THE LEFSCHETZ PRINCIPLE, FIXED POINT THEORY, AND INDEX THEORY November 5, 2009

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Abstract. This is a rough historical account of some uses of the Lefschetz Principle in fixed point theory and index theory. The Lefschetz Principle states that the alternating sum of the traces on cohomology (a global and rigid invariant) is equal to the alternating sum of the traces on the underlying cochain complex (a local and far less rigid invariant). The original Lefschetz Theorem for compact polyhedra then follows easily. The Lefschetz Principle extends readily to index theory and general fixed point theory on compact manifolds, where it is more commonly known as the heat equation method. We outline the proofs of the Atiyah-Singer Index Theorem and the Atiyah-Bott Fixed Point Theorem using this method.

Some Bott Magic? "No! Just physics!"

The above photo was taken at the Bott house on Martha's Vinyard, probably in the summer of 1983, and probably by Paul Schweitzer, SJ. Pictured with the Master are the author in the middle, and Lawrence Conlon (a Bott student). Note the charcoal starter, home made from a stove pipe and complete with floppy wooden handles. Someone asked Raoul if it worked by magic. His answer was typical of the man–succinct and to the point.

This is an expanded write up of the talk I gave at the conference "A Celebration of Raoul Bott's Legacy in Mathematics", held June 9-13, 2008, at the Centre de Recherches Mathématiques, Université de Montréal. Its purpose was to illustrate some of the magic that Raoul Bott worked in mathematics. It covers some very deep results, so time constraints dictated that proofs be outlined only in the broadest terms. They should not be taken literally. Readers interested in the details of the proofs should consult the references.

1. Lefschetz Fixed Point Theorem

The classical Lefschetz Fixed Point Theorem gives a cohomological criterion for a continuous map to have a fixed point. In its simplest form its proof is almost obvious once the Lefschetz Principle is proven. The material in this section follows closely a lecture given by Raoul Bott in 1984.

Theorem 1.1 (Lefschetz Fixed Point Theorem)**.** *Let X be a compact polyhedron, and $f : X \to X$ a continuous map. Set*

$$L(f) = \sum_k (-1)^k \operatorname{tr}\Big(f^* : H^k(X) \to H^k(X)\Big).$$

If $L(f) \neq 0$, then f has a fixed point.

Since the map f^* only depends on the homotopy class of f, $L(f)$ is a homotopy invariant. Also note that $L(f)$ is a generalization of the Euler number $\chi(X)$ of X, since $L(I) = \chi(X)$, where $I : M \to M$ is the identity map.

A good example to keep in mind is the antipodal map $A : S^2 \to S^2$, which has no fixed points, so $L(A)$ must be zero. In particular, $\operatorname{tr}\Big(A^* : H^0(S^2) \to H^0(S^2)\Big) = 1$, since S^2 is connected, and because A is orientation reversing, $\operatorname{tr}\Big(A^* : H^2(S^2) \to H^2(S^2)\Big) = -1$, so indeed $L(A) = 0$.

Proof. Our method of proof is to assume that f has no fixed points and then show that $L(f) = 0$. The most naive approach would be to attempt to show that all the individual $\operatorname{tr}\Big(f^* : H^k(X) \to H^k(X)\Big)$ are zero. As the example above shows, this is a vain hope, but it does contain the essential idea of the proof, once we realize that the reason it doesn't work is that the spaces $H^k(X)$ are too small. If we are willing to expand the domain of the f^* the proof becomes obvious. To do this we need

Proposition 1.2 (The Lefschetz Principle)**.**

$$L(f) = \sum_k (-1)^k \operatorname{tr}\Big(f^* : C^k(X) \to C^k(X)\Big).$$

Here we are using simplicial cochains $C^k(X)$ to compute the cohomology of X, and we may use simplices as small as we like to do so. Since f has no fixed points and X is compact, f must move points at least a fixed positive distance, say δ. Then we use simplices which have diameter less than $\delta/100$ and we approximate f by a simplicial map g so that pointwise g is within $\delta/10$ of f. We may assume that δ is so small that g is homotopic to f, so we may as well assume that $g = f$. Because f moves points at least δ and our simplices are at most $\delta/100$ in diameter, it is impossible for f to map any simplex to itself, so it is immediate that for all k, $\operatorname{tr}\Big(f^* : C^k(X) \to C^k(X)\Big) = 0$. $\square$

Proof of The Lefschetz Principle. Denote by Z^k the cocycles in $C^k = C^k(X)$, and by B^k the coboundaries. Then we have the following commutative diagrams of finite dimensional vector spaces.

$$(*) \quad \begin{array}{ccccccccc} 0 & \longrightarrow & Z^k & \longrightarrow & C^k & \longrightarrow & B^{k+1} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & Z^k & \longrightarrow & C^k & \longrightarrow & B^{k+1} & \longrightarrow & 0 \end{array} \qquad (**) \quad \begin{array}{ccccccccc} 0 & \longrightarrow & B^k & \longrightarrow & Z^k & \longrightarrow & H^k & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & B^k & \longrightarrow & Z^k & \longrightarrow & H^k & \longrightarrow & 0 \end{array}$$

where all the vertical maps are restriction of f^*. Note that the rows of both diagrams are short exact sequences. Thus for all k,

$$\operatorname{tr}\Big(f^* \,|\, Z^k\Big) + \operatorname{tr}\Big(f^* \,|\, B^{k+1}\Big) = \operatorname{tr}\Big(f^* \,|\, C^k\Big)$$

and

$$\operatorname{tr}\Big(f^* \,|\, B^k\Big) + \operatorname{tr}\Big(f^* \,|\, H^k\Big) = \operatorname{tr}\Big(f^* \,|\, Z^k\Big).$$

