

# LEAFWISE HOMOTOPY EQUIVALENCES AND LEAFWISE SOBOLOV SPACES

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ABSTRACT. We prove that a leafwise homotopy equivalence between compact foliated manifolds induces a well defined bounded operator between all Sobolev spaces of leafwise (for the natural foliations of the graphs of the original foliations) differential forms with coefficients in a leafwise flat bundle. We further prove that the associated map on the leafwise reduced  $L^2$  cohomology is an isomorphism which only depends on the leafwise homotopy class of the homotopy equivalence.

## 1. INTRODUCTION

Let  $(M, F)$  and  $(M', F')$  be smooth oriented foliations on closed oriented manifolds  $M$  and  $M'$  and let

$$f : (M, F) \longrightarrow (M', F'),$$

be an orientation-preserving leafwise homotopy equivalence. So, there exists an orientation-preserving leafwise map  $g : (M', F') \longrightarrow (M, F)$  such that  $g \circ f$  and  $f \circ g$  are homotopic to the identity maps through leaf-preserving maps. Let  $E'$  be a leafwise flat Hermitian bundle over  $M'$ , and set  $E = f^*(E')$ . We then prove in the present paper the following results:

- (Theorem 3.3).  $f$  induces an isomorphism on the (reduced) Haefliger cohomologies of  $F$  and  $F'$ .
- $f$  induces a well defined leafwise map on the homotopy (and holonomy) groupoids of the foliations which is leafwise uniformly proper for the induced (source and target) foliations of the groupoids. This immediately implies that it induces a well defined map on the cohomology with compact supports of the homotopy (and holonomy) covers of each leaf of the foliation.
- (Theorem 3.9). For each  $x \in M$ , we may form the Sobolev spaces obtained from the differential forms, with compact support and with coefficients in the pull back of  $E$ , on the monodromy (or holonomy) covering space of the leaf of  $F$  through  $x$ . Similarly for  $x' \in M'$  and  $E'$ . Then we prove that  $f$  induces a well defined uniformly bounded operator  $\tilde{f}^*$  between such twisted leafwise Sobolev spaces of the same Sobolev degree.
- (Theorem 3.12). Finally, we show that the uniformly bounded operator  $\tilde{f}^*$  associated with the leafwise flat Hermitian bundle  $E'$  induces an isomorphism on reduced  $L^2$  cohomology, which is compatible with twisted wedge products.

Regarding the induced map on Haefliger cohomologies, the result is classical and we outline the proof for completeness. The situation for the induced operator on the leafwise twisted cohomologies is more involved. When the foliations are for instance top dimensional each with one leaf and the bundles are trivial line bundles, the obvious pull-back map defined by  $f$  on smooth forms yields an unbounded map on the Sobolev forms of a given Sobolev degree which, in general, is not even a closable operator. Our method to define an appropriate pull-back map in the general case of foliations with flat bundles relies on two techniques and hence produces two definitions which eventually induce the same operator on Sobolev cohomologies. The first one is reduction to the case of submersions [HiS92] and we show that the resulting operator is Sobolev bounded and induces an operator between (reduced twisted) cohomologies which does not depend on the reduction process. The second technique that we use exploits the Whitney isomorphism and allows us to prove the compatibility with wedge products. Using these two descriptions of the induced operator on twisted  $L^2$  cohomologies, we then prove the isomorphism property. The results in this paper are

crucial in the differential geometric approach to the Baum-Connes Novikov conjecture for foliations, using Haefliger cohomology and characteristic classes of transversely smooth idempotents, see [BH04, BH11].

## 2. SOME NOTATION

If  $V \rightarrow N$  is a vector bundle over a manifold  $N$ , we denote the space of smooth sections by  $C^\infty(V)$  or by  $C^\infty(N; V)$  if we want to emphasize the base space of the bundle. The compactly supported sections are denoted by  $C_c^\infty(V)$  or  $C_c^\infty(N; V)$ . The space of differential  $k$  forms on  $N$  is denoted  $\mathcal{A}^k(N)$ , and we set  $\mathcal{A}^*(N) = \bigoplus_{k \geq 0} \mathcal{A}^k(N)$ . The space of compactly supported  $k$  forms is denoted  $\mathcal{A}_c^k(N)$ , and  $\mathcal{A}_c^*(N) = \bigoplus_{k \geq 0} \mathcal{A}_c^k(N)$ . The tangent and cotangent bundles of  $N$  will be denoted  $TN$  and  $T^*N$ .

