond fraction negative, $1 - u^*(t) < 0$. The $u^*(t)$ non-monotonic behavior is very interesting compared to the constant behavior in the constant parameter model in [12] or Merton's [25] mainly pure diffusion results.

In Figure 16.7 on the left, the optimal, expected, discounted utility of terminal wealth and cumulative consumption, $v^*(t,w)$, is displayed in three dimensions. The behavior of $v^*(t,w)$ for fixed time t reflects the CRRA utility of function $\mathcal{U}(w)$ template of the separable canonical solution form in (16.32), while the decay in time toward the final time $t_f = 16.5$ and final value $v^*(t_f, w) = 0$ for fixed wealth w derives from the separable time function $v_0(t)$. The optimal value function $v^*(t,w)$ results, and the following optimal consumption policy $c^*(t,w)$ results in Fig. 16.7 on the right, in this computational example, are qualitatively similar to that of the time-independent log-normal jump parameter case in [12] and the time-independent log-uniform jump parameter case in [15] computational results. Note that the wealth grid uses a specially constructed transformation tailored to the CRRA utility to capture the non-smooth behavior as $w \to 0^+$.



(a) Regular stock fraction policy $u_{\rm reg}(t)$. (b) Optimal stock fraction policy, $u^*(t)$.

Figure 16.6. Regular and optimal portfolio stock fraction policies, $u_{\rm reg}(t)$ and $u^*(t)$ on $t \in [1988, 2004.5]$, the latter subject to the control constraints set $[U^{(0)}_{\rm min}, U^{(0)}_{\rm max}] = [-18, 12]$.



(a) Optimal portfolio value $v^*(t, w)$. (b) Optimal consumption policy $c^*(t, w)$.

Figure 16.7. Optimal portfolio value $v^*(t, w)$ and optimal consumption policy $c^*(t, w)$ for $(t, w) \in [1988, 2004.5] \times [0, 100]$.

5. Conclusions

The main contributions of this work are the introduction of the log-double-uniformly distributed jump-amplitude into the jump-diffusion stock price model and the development of time-dependent jump-diffusion parameters. In particular, a significant effect on the variation of the instantaneous stock fraction policy is seen to be due to variations in the interest and discount rates. The double-uniformly distributed jump-amplitude feature of the model is a reasonable assumption for rare, large jumps, crashes or buying-frenzies, when there is only a sparse population of isolated jumps in the tails of the market distribution. Additional realism in the jump-diffusion model is given by the introduction of time dependence in the distribution and in the associated parameters. Finally, the large difference in the severity of the limits on borrowing and short-selling is made clear for the bounds on the instantaneous stock fraction with respect to compact support and non-compact support models of jump-amplitudes.

Further improvements, but with greater computational complexity, would be to estimate the double-uniform distribution limits $[a, b]$ by fitting the theoretical distribution to real market distributions, using longer and overlapping (moving-average) partitioning of the market data to reduce the effects of small sample sizes.

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