We may combine all this information as follows. For $t \in \mathbb{R}$, set

$$B_t = \sum_k t^k \operatorname{tr}\Big(f^* \mid B^k\Big), \quad C_t = \sum_k t^k \operatorname{tr}\Big(f^* \mid C^k\Big), \quad H_t = \sum_k t^k \operatorname{tr}\Big(f^* \mid H^k\Big),$$

and

$$Z_t = \sum_k t^k \operatorname{tr}\Big(f^* \mid Z^k\Big).$$

Then $(*)$ gives the equation

$$C_t \;=\; Z_t + \frac{1}{t} B_t,$$

while $(**)$ gives

$$H_t \;=\; Z_t - B_t.$$

Subtracting we get

$$C_t - H_t \;=\; \frac{1+t}{t} B_t,$$

and setting $t = -1$ gives

$$C_{-1} \;=\; H_{-1},$$

which is the Lefschetz Principle. □

This seemingly simple theorem has many wonderful and varied applications. We give but a few.

Corollary 1.3. *Let X be a compact, contractible polyhedron, and $f : X \to X$ a continuous map. Then f has a fixed point.*

Compactness is essential here, as the map $x \to x + 1$ of $\mathbb{R}$ to itself has no fixed point.

Corollary 1.4 (Brouwer Fixed Point Theorem)**.** *Any continuous map of the closed unit disc in $\mathbb{R}^n$ must have a fixed point.*

Corollary 1.5. *Let X be a compact polyhedron with $\chi(X) \neq 0$. Then any flow on X has a fixed point.*

Corollary 1.6. *For all $k > 0$, there is no continuous map $f : S^{2k} \to S^{2k}$ so that $x \perp f(x)$ for all $x \in S^{2k}$.*

For a smooth manifold M, the Lefschetz Theorem can be refined. At each fixed point x of f, we have the self map $f_{*,x} : TM_x \to TM_x$. A fixed point x is non-degenerate provided that the determinant

$$\det(1 - f_{*,x}) \neq 0.$$

Such fixed points are isolated. The graph of f is transversal to the diagonal ΔM of $M \times M$ if and only if all of its fixed points are non-degenerate. If f is transversal, then compactness of M implies that there are only a finite number of fixed points. Then the Lefschetz number of f is the sum of the signs of the determinents of the linear maps $(1 - f_{*,x})$, that is

Theorem 1.7 (H. Hopf)**.** *Suppose that $f : M \to M$ is a smooth map of a compact smooth manifold whose graph is transversal to the diagonal $\Delta M \subset M \times M$. Then*

$$L(f) \;=\; \sum_{f(x)=x} \frac{\det(1 - f_{*,x})}{|\det(1 - f_{*,x})|}.$$

A wonderful generalization of this to holomorphic vector fields was given by Bott in [B67a], see also [B67b]. Let M a compact, complex n dimensional manifold. Denote by c_j the j-th Chern polynomial, and by $c_j(M) \in H^{2j}(M)$ the j-th Chern class of M. Suppose that Y a holomorphic vector field on M with isolated non-degenerate zeros. In local coordinates Y can be written as

$$Y(z) = \sum_k a_k(z) \partial / \partial z_k.$$

For $z \in M$, set

$$A_z = \Big[\frac{\partial a_k}{\partial z_\ell}(z)\Big].$$

The fact that Y has non-degenerate zeros is equivalent to the fact that at any zero z of Y, $\det(A_z) \neq 0$. Since $c_n(A_z)$ is a multiple of $\det(A_z)$, this is also equivalent to $c_n(A_z) \neq 0$. Now let $j_1, \dots, j_\ell$ be non-negative integers with $j_1 + ... + j_\ell \leq n$, and set

$$C_J = c_{j_1} \cdots c_{j_\ell}.$$

Theorem 1.8 (Bott)**.**

$$\int_M C_J(M) \;=\; \sum_{Y(z)=0} \frac{C_J(A_z)}{c_n(A_z)}.$$

As the proof does not use the Lefschetz Principle, we will omit it. The reader should note that if $j_1 + \cdots + j_\ell < n$, then the right hand side vanishes, e.g. $\displaystyle\sum_{Y(z)=0} \frac{1}{c_n(A_z)} = 0$.

2. Index Theory

Our next application of the Lefschetz Principle occurs in index theory. The Atiyah-Singer Index Theorem [ASI] was one of the watershed results of the last century. For an elliptic differential operator on a compact manifold, this theorem establishes the equality of the analytical index of the operator (the dimension of the space of solutions of the operator minus the dimension of the space of solutions of its adjoint) and the topological index (which is defined in terms of characteristic classes associated to the operator and the manifold it is defined over). It subsumes many other important theorems (e. g. the Signature Theorem, the Riemann-Roch Theorem) as special cases, and it has many far reaching extensions: to families of operators; to operators on covering spaces; to operators defined along the leaves of foliations; and to operators defined purely abstractly.

In order to state the theorem, we need some notation. Let M be closed, oriented, n-dim, Riemannian manifold, and let $E_0, E_1, \dots, E_k$ be Hermitian vector bundles defined over M. For each i suppose that we have a first order differential operator $d_i : C^\infty(E_i) \longrightarrow C^\infty(E_{i+1})$, and that $d_{i+1} \circ d_i = 0$. To say that each d_i is a first order differential operator means that at each point of M there is a coordinate chart $U, x_1, \dots, x_n$, so that the E_i are trivial over U, and with respect to these trivializations, d_i may be written as

$$d_i = A_0^i(x) + \sum_{i=1}^n A_j^i(x)\partial/\partial x_j,$$

where the $A_j^i(x)$ are $\dim E_i$ by $\dim E_{i+1}$ matrices. So we may think of the $A_j^i(x)$ as linear maps between the finite dimensional spaces $A_j^i(x) : E_{i,x} \to E_{i+1,x}$.