Let  $M$  be a compact  $n$ -dimensional Riemannian manifold with oriented foliation  $F$  of dimension  $p$  and codimension  $q$ . We denote a leaf of  $F$  by  $L$ . The leaf through the point  $x \in M$  is denoted  $L_x$ , so  $\dim L_x = p$ . We will be working on the homotopy groupoids (also called the monodromy groupoids) of our foliations, but our results extend to the holonomy groupoid, as well as any groupoids between these two extremes. Recall that the homotopy groupoid  $\mathcal{G}$  of  $F$  consists of equivalence classes  $[\gamma]$  of paths  $\gamma : [0, 1] \rightarrow M$  such that the image of  $\gamma$  is contained in a leaf of  $F$ . Two such paths are equivalent if they are in the same leaf and homotopy equivalent (with endpoints fixed) in that leaf. The source and range maps  $s, r : \mathcal{G} \rightarrow M$  are given by  $s([\gamma]) = \gamma(0)$  and  $r([\gamma]) = \gamma(1)$ . These give rise to the two natural transverse foliations  $F_s$  and  $F_r$  whose leaves are respectively  $\tilde{L}_x = s^{-1}(x)$ , and  $\tilde{L}^x = r^{-1}(x)$ , for each  $x \in M$ . Note that  $r : \tilde{L}_x \rightarrow L_x$  is the simply connected covering of  $L$ . We will work with the foliation  $F_s$ .

The basic open sets defining of the manifold structure of  $\mathcal{G}$  are given as follows. Let  $\mathcal{U}$  be a finite good cover of  $M$  by foliation charts as defined in [HL90]. Given  $U$  and  $V$  in this cover and a leafwise path  $\gamma$  starting in  $U$  and ending in  $V$ , we define  $(U, \gamma, V)$  to be the set of equivalence classes of leafwise paths starting in  $U$  and ending in  $V$  which are homotopic to  $\gamma$  through a homotopy of leafwise paths whose end points remain in  $U$  and  $V$  respectively. It is easy to see, using the holonomy defined by  $\gamma$  from a transversal in  $U$  to a transversal in  $V$ , that if  $U, V \simeq \mathbb{R}^p \times \mathbb{R}^q$ , then  $(U, \gamma, V) \simeq \mathbb{R}^p \times \mathbb{R}^p \times \mathbb{R}^q$ . Note that the intersection of any leaf  $\tilde{L}_x$  and any basic open set  $(U, \gamma, V)$  consists of at most one plaque of the foliation  $F_s$  in  $(U, \gamma, V)$ , i.e. each  $\tilde{L}_x$  passes through any  $(U, \gamma, V)$  at most once.

The (reduced) Haefliger cohomology of  $F$ , [Ha80] is given as follows. For each  $U_i \in \mathcal{U}$ , let  $T_i \subset U_i$  be a transversal and set  $T = \bigcup T_i$ . We may assume that the closures of the  $T_i$  are disjoint. Let  $\mathcal{H}$  be the holonomy pseudogroup induced by  $F$  on  $T$ . Give  $\mathcal{A}_c^k(T)$  the usual  $C^\infty$  topology, and denote the exterior derivative by  $d_T : \mathcal{A}_c^k(T) \rightarrow \mathcal{A}_c^{k+1}(T)$ . The usual Haefliger cohomology is defined using the quotient of  $\mathcal{A}_c^k(T)$  by the vector subspace  $L^k$  generated by elements of the form  $\alpha - h^* \alpha$  where  $h \in \mathcal{H}$  and  $\alpha \in \mathcal{A}_c^k(T)$  has support contained in the range of  $h$ . The (reduced) Haefliger cohomology uses the quotient of  $\mathcal{A}_c^k(T)$  by the closure  $\overline{L^k}$  of  $L^k$ . This closure is taken in the following sense.  $\overline{L^k}$  consists of all elements in  $\omega \in \mathcal{A}_c^k(T)$ , so that there are sequences  $\{\omega_n\}, \{\hat{\omega}_n\} \subset L^k$  with  $\|\omega - \omega_n\| \rightarrow 0$  and  $\|d_T(\omega) - \hat{\omega}_n\| \rightarrow 0$ . The norm  $\|\cdot\|$  is the sup norm, that is  $\|\omega\| = \sup_{x \in T} \|\omega(x)\|_x$ , where  $\|\cdot\|_x$  is the norm on  $(\wedge^k T^*T)_x$ . Set  $\mathcal{A}_c^k(M/F) = \mathcal{A}_c^k(T)/\overline{L^k}$ . The exterior derivative  $d_T$  induces a continuous differential  $d_H : \mathcal{A}_c^k(M/F) \rightarrow \mathcal{A}_c^{k+1}(M/F)$ . Note that  $\mathcal{A}_c^k(M/F)$  and  $d_H$  are independent of the choice of cover  $\mathcal{U}$ . In this paper, the complex  $\{\mathcal{A}_c(M/F), d_H\}$  and its cohomology  $H_c^*(M/F)$  will be called, respectively, the Haefliger forms and Haefliger cohomology of  $F$ .

## 3. LEAFWISE MAPS

Let  $M$  and  $M'$  be compact Riemannian manifolds with oriented foliations  $F$  and  $F'$ . Let  $f : M \rightarrow M'$  be a smooth leafwise homotopy equivalence which preserves the leafwise orientations. (We need only assume transverse smoothness, and leafwise continuity. A standard argument then allows  $f$  to be approximated by a smooth map.) Let  $g : M' \rightarrow M$  be a leafwise homotopy inverse of  $f$ . Then there are leafwise homotopies  $h : M \times I \rightarrow M$  and  $h' : M' \times I \rightarrow M'$  with  $I = [0, 1]$ , so that for all  $x \in M, x' \in M'$

$$h(x, 0) = x, \quad h(x, 1) = g \circ f(x), \quad h'(x', 0) = x', \quad \text{and} \quad h'(x', 1) = f \circ g(x').$$

We begin by recalling two results on such leafwise maps from [HL91].

**Lemma 3.1** (Lemma 3.17 of [HL91]). *Given finite coverings of  $M$  and  $M'$  by foliation charts, there is a number  $N$  such that for each plaque  $Q$  of  $M'$ , there are at most  $N$  plaques  $P$  of  $M$  such that  $f(\overline{P}) \cap \overline{Q} \neq \emptyset$ .*

Thus  $f$  is leafwise uniformly proper and so induces a well defined map  $f^* : H_c^*(L'_{f(x)}; \mathbb{R}) \rightarrow H_c^*(L_x; \mathbb{R})$ . In general this map does not extend to the leafwise  $L^2$  forms, as shown by simple examples.

**Lemma 3.2** (Lemma 3.16 of [HL91]). *For any finite cover of  $M$  by foliation charts there is a number  $N$  such that for each plaque  $P$  of  $M$ , there are at most  $N$  plaques  $Q$  such that  $h(\overline{Q} \times I) \cap \overline{P} \neq \emptyset$ .*

Note that this lemma implies that there is a global bound on the leafwise distance that  $h$  moves points, i. e. there is a global bound on the leafwise lengths of all the curves  $\{\gamma_x \mid x \in M\}$ , where  $\gamma_x(t) = h(x, t)$ .

We remark that since  $f$  is a homotopy equivalence between  $M$  and  $M'$ , the dimensions of  $M$  and  $M'$  are the same.

**Theorem 3.3.**  *$f$  induces an isomorphism  $f^* : H_c^*(M'/F') \rightarrow H_c^*(M/F)$  with inverse  $g^*$ .*

*Proof.* The map  $f$  induces a map  $\widehat{f}$  on transversals. In particular, suppose that  $U$  and  $U'$  are foliation charts of  $M$  and  $M'$  respectively, and that  $f(U) \subset U'$ . If  $T$  and  $T'$  are transversals of  $U$  and  $U'$ , then  $f$  induces the map  $\widehat{f} : T \rightarrow T'$ .