For each co-tangent vector $\xi = (\xi_1, ..., \xi_n) \in T^*M_x$, we may form the linear map

$$\sigma(d_i, \xi) = \sum_{i=1}^n \xi_j A_j^i(x) : E_{i,x} \to E_{i+1,x},$$

where the reader should note that we have discarded $A_0^i(x)$, the order zero part of d_i. Then $(E, d) = (\{E_i\}, \{d_i\})$ is an **elliptic complex** provided that for each non-zero ξ, the symbol sequence

$$0 \longrightarrow E_{0,x} \overset{\sigma(d_0,\xi)}{\longrightarrow} E_{1,x} \overset{\sigma(d_1,\xi)}{\longrightarrow} \cdots \overset{\sigma(d_{k-2},\xi)}{\longrightarrow} E_{k-1,x} \overset{\sigma(d_{k-1},\xi)}{\longrightarrow} E_{k,x} \longrightarrow 0$$

is exact. This means that the complex (E, d) defines a K-theory class $\sigma(E, d) \in K^*(T^*M)$. Using the Riemannian structure on M, we can (and will) identify the tangent bundle TM and the co-tangent bundle T^*M.

2.1. **Two examples.** We briefly recall two classical examples of elliptic complexes, namely the de Rham and the Signature complexes. There are two other classical elliptic complexes, the Dolbeault and Spin complexes. The Dolbeault complex is briefly discussed in Section 3. For a discussion of the Spin complex, and more complete discussions of the other complexes, the reader is referred to [ASIII, BGV92, Gi84, LM89]. As explained in those references, many more elliptic complexes can be constructed out of these four complexes by twisting them by an arbitrary Hermitian bundle over M.

• The de Rham Complex.

The de Rham complex of a manifold M consists of the $\mathbb{C}$ valued exterior differential forms. So in this case, $E_i = \wedge^i T^*M \otimes \mathbb{C}$, that is the i-th exterior power of the complexified cotangent bundle, and d_i is just the usual exterior derivative. The symbol maps are then $\sigma(d_i, \xi) = \wedge\xi$, and it is a classical result that the symbol sequence is exact for any non-zero ξ. The index of this complex is just the Euler number of M, a fairly simple invariant.

• The Signature Complex.

A far more interesting invariant is provided by the signature complex. For this, we assume that M is oriented, that its dimension is even, say $n = 2\ell$, and we choose a Riemannian metric on M. Recall (see for instance [W]) that we then have a bundle map $*$ from $\wedge^i T^*M \otimes \mathbb{C}$ to $\wedge^{n-i} T^*M \otimes \mathbb{C}$, which satisfies

$$*^2 = (-1)^{i(n-i)}.$$

If we define the map

$$\tau = \sqrt{-1}^{\,i(i-1)+\ell} * : \wedge^i T^*M \otimes \mathbb{C} \to \wedge^{n-i} T^*M \otimes \mathbb{C},$$

then one sees easily that $\tau^2 = 1$. Thus τ is an involution on the bundle

$$E = \bigoplus_{i=0}^{n} \wedge^i T^*M \otimes \mathbb{C},$$

so E splits as the sum of the ± 1 eigenspaces of τ,

$$E = E_+ \oplus E_-.$$

The adjoint $d^* : C^\infty(\wedge^i T^*M \otimes \mathbb{C}) \to C^\infty(\wedge^{i-1} T^*M \otimes \mathbb{C})$ of the exterior derivative d is just $d^* = - * d *$. If we set $D = d + d^*$, then it is not difficult to show that $D\tau = -\tau D$, so $D : C^\infty(E) \to C^\infty(E)$ maps $C^\infty(E_+)$ to $C^\infty(E_-)$, and vice-versa. The signature complex then is

$$D_+ : C^\infty(E_+) \to C^\infty(E_-).$$

It is called the signature complex because the index of D_+ is the signature of the quadratic form on $H^\ell(M;\mathbb{R})$ given by the cup product (which is in fact zero if ℓ is odd). See again [ASIII, BGV92, Gi84, LM89].

Returning now to our general situation of the elliptic complex (E, d), we set $H^i(E, d) = \ker d_i /$ image d_{i-1}. The facts that M is compact and (E, d) is elliptic imply that $\dim H^i(E, d) < \infty$. We then define

$$\mathrm{Ind}(E, d) \;=\; \sum_{i=0}^{k} (-1)^i \dim H^i(E, d).$$

Recall, see [MS89], that associated to any manifold such as M, there is a characteristic cohomology class, the Todd class, $Td(TM \otimes \mathbb{C}) \in H^*(TM;\mathbb{R})$, and that there is a natural map, the Chern character, $\mathrm{ch} : K^*(T^*M) \to H^*(T^*M;\mathbb{R}) = H^*(TM;\mathbb{R})$.

Theorem 2.1 (Atiyah-Singer)**.**

$$\mathrm{Ind}(E, d) = (-1)^n \int_{TM} Td(TM \otimes \mathbb{C})\mathrm{ch}\big(\sigma(E, d)\big).$$

Proof. The original proof of Atiyah and Singer outlined in [AS63] is based on cobordism theory, and a proof along these lines appeared in [P65]. The proof in [ASI] uses psuedodifferential operators and K-theory, techniques which generalize to many interesting cases. The proof outlined below, given by Atiyah, Bott, and Patodi in [ABP73] and independently by Gilkey in [Gi73], is based on the heat equation, which is a variation of the zeta-function argument due to Atiyah and Bott.