**Lemma 3.4.**  *$\widehat{f} : T \rightarrow T'$  is an immersion.*

*Proof.* Being an immersion is a local property, so by reducing the size of our charts if necessary, we may assume that  $g(U') \subset U_1$ , where  $U_1$  is a foliation chart for  $F$  with transversal  $T_1$ . Then  $\widehat{g} : T' \rightarrow T_1$ . The leafwise homotopy  $h$  induces a map  $\widehat{h} : T \rightarrow T_1$ . In particular this is the map induced on transversals by the map  $x \rightarrow h(x, 1)$ . Since  $h$  is continuous and leafwise, it is easy to see that  $\widehat{h} = h_\gamma$  where  $h_\gamma$  is the holonomy along the leafwise path  $\gamma_x(t) = h(x, t)$ , where  $x \in T$ . Thus  $\widehat{h}$  is locally invertible. Since  $h$  is a homotopy of  $gf$  to the identity, the composition,  $\widehat{h}^{-1}\widehat{g}\widehat{f} : T \rightarrow T$  is the identity, so  $\widehat{f}$  must be an immersion.  $\square$

Since  $\widehat{g}$  must also be an immersion, it follows immediately that the codimensions of  $F$  and  $F'$  are the same, and so the dimensions of  $F$  and  $F'$  are also the same.

To construct the map  $f^* : H_c^*(M'/F') \rightarrow H_c^*(M/F)$ , we proceed as follows. Let  $\mathcal{U}$  and  $\mathcal{U}'$  be finite good covers of  $M$  and  $M'$  respectively. We may assume that for each  $U \in \mathcal{U}$ , we have chosen a  $U' \in \mathcal{U}'$  so that  $f(U) \subset U'$  and that the induced map on transversals  $\widehat{f} : T \rightarrow T'$  is a diffeomorphism onto its image. Let  $\alpha' \in H_c^*(M'/F')$ . Since  $f$  is onto, we may choose a Haefliger form  $\phi' = \sum_{U \in \mathcal{U}} \phi'_U$  in  $\alpha'$  so that  $\phi'_U$  has support in  $\widehat{f}(T)$  where  $T$  is a transversal in  $U$ . We then define  $\widehat{f}^*(\alpha')$  to be the class of the Haefliger form  $\sum_{U \in \mathcal{U}} \widehat{f}^*(\phi'_U)$ .

The question of whether  $\widehat{f}^*$  is well defined reduces to showing the following.

**Lemma 3.5.** *Suppose that  $U_1$  and  $U_2$  are foliation charts on  $M$  with transversals  $T_1$  and  $T_2$ . Suppose further that  $\phi'$  is a Haefliger form on  $M'$  with support contained in  $\widehat{f}(T_1) \cap \widehat{f}(T_2)$ . Then as Haefliger forms on  $M$ ,  $[\widehat{f}|_{T_1}]^*(\phi') = [\widehat{f}|_{T_2}]^*(\phi')$ .*

*Proof.* Set  $\widehat{f}_i = \widehat{f}|_{T_i}$ . By writing  $\phi'$  as a sum of Haefliger forms and reducing the size of their supports, we may assume that the support of  $\phi'$  is contained in a transversal  $T'$ , that  $\widehat{g}(T')$  is contained in a transversal  $T$  of  $M$ , and that the holonomy maps  $h_i : T_i \rightarrow T$  determined by the paths  $\gamma_i(t) = h(x_i, t)$ , for  $x_i \in T_i$ , are defined on the supports of  $\widehat{f}_i^*(\phi')$ , respectively. Further, we may suppose that all the maps  $\widehat{f}_1, \widehat{f}_2, h_1, h_2$  and  $\widehat{g}|_{T'}$  are diffeomorphisms onto their images. Since  $h$  is a homotopy of  $gf$  to the identity,  $\widehat{f}_1 = \widehat{g}^{-1} \circ h_1$  and  $\widehat{f}_2 = \widehat{g}^{-1} \circ h_2$ , so  $\widehat{f}_1^*(\phi') = h_1^* \circ (\widehat{g}^{-1})^*(\phi')$  and  $\widehat{f}_2^*(\phi') = h_2^* \circ (\widehat{g}^{-1})^*(\phi')$ . Thus  $\widehat{f}_1^*(\phi') = h_1^* \circ (h_2^{-1})^*(\widehat{f}_2^*(\phi'))$ , so as Haefliger forms  $\widehat{f}_1^*(\phi') = \widehat{f}_2^*(\phi')$ .  $\square$

It now follows easily that the induced map on Haefliger cohomology  $f^* : H_c^*(M'/F') \rightarrow H_c^*(M/F)$  is an isomorphism with inverse  $g^*$ .  $\square$

We now show that  $f$  induces a well defined map on the homotopy groupoids of  $F$  and  $F'$ . Given  $[\gamma] \in \mathcal{G}$ , set  $\check{f}([\gamma]) = [f \circ \gamma]$ . Recall that  $\check{f}$  is leafwise uniformly proper if for any  $C_0$ , there is  $C_1$  so that if the leafwise distance from  $\check{f}(z_0)$  to  $\check{f}(z_1)$  is less than  $C_0$ , then the leafwise distance from  $z_0$  to  $z_1$  is less than  $C_1$ .

**Lemma 3.6.**  *$\check{f} : \mathcal{G} \rightarrow \mathcal{G}'$  is a well defined smooth leafwise map, which is leafwise uniformly proper.*

*Proof.* That  $\check{f}$  is well defined and smooth is clear. Similarly, set  $\check{g}([\gamma']) = [g \circ \gamma']$ .

Let  $\mathcal{U}$  be a finite good cover of  $M$ . Since  $M$  is compact, there is a bound  $m(P)$  on the diameter of any plaque in the cover  $\mathcal{U}$ . Then  $m(P)$  is also a bound for any plaque of  $F_s$  in the corresponding cover of  $\mathcal{G}$ . Let  $\mathcal{U}'$  be a finite good cover of  $M'$ , such that for each  $U' \in \mathcal{U}'$  there is  $U \in \mathcal{U}$  so that  $g(U') \subset U$ . Given  $(U', \gamma', V')$  in the cover of  $\mathcal{G}'$  corresponding to  $\mathcal{U}'$ , choose  $U, V \in \mathcal{U}$  with  $g(U') \subset U$  and  $g(V') \subset V$ . If we set  $\gamma = g \circ \gamma'$ , then  $\check{g}(U', \gamma', V') \subset (U, \gamma, V)$ . Because  $\mathcal{U}'$  is a good cover, there is  $\epsilon > 0$  so that if  $z'_0, z'_1 \in \tilde{L}'$  with the leafwise distance  $d_{\tilde{L}'}(z'_0, z'_1) < \epsilon$ , then there is a  $(U', \gamma', V')$  with  $z'_0, z'_1 \in (U', \gamma', V')$ , so  $\check{g}(z'_0), \check{g}(z'_1) \in (U, \gamma, V)$ . Since  $\check{g}(\tilde{L}') \cap (U, \gamma, V)$  consists of at most one plaque of  $\check{g}(\tilde{L}')$ , it follows that  $d_{\tilde{L}}(\check{g}(z'_0), \check{g}(z'_1)) < m(P)$ . Thus if  $z'_t$  is a path in  $\tilde{L}'$  of length less than  $C$ , then  $\check{g} \circ z'_t$  is a path in  $\check{g}(\tilde{L}')$  of length less than  $m(P)C/\epsilon$ .

Suppose that  $f(x) = x'$  and let  $A' \subset \tilde{L}'_{x'}$  have diameter  $\text{dia}(A') \leq C$ . Let  $z_0, z_1 \in \tilde{L}_x$  with  $\check{f}(z_i) = z'_i \in A'$ , and choose a path  $z'_t$  in  $\tilde{L}'_{x'}$  of length less than  $C$  between  $z'_0$  and  $z'_1$ . Then  $\check{g} \circ z'_t$  is a path in  $\tilde{L}_{gf(x)}$  of length less than  $m(P)C/\epsilon$ . Composition on the right by the path  $\gamma_x(t) = h(x, t)$  is an isometry from  $\tilde{L}_{gf(x)}$  to  $\tilde{L}_x$ . So  $(\check{g} \circ z'_t) \cdot \gamma_x$  is a path in  $\tilde{L}_x$  of length less than  $m(P)C/\epsilon$ . Thus

$$d_{\tilde{L}_x}([\check{g} \circ z'_0] \cdot \gamma_x, [\check{g} \circ z'_1] \cdot \gamma_x) \leq m(P)C/\epsilon.$$