Using the metrics on M and the E_i, define the adjoint maps

$$d_{i-1}^* : C^\infty(E_i) \longrightarrow C^\infty(E_{i-1}).$$

The associated Laplacians are given by

$$\Delta_i = d_{i-1} d_{i-1}^* + d_i^* d_i : C^\infty(E_i) \longrightarrow C^\infty(E_i).$$

The Δ_i are self adjoint, non-negative operators. The ellipticity of (E,d) and the compactness of M imply that the Δ_i have some very nice properties. In particular they have discrete, real eigenvalues $0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots$, which march off to infinity rather quickly. In addition the eigenspaces $E_i(\lambda_j) \subset C^\infty(E_i)$ associated to λ_j are finite dimensional, and $L^2(E_i) = \bigoplus_{j=0}^{\infty} E_i(\lambda_j)$. Thus we may think of Δ_i as the infinite diagonal matrix

$$\Delta_i = \mathrm{Diag}(0,\ldots,0,\lambda_1,\ldots,\lambda_1,\lambda_2,\ldots,\lambda_2,\ldots),$$

where each λ_i occurs only a finite number of times. The associated heat operator is the infinite diagonal matrix

$$e^{-t\Delta_i} = \mathrm{Diag}(1,\ldots,1,e^{-t\lambda_1},\ldots,e^{-t\lambda_1},e^{-t\lambda_2},\ldots).$$

The λ_i go to infinity so fast that this operator is of trace class, that is

$$\mathrm{tr}\, e^{-t\Delta_i} \;=\; \sum_{j=0}^{\infty} e^{-t\lambda_j} \dim E_i(\lambda_j) < \infty.$$

One of the fundamental steps in the heat equation proof of the index theorem is the following.

Proposition 2.2 (The Heat Equation Lefschetz Principle)**.** *For all $t > 0$,*

$$\mathrm{Ind}(E,d) \;=\; \sum_{i=0}^{k} (-1)^i \mathrm{tr}\; e^{-t\Delta_i}.$$

Proof. It is a fairly routine calculation to prove that for each *positive* λ_j, the sequence

$$0 \longrightarrow E_0(\lambda_j) \xrightarrow{d_0} E_1(\lambda_j) \xrightarrow{d_1} \cdots \xrightarrow{d_{k-1}} E_k(\lambda_j) \longrightarrow 0,$$

is exact. So for all $\lambda_j > 0$,

$$\sum_{i=0}^{k} (-1)^i \dim E_i(\lambda_j) = 0.$$

Thus we have that for all $t > 0$ (recall $\lambda_0 = 0$),

$$\sum_{i=0}^{k} (-1)^i \mathrm{tr}\; e^{-t\Delta_i} \;=\; \sum_{i=0}^{k} \left[\sum_{j=0}^{\infty} (-1)^i e^{-t\lambda_j} \dim E_i(\lambda_j) \right] \;=$$

$$\sum_{j=0}^{\infty} e^{-t\lambda_j} \left[\sum_{i=0}^{k} (-1)^i \dim E_i(\lambda_j) \right] \;=\; \sum_{i=0}^{k} (-1)^i \dim E_i(\lambda_0).$$

Now Hodge Theory tells us that $E_i(\lambda_0) \simeq H^i(E,d)$, so

$$\sum_{i=0}^{k} (-1)^i \dim E_i(\lambda_0) \;=\; \sum_{i=0}^{k} (-1)^i \dim H^i(E,d) \;=\; \mathrm{Ind}(E,d).$$

□

The heat operator $e^{-t\Delta_i}$ is much more than trace class. In fact it is a smoothing operator, so there is a smooth section $k_t^i(x,y)$ of $\mathrm{Hom}(\pi_2^* E_i, \pi_1^* E_i)$ over $M \times M$, (where $\pi_j : M \times M \to M$ are the projections), so that for $s \in C^\infty(E_i)$,

$$e^{-t\Delta_i}(s)(x) = \int_M k_t^i(x,y) s(y) dy.$$

In particular, if $\xi^i_{j,\ell}$ is an orthonormal basis of $E_i(\lambda_j)$, we have

$$k_t^i(x,y) = \sum_{j,\ell} e^{-t\lambda_j} \xi^i_{j,\ell}(x) \otimes \xi^i_{j,\ell}(y),$$

where the action of $\xi^i_{j,\ell}(x)\otimes\xi^i_{j,\ell}(y)$ on $s(y)$ is

$$\xi^i_{j,\ell}(x)\otimes\xi^i_{j,\ell}(y)(s(y)) \;=<\xi^i_{j,\ell}(y),s(y)>\xi^i_{j,\ell}(x),$$

and $<\cdot,\cdot>$ is the inner product on $E_{i,y}$. It follows fairly easily that

$$\text{tr } e^{-t\Delta_i} = \int_M \text{tr}\Big(k^i_t(x,x)\Big)dx,$$

so we have that

$$\text{Ind}(E,d) \;=\; \sum_{i=0}^k(-1)^i\int_M \text{tr}\Big(k^i_t(x,x)\Big)dx \;=\; \int_M\sum_{i=0}^k(-1)^i\text{tr}\Big(k^i_t(x,x)\Big)dx,$$

which is independent of t. For t near zero, the heat operator is essentially a local operator and so is subject to local analysis. It is a classical result, see for instance [BGV92] and [Gi84], that it has an asymptotic expansion as $t\to 0$. In particular, for t near 0,

$$\sum_{i=0}^k(-1)^i\text{tr}\Big(k^i_t(x,x)\Big) \;\sim\; \sum_{j\geq -n} t^{j/2}a_j(x),$$

where the $a_j(x)$ can be computed locally (that is, in any coordinate system and relative to any local framings) from the Δ_i. Each $a_j(x)$ is a complicated expression in the derivatives of the Δ_i, up to a finite order which depends on j. Now we have,