By Lemma 3.2, the path  $\gamma_y$  has length bounded by say  $B$ , for all  $y \in M$ . Set  $y_i = r(z_i)$ , and note that  $[\gamma_{y_i}^{-1} \cdot (\check{g} \circ z'_i) \cdot \gamma_x] = z_i$ , since  $h$  is a leafwise homotopy equivalence between  $g \circ f$  and the identity. As

$$d_{\tilde{L}_x}(z_i, [(\check{g} \circ z'_i) \cdot \gamma_x]) = d_{\tilde{L}_x}([\gamma_{y_i}^{-1} \cdot (\check{g} \circ z'_i) \cdot \gamma_x], [(\check{g} \circ z'_i) \cdot \gamma_x]) \leq \text{length}(\gamma_{y_i}) \leq B,$$

we have

$$d_{\tilde{L}_x}(z_0, z_1) \leq 2B + m(P)C/\epsilon.$$

Thus  $\text{dia}(\check{f}^{-1}(A')) \leq 2B + m(P) \text{dia}(A')/\epsilon$ , and  $\check{f}$  is leafwise uniformly proper.  $\square$

Thus  $\check{f}$  induces a well defined map  $\check{f}^* : \mathbf{H}_c^*(\tilde{L}'_{f(x)}; \mathbb{R}) \rightarrow \mathbf{H}_c^*(\tilde{L}_x; \mathbb{R})$ . As noted above, in general this map does not induce a well defined map on leafwise  $L^2$  forms. We will use two different constructions to deal with this problem. First we adapt the construction of the  $L^2$  pull-back map of Hilsun-Skandalis in [His92] to our setting. This has the advantage that it is transversely smooth. However, the properties of this map are not obvious, so we will also use the construction in [HL91], which is based on results of Dodziuk, [D77]. We assume the reader is familiar with the theory of Sobolev spaces of sections of a vector bundle over a manifold.

Suppose that  $E' \rightarrow M'$  is a complex Hermitian bundle over  $M'$  which is leafwise flat, and set  $E = f^*(E')$ . We denote also by  $E$  its pull back by  $r$  to  $\mathcal{G}$ . The context should make it clear which bundle we are using. We do **not** assume that the leafwise flat structure on  $E$  preserves the inner product on  $E$ . We will denote by  $\mathcal{A}_c^*(F_s, E)$  the field of spaces over  $M$  given by  $\mathcal{A}_c^*(F_s, E)_x = \mathcal{A}_c^*(\tilde{L}_x, E)$ , the differential forms on  $\tilde{L}_x$  with compact support and with coefficients in  $E|_{\tilde{L}_x}$ . For  $a \in \mathbb{Z}$ , we denote by  $W_a^*(F_s, E)$  the field of Hilbert spaces over  $M$  given by  $W_a^*(F_s, E)_x = W_a^*(\tilde{L}_x, E)$ , the  $a$ -th Sobolev space of differential forms on  $\tilde{L}_x$  with coefficients in  $E|_{\tilde{L}_x}$ . Just as it does for the leafwise  $L^2$  forms, the compactness of  $M$  implies that these spaces do not depend on our choice of Riemannian structure.