$$\text{Ind}(E,d) \;=\; \int_M\sum_{i=0}^k(-1)\text{tr}\Big(k^i_t(x,x)\Big)dx \;=\; \lim_{t\to 0}\sum_{j\geq -n}t^{j/2}\int_M a_j(x) \;=\; \int_M a_0(x),$$

since the quantity $\displaystyle\int_M\sum_{i=0}^k(-1)^i\text{tr}\Big(k^i_t(x,x)\Big)dx$ is independent of t. It was the hope, first raised explicitly by McKean and Singer [McS67], that there might be some "miraculous" cancellations in the complicated expression for $a_0(x)$ that would yield the Atiyah-Singer integrand, that is there would be a local index theorem. Atiyah-Bott-Patodi and Gilkey showed that this was indeed the case, at least for Dirac operators twisted by Hermitian bundles. For a particularly succinct proof, which shows that the cancellations are not at all "miraculous", but rather natural, see [Ge88]. Standard arguments in K-theory, then lead directly to the full Atiyah-Singer Index Theorem. □

In [ASIV], Atiyah and Singer proved the index theorem for families of compact manifolds. For a heat equation proof of this result see [Bi86], and for a heat equation proof in the case of foliations, (a theorem due to Connes [C94]), see [HL99].

We end this section with an outline of this extension to foliations. One major problem is that, in general, a foliation F of a compact Riemannian manifold M will have both compact and non-compact leaves. This introduces a number of difficulties, for example non-compact leaves can limit on compact ones, causing fearsome problems with the transverse smoothness of the heat operators. Some of these difficulties can be solved by working on the graph $\mathcal{G}$ of F instead of F itself. $\mathcal{G}$ is constructed by associating to each point in M the holonomy cover of the leaf through that point, so $\mathcal{G}$ has a natural foliation F_s, and there is a natural covering map $\mathcal{G}\to M$ which takes leaves of F_s to leaves of F. The possible non-compactness of the leaves of F_s causes problems with the spectrums of the leafwise Laplacians, since on even the simplest non-compact manifold, namely $\mathbb{R}$, the spectrum of the usual Laplacian is the interval $[0,\infty)$. Thus we can not think of the heat operators as nice infinite dimensional diagonal matrices with entries going quickly to zero. However, these heat operators are still smoothing, so have nice smooth Schwartz kernels when restricted to any leaf. If $\mathcal{G}$ is Hausdorff, then it is almost (but not quite) a fiber bundle, and this implies that Duhamel's formula for the derivative of a family of heat kernels extends to heat kernels defined on the leaves of F_s, see [He95]. The heat kernel we are interested in is $e^{-B_t^2}$, where B_t is the Bismut superconnection obtained using the metric on M scaled by the factor $1/t$. Suppose that $D: C^\infty_c(E)\to C^\infty_c(E)$ is a generalized Dirac operator defined along the leaves of F, and ∇ is a connection on the bundle E over M. Then in simple cases, $B_t = \sqrt{t}D + \nabla$ pulled back to $\mathcal{G}$ by the natural map $\mathcal{G} \to M$.

One of the major results of [He95] is that the Schwartz kernel of $e^{-B_t^2}$ is smooth in all its variables, both leafwise and in directions transverse to the leaves. This allows us to define a Chern character which takes values in the "de Rham cohomology of the space of leaves of F." This is in quotes because the space of leaves is usually a badly behaved space, so has no de Rham cohomology in the usual sense. Fortunately, Haefliger [H80] has defined a de Rham theory for foliations which plays this role rather well. The Chern character is then defined using a (super) trace Tr_s on Schwartz kernels of leafwise smoothing operators, and this trace takes values in the Haefliger forms.

The proof of the families index theorem for foliations then has three steps. The first is to show that $\mathrm{Tr}_s(e^{-B_t^2})$ is a closed Haefliger form and its cohomology class is independent of t, that is the Lefschetz Principle still holds. This is the main result of [He95]. The fact that it is closed relies heavily on Duhamel's formula and the trace property of Tr_s, while the independence from the metric is a fairly standard argument. The second step is to compute the limit as $t \to 0$ of $\mathrm{Tr}_s(e^{-B_t^2})$. The calculation for families of compact manifolds in [Bi86] works just as well for foliations because the operator $e^{-B_t^2}$ becomes very Gaussian along the diagonal as $t \to 0$, so the result is purely local, and locally the foliation case looks just like the compact families case. Of course the final step is to compute the limit as $t \to \infty$ of $\mathrm{Tr}_s(e^{-B_t^2})$, which is the main result of [HL99]. To do this, we adapt an argument of [BGV92], and split the spectrum of D into three pieces, namely 0 and the intervals $(0, t^{-a})$ and $[t^{-a}, \infty)$, for judicious choice of $a > 0$. In [BGV92], they are dealing with the compact families case, and so for t large enough, the spectrum in the interval $(0, t^{-a})$ is the empty set. We are not so lucky. To handle this interval, we must make some assumptions. The first is that the spectral projections P_0 onto the kernel of D, and P_t associated to the interval $(0, t^{-a})$ are transversely smooth, that is have Schwartz kernels which are differentiable in all directions, both leafwise and transversely to the leaves of F_s. The second is that the "density" of the spectrum of D in the interval $(0, t^{-a})$ is not too great, in particular, we assume that $\mathrm{Tr}(P_t)$ is $\mathcal{O}(t^{-\beta})$ for sufficiently large β. The interval $[t^{-a}, \infty)$ is somewhat easier to handle as here $\mathrm{Tr}_s(e^{-B_t^2})$ is decaying very rapidly as $t \to \infty$. Then a rather lengthy and quite complicated argument shows that

$$\lim_{t\to\infty} \mathrm{Tr}_s(e^{-B_t^2}) = \mathrm{Tr}_s(e^{-(P_0\nabla P_0)^2}).$$

The proof is finished by noting that $P_0\nabla P_0$ is a "connection" on the "index bundle", that is on the kernel of D minus the cokernel of D, and that $\mathrm{Tr}_s(e^{-(P_0\nabla P_0)^2})$ is just the Chern character of this index bundle.