Let  $i : M' \hookrightarrow \mathbb{R}^k$  be an imbedding of the compact manifold  $M'$  in some Euclidean space  $\mathbb{R}^k$ , and identify  $M'$  with its image. For  $x' \in M'$  and  $t \in \mathbb{R}^k$ , define  $p(x', t)$  to be the projection of the tangent vector  $X^t = \frac{d}{ds} \big|_{s=0} (x' + st)$  at  $x'$  determined by  $t$ , to the leaf  $L'_{x'}$  in  $(M', F') \subset \mathbb{R}^k$ . In particular, first project  $X^t$  to  $TF'_{x'}$ , and then exponentiate it to  $L'_{x'}$ , thinking of  $L'_{x'}$  as a Riemannian manifold in its own right. Since  $M'$  is compact, we may choose a ball  $B^k \subset \mathbb{R}^k$  so small that the restriction of the smooth map

$p_f = p \circ (f, \text{id}) : M \times B^k \rightarrow M'$  to any  $p_f : L_x \times B^k \rightarrow L'_{f(x)}$  is a submersion. Lifting this map to the groupoids, we get

$$p_f : \mathcal{G} \times B^k \longrightarrow \mathcal{G}',$$

which is a leafwise map if  $\mathcal{G} \times B^k$  is endowed with the foliation  $F_s \times B^k$ . Note that  $p_f : \tilde{L}_x \times B^k \rightarrow \tilde{L}'_{f(x)}$  is the map induced on the coverings by  $p_f : L_x \times B^k \rightarrow L'_{f(x)}$ . In particular,  $p_f([\gamma], t)$  is the composition of leafwise paths  $P_f(\gamma, t)$  and  $f \circ \gamma$ ,

$$p_f([\gamma], t) = [P_f(\gamma, t) \cdot (f \circ \gamma)],$$

where  $P_f(\gamma, t) : [0, 1] \rightarrow L'_{f(r(\gamma))}$  is

$$P_f(\gamma, t)(s) = p_f(r(\gamma), st).$$

To see that this is a smooth map, let  $(U, \gamma, V) \times B^k$  and  $(U', f \circ \gamma, V')$  be local coordinate charts on  $\mathcal{G} \times B^k$  and  $\mathcal{G}'$ , respectively, with coordinates  $(w, y, z, t)$  and  $(w', y', z')$ . Then in these coordinates,

$$p_f(w, y, z, t) = (w'(f(w, y)), y'(f(w, y)), z'(p_f(y, z, t))),$$

where the second  $p_f$  is the map  $p_f : V \times B^k \rightarrow V'$ .

The crucial fact about  $p_f$  is that it has all the same essential properties of the projection  $\pi_1 : \mathcal{G} \times B^k \rightarrow \mathcal{G}$ . First note that, because  $f$  and  $\tilde{f}$  are leafwise uniformly proper and  $M \times B^k$  is compact, both the maps denoted  $p_f$  are also leafwise uniformly proper. Second, we may assume that the metric on each  $L_x \times B^k$  (respectively  $\tilde{L}_x \times B^k$ ) is the product of a fiberwise metric for the submersion  $p_f$  and the pull-back under  $p_f$  of the metric on  $L'_{f(x)}$  (respectively  $\tilde{L}'_{f(x)}$ ). To see this, give  $L \times B^k$  the product metric, using the standard metric on  $B^k$ . The induced metric on  $\tilde{L} \times B^k$  is then the product metric. The fibers of both submersions  $p_f$  inherit a Riemannian metric, and we denote by  $d \text{vol}_{\text{vert}}$  the canonical  $k$  form on both  $L \times B^k$  and  $\tilde{L} \times B^k$  whose restriction to the oriented fibers of  $p_f$  is the volume form. Denote by  $*$  the Hodge operator on both  $L \times B^k$  and  $\tilde{L} \times B^k$ , and similarly for  $*$  on  $L'$  and  $\tilde{L}'$ . Consider the sub-bundle  $p_f^* T^* F' \subset T^*(F \times B^k)$  and its orthogonal complement  $p_f^* T^* F'^{\perp}$ . Define a new metric on  $T^*(F \times B^k) = p_f^* T^* F' \oplus p_f^* T^* F'^{\perp}$  (and so also on  $T^*(F_s \times B^k)$ ) by declaring that these sub-bundles are still orthogonal, and the new metric on  $p_f^* T^* F'^{\perp}$  is the same as the original, while the new metric on  $p_f^* T^* F'$  is the pullback of the metric on  $T^* F'$ . Denote the leafwise Hodge operator of the new metric by  $\hat{*}$ . As remarked above, this change of metric does not alter any of our Sobolev spaces. In particular, note that for any non-zero  $\alpha \in \wedge^{\ell} T^*(F \times B^k)$  and any  $c \in \mathbb{R}_+^*$ ,