For an extension of this result, which significantly reduces the assumptions needed by using a more complicated operator of heat type, see [BHII].

3. Atiyah-Bott Fixed Point Theorem

Our final application of the Lefschetz Principle is to the proof of the very general Atiyah-Bott Fixed Point Theorem, [AB67], for elliptic complexes.

Let (E, d) be an elliptic complex over a compact manifold M. An **endomorphism** T of (E, d) is a collection of maps $T_i : C^\infty(E_i) \to C^\infty(E_i)$, so that $T_{i+1} \circ d_i = d_i \circ T_i$. Then each T_i induces $T_i^* : H^i(E, d) \to H^i(E, d)$, and we set

$$L(T) \;=\; \sum_{i=0}^{k} (-1)^i \mathrm{tr}(T_i^*).$$

We will be concerned only with the so-called **geometric endomorphisms** associated with a smooth map $f : M \to M$. Now the problem with f is that, in general, it does not induce a map from sections of E_i to sections of E_i, but rather from E_i to the pull-back f^*E_i of E_i. To correct for this, we assume that we have bundle maps $A_i : f^*E_i \to E_i$ so that if we define $T_i : C^\infty(E_i) \to C^\infty(E_i)$ to be the composition

$$C^\infty(E_i) \xrightarrow{f^*} C^\infty(f^*E_i) \xrightarrow{A_i} C^\infty(E_i),$$

then $T_{i+1} \circ d_i = d_i \circ T_i$. This is not a very strong restriction as the examples below will show.

At a fixed point x of f, the fibers of f^*E_i and E_i agree, so $A_{i,x} : f^*E_{i,x} = E_{i,x} \to E_{i,x}$ has a trace.

Theorem 3.1 (Atiyah-Bott)**.** *Let (E, d) be an elliptic complex over the compact manifold M. Suppose that the graph of $f : M \to M$ is transversal to the diagonal $\Delta M \subset M \times M$. Let T be a geometric endomorphism associated to f, derived from bundle maps $A_i : f^*E_i \to E_i$. Then*

$$L(T) \;=\; \sum_{f(x)=x} \frac{\sum_{i=0}^{k} (-1)^i \operatorname{tr}(A_{i,x})}{|\det(1 - f_{*,x})|}.$$

Proof. Recall the spectral decompositions $L^2(E_i) = \oplus_j E_i(\lambda_j)$ associated to the heat operators of the elliptic complex (E, d), and denote by P_i^j, the projection of $L^2(E_i)$ onto the λ_j eigenspace $E_i(\lambda_j)$. As $\lambda_0 = 0$, $E_i(\lambda_0) \simeq H^i(E, d)$, and we have the commutative diagram

$$\begin{tikzcd}
E_i(\lambda_0) \arrow[r, "P_i^0 T_i P_i^0"] \arrow[d] & E_i(\lambda_0) \arrow[d] \\
H^i(E,d) \arrow[r, "T_i^*"] & H^i(E,d)
\end{tikzcd}$$

Thus, for each i, $\operatorname{tr}(T_i^*) = \operatorname{tr}\Big(P_i^0 T_i P_i^0\Big)$, so

$$L(T) \;=\; \sum_{i=0}^{k} (-1)^i \operatorname{tr}(T_i^*) \;=\; \sum_{i=0}^{k} (-1)^i \operatorname{tr}\Big(P_i^0 T_i P_i^0\Big).$$

Once again we bring the Lefschetz Principle to bear in the form

Proposition 3.2 (General Fixed Point Lefschetz Principle)**.** *For all $t > 0$,*

$$L(T) \;=\; \sum_{i=0}^{k} (-1)^i \operatorname{tr}\Big(e^{-t\Delta_i} T_i e^{-t\Delta_i}\Big).$$

Proof. The spectral projections $P_i^j : L^2(E_i) \to E_i(\lambda_j)$ satisfy $P_{i+1}^j\, d_i = d_i\, P_i^j$, so for each *positive* eigenvalue λ_j, we have the commutative diagram, with exact rows

$$\begin{tikzcd}
0 \arrow[r] & E_0(\lambda_j) \arrow[r, "d_0"] \arrow[d, "P_0^j T_0 P_0^j"'] & E_1(\lambda_j) \arrow[r, "d_1"] \arrow[d, "P_1^j T_1 P_1^j"'] & \cdots \arrow[r, "d_{k-1}"] & E_k(\lambda_j) \arrow[r] \arrow[d, "P_k^j T_k P_k^j"'] & 0 \\
0 \arrow[r] & E_0(\lambda_j) \arrow[r, "d_0"] & E_1(\lambda_j) \arrow[r, "d_1"] & \cdots \arrow[r, "d_{k-1}"] & E_k(\lambda_j) \arrow[r] & 0.
\end{tikzcd}$$

Thus, for each $\lambda_j > 0$,

$$\sum_{i=0}^{k} (-1)^i \operatorname{tr}\Big(P_i^j T_i P_i^j\Big) \;=\; 0.$$

As above, we then have

$$\sum_{i=0}^{k} (-1)^i \operatorname{tr}\Big(e^{-t\Delta_i} T_i e^{-t\Delta_i}\Big) \;=\; \sum_{i=0}^{k} (-1)^i \Big[\sum_{j=0}^{\infty} e^{-2t\lambda_j} \operatorname{tr}\Big(P_i^j T_i P_i^j\Big)\Big] \;=$$

$$\sum_{j=0}^{\infty} e^{-2t\lambda_j} \Big[\sum_{i=0}^{k} (-1)^i \operatorname{tr}\Big(P_i^j T_i P_i^j\Big)\Big] \;=\; \sum_{i=0}^{k} (-1)^i \operatorname{tr}\Big(P_i^0 T_i P_i^0\Big) \;=\; L(T).$$

$\square$

Now $\operatorname{tr}\Big(e^{-t\Delta_i}T_ie^{-t\Delta_i}\Big) = \operatorname{tr}\Big(T_ie^{-2t\Delta_i}\Big)$, so

$$L(T) \;=\; \lim_{t\to 0}\sum_{i=0}^{k}(-1)^i\operatorname{tr}\Big(e^{-t\Delta_i}T_ie^{-t\Delta_i}\Big) \;=$$

$$\lim_{t\to 0}\sum_{i=0}^{k}(-1)^i\operatorname{tr}\Big(T_ie^{-2t\Delta_i}\Big) \;=\; \lim_{t\to 0}\sum_{i=0}^{k}(-1)^i\operatorname{tr}\Big(T_ie^{-t\Delta_i}\Big).$$

As $t \to 0$, the Schwartz kernel $k_t^i(x,y)$ of $e^{-t\Delta_i}$ becomes very Gaussian shaped along the diagonal $\Delta M \subset M\times M$. This means that given any neighborhood of ΔM, we may choose t so small that $k_t^i(x,y)$ is essentially supported inside that neighborhood. This is what we meant in the previous section when we said that for small t, $e^{-t\Delta_i}$ is essentially a local operator. The operator $T_ie^{-t\Delta_i}$ also has smooth Schwartz kernel $k_t^{T,i}(x,y)$, given by

$$k_t^{T,i}(x,y) \;=\; A_{i,x}k_t^i(f(x),y).$$

Thus, as $t \to 0$, $k_t^{T,i}(x,y)$ becomes very Gaussian shaped along the graph of f, $Gr(f) \subset M \times M$. Now,

$$\operatorname{tr}\Big(T_ie^{-t\Delta_i}\Big) \;=\; \int_M \operatorname{tr}\Big(k_t^{T,i}(x,x)\Big)\,dx \;=\; \int_{\Delta M\subset M\times M}\operatorname{tr}\Big(k_t^{T,i}(x,x)\Big)\,dx.$$

By taking t sufficiently small we may force the support of $\operatorname{tr}\Big(k_t^{T,i}(x,y)\Big)$ to be essentially contained in any neighborhood of the graph of f we choose. Thus, in order to compute

$$L(T) \;=\; \sum_{i=0}^{k}(-1)^i\Big[\lim_{t\to 0}\int_{\Delta M\subset M\times M}\operatorname{tr}\Big(k_t^{T,i}(x,x)\Big)\,dx\Big],$$

we may restrict to any neighborhood of $Gr(f)$ intersected with ΔM, provided that t is sufficiently small. But a neighborhood of $Gr(f)$ intersected with ΔM is just a neighborhood of the fixed points of f. The theorem now follows by a direct local computation. $\square$

This result has a large number of interesting and deep applications. For the classical complexes, it has the following beautiful specializations. This material is taken from [AB68], and for details the reader should consult that paper.

• The de Rham complex.

In this case, we immediately recover Theorem 1.7, the classical Hopf Theorem. This result extends to the tensor product of the de Rham complex $(\wedge T^*M, d)$ with any flat vector bundle E over M. This yields the elliptic complex $(E\otimes\wedge T^*M, 1\otimes d)$. If f is a transversal map, and $A : f^*E \to E$ is a bundle map preserving the flat structure, then the Lefschetz number of the resulting endomorphism T is

$$L(T) \;=\; \sum_{f(x)=x}\operatorname{tr}(A_{i,x})\Big(\frac{\det(1-f_{*,x})}{|\det(1-f_{*,x})|}\Big).$$

• The Signature complex.

Suppose that f is an isometry of a compact, oriented, Riemannian, $2n$ dimensional manifold M, and that f is transversal to ΔM. Then at each fixed point $x \in M$, $f_{*,x} : TM_x \to TM_x$ is an isometry, so TM_x decomposes into an orthogonal direct sum of 2 dimensional sub-spaces,

$$TM_x = E_1 \oplus E_2 \oplus \cdots \oplus E_n,$$

which are preserved by $f_{*,x}$. The action of $f_{*,x}$ on $E_{i,x}$ is given by rotation through the angle θ_i^x, and the collection $\theta_1^x, ..., \theta_n^x$ is called a coherent system of angles for $f_{*,x}$. The Atiyah-Bott Theorem in this case takes the form

$$L(f) \;=\; \sum_{f(x)=x} i^{-n}\prod_{k=1}^{n}\cot(\theta_k^x/2).$$

As interesting applications of this result we have the following.

Theorem 3.3 (Atiyah-Bott)**.** *Let M be a compact, connected, oriented manifold (of positive dimension), and let $f : M \to M$ be an automorphism of prime power p^ℓ with p odd. Then f cannot have just one fixed point.*

Theorem 3.4 (Atiyah-Bott, Milnor)**.** *Let G be a compact Lie group of diffeomorphisms of a homology sphere M with fixed points x and y. Assume that the action is free except at x and y. Then the induced representations of G on TM_x and TM_y are isomorphic.*

• The Spin complex.

Suppose that f is an isometry of a compact, oriented, Riemannian, $2n$ dimensional manifold M, and that f is transversal to ΔM. For each fixed point x of f, denote by $\theta_1^x, ..., \theta_n^x$ a coherent system of angles for $f_{*,x}$. Suppose further that M admits a Spin structure, and that f admits a lifting $\widehat{f}$ to this Spin structure. The Spin number $\mathrm{Spin}(\widehat{f}, M)$ is then given by

$$\mathrm{Spin}(\widehat{f}, M) \;=\; \sum_{f(x)=x} \epsilon(\widehat{f}, x)(i/2)^n \prod_k \csc(\theta_k^x/2),$$

where $\epsilon(\widehat{f}, x) = \pm 1$, depending on the particular lifting $\widehat{f}$.

• The Dolbeault complex.

For a compact, complex analytic manifold M, we actually have a family of elliptic complexes. The complexified cotangent bundle $T^*M \otimes_{\mathbb{R}} \mathbb{C}$ splits naturally into two complex sub-bundles,

$$T^*M \otimes_{\mathbb{R}} \mathbb{C} \;=\; T^{1,0} \oplus T^{0,1}.$$

In local holomorphic coordinates, $T^{1,0}$ is spanned by the dz_i, while $T^{0,1}$ is spanned by the $d\bar{z}_i$, so $T^{1,0}$ has a holomorphic structure, while $T^{0,1}$ has an anti-holomorphic structure. Set

$$\wedge^{p,q} \;=\; \wedge^p T^{1,0} \oplus \wedge^q T^{0,1}.$$

The operator $d \otimes 1$ on $C^\infty(\wedge^* T^*M \otimes_{\mathbb{R}} \mathbb{C}) = C^\infty(\wedge^{*,*})$ splits naturally as

$$d \otimes 1 \;=\; \partial + \overline{\partial},$$

where ∂ maps $C^\infty(\wedge^{p,q})$ to $C^\infty(\wedge^{p+1,q})$, and $\overline{\partial}$ maps $C^\infty(\wedge^{p,q})$ to $C^\infty(\wedge^{p,q+1})$. For each $p = 1, ..., n = \dim_{\mathbb{C}} M$,

$$0 \longrightarrow C^\infty(\wedge^{p,0}) \xrightarrow{\overline{\partial}} C^\infty(\wedge^{p,1}) \xrightarrow{\overline{\partial}} \cdots \xrightarrow{\overline{\partial}} C^\infty(\wedge^{p,n}) \longrightarrow 0,$$

is an elliptic complex. The cohomology groups of this complex are denoted $H^{p,*}(M)$. If $f : M \to M$ is a holomorphic map, then f^* commutes with $\overline{\partial}$, so it induces $f^{p,*} : H^{p,*}(M) \to H^{p,*}(M)$, and for each $p = 1, ..., n$, we have the Lefschetz number $L(f^{p,*})$.

As a real vector space, $T^*M_x \simeq T^{1,0}M_x$, so it has a complex structure. If A is any $\mathbb{C}$ linear map of T^*M_x, we may compute its $\mathbb{C}$ trace, $\mathrm{tr}_{\mathbb{C}}(A)$ and $\mathbb{C}$ determinant $\det_{\mathbb{C}}(A)$. Suppose that f is transversal to ΔM. At a fixed point x of f, $f_x^* : T^*M_x \to T^*M_x$ is just such a $\mathbb{C}$ linear map (in fact an isomorphism), as is $\wedge^p f_x^* : \wedge^p T^*M_x \to \wedge^p T^*M_x$. With this in mind, the Atiyah-Bott fixed point formula now takes the form

$$L(f^{p,*}) \;=\; \sum_{f(x)=x} \frac{\mathrm{tr}_{\mathbb{C}}(\wedge^p f_x^*)}{\det_{\mathbb{C}}(1 - f_x^*)}.$$

More generally, if E is any holomorphic vector bundle over M, there is the associated elliptic complex (note that $p = 0$ here)

$$0 \longrightarrow C^\infty(E \otimes_{\mathbb{C}} \wedge^{0,0}) \xrightarrow{1\otimes\overline{\partial}} C^\infty(E \otimes_{\mathbb{C}} \wedge^{0,1}) \xrightarrow{1\otimes\overline{\partial}} \cdots \xrightarrow{1\otimes\overline{\partial}} C^\infty(E \otimes_{\mathbb{C}} \wedge^{0,n}) \longrightarrow 0.$$

If $f : M \to M$ is a holomorphic map and $A : f^*E \to E$ is a holomorphic bundle map, then there is the associated endomorphism T of this complex, and we have the Lefschetz number $L(T)$. If f is transversal to ΔM, then

$$L(T) \;=\; \sum_{f(x)=x} \frac{\mathrm{tr}_{\mathbb{C}}(A_x)}{\det_{\mathbb{C}}(1 - f_x^*)},$$

where for a fixed point x, $A_x : f^*E_x = E_x \to E_x$.

As interesting applications of the Atiyah-Bott Dolbeault fixed point formula we have: any holomorphic self map of a rational algebraic manifold must have a fixed point; when applied to S^1 actions, it implies the Weyl character formula; if S^1 acts non-trivially on a spin manifold M, then the $\widehat{A}(M) = 0$. This last is due to Atiyah and Hirzebruch, [AH70], and uses the extension mentioned below.

Note that Theorem 3.1 extends to more general fixed point sets N. The map $f_{*,N} : TM/TN \to TM/TN$ is required to satisfy $\det(I - f_{*,N}) \neq 0$. The identity map satisfies this (vacously), so this result contains the Atiyah-Singer Index Theorem as a special case. For a discussion of the history of this result, see [Gi84], and for its extension to foliations, see [HL90].

4. Afterword

When I originally wrote this talk, my intention was to give the audience a feeling for some of the wonderful mathematics of Raoul Bott in a way that was both informative and entertaining (and as close to his style as I could). This necessitated a bit loose play, sometimes called "fictionalized history", but nothing too egregious, I hoped. I was honored to have Sir Michael Atiyah in the audience. At the end of the talk, he murmured, "Very nice. But it didn't happen quite that way." My only defense was to reply, "But it makes for such a good story this way."

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