$$0 < \frac{c\alpha \wedge \hat{*}c\alpha}{c\alpha \wedge *c\alpha} = \frac{\alpha \wedge \hat{*}\alpha}{\alpha \wedge *\alpha},$$

so the compactness of the sphere bundle  $(\wedge^{\ell} T^*(F \times B^k) - \{0\})/\mathbb{R}_+^*$  implies that there are  $0 < C_1 < C_2$ , so that for all  $\alpha \in \wedge^{\ell} T^*(F \times B^k)$ ,

$$C_1 \alpha \wedge *\alpha \leq \alpha \wedge \hat{*}\alpha \leq C_2 \alpha \wedge *\alpha,$$

where we identify the oriented volume elements of  $L \times B^k$  at a point with  $\mathbb{R}_+^*$ . This property is inherited by the two induced metrics on  $T^*(F_s \times B^k)$ , so the two norms used to define the Sobolev spaces  $W_a^{\ell}(F_s, E)$  are comparable. Thus we can substitute the second metric for the first, or what is more notationally convenient, assume that the first metric satisfies the same pull back property as the second.

Simple computations give two immediate consequences of this assumption. Namely, for any  $\alpha_1, \alpha_2 \in \wedge^{\ell} T^* F'_s$ ,

$$\mathbf{3.7.} \quad p_f^* \alpha_1 \wedge * p_f^* \alpha_2 = d \text{vol}_{\text{vert}} \wedge p_f^* (\alpha_1 \wedge *' \alpha_2),$$

and

$$\mathbf{3.8.} \quad d \text{vol}_{\text{vert}} \wedge p_f^* \alpha_1 \wedge *(d \text{vol}_{\text{vert}} \wedge p_f^* \alpha_2) = d \text{vol}_{\text{vert}} \wedge p_f^* (\alpha_1 \wedge *' \alpha_2).$$

Denote by  $\pi_2 : \mathcal{G} \times B^k \rightarrow B^k$  the projection, and choose a smooth compactly supported  $k$ -form  $\omega$  on  $B^k$  whose integral is 1. We shall refer to such a form as a Bott form on  $B^k$ . Denote by  $e_{\omega}$  the exterior

multiplication by the differential  $k$ -form  $\pi_2^* \omega$  on  $\mathcal{G} \times B^k$ . For  $\xi \in \mathcal{A}_c^*(F'_s, E')$ , we define  $f^{(i, \omega)}(\xi) \in \mathcal{A}_c^*(F_s, E)$  as

$$f^{(i, \omega)}(\xi) = (\pi_{1, *}) \circ e_\omega \circ p_f^*(\xi).$$

The map  $p_f : \mathcal{G} \times B^k \rightarrow \mathcal{G}'$  is a leafwise (for  $F_s \times B^k$ ) submersion extending  $\check{f}$ , so  $p_f^*(\xi)$  is a leafwise form on  $\mathcal{G} \times B^k$  with coefficients in the bundle  $p_f^* E'$ . The map  $\pi_{1, *}$  is integration over the fiber of the projection  $\pi_1 : \mathcal{G} \times B^k \rightarrow \mathcal{G}$  of such forms. In general, the fiber of  $p_f^* E'$  is not constant on fibers of the fibration  $\pi_1 : \mathcal{G} \times B^k \rightarrow \mathcal{G}$ . To correct for this, we use the parallel translation given by the flat structure of  $p_f^* E'$  to identify all the fibers of  $p_f^* E' | z \times B^k$  with  $(p_f^* E')_{(z, 0)} = (\check{f}^* E')_z = (f^* E')_{r(z)}$ . This is well defined because the ball  $B^k \subset \mathbb{R}^k$  is contractible, so parallel translation is independent of the path taken from  $(z, 0)$  to  $(z, t)$  in  $z \times B^k$ .

**Theorem 3.9.** *For any  $a \in \mathbb{Z}$ ,  $f^{(i, \omega)}$  extends to a bounded operator from  $W_a^*(F'_s, E')$  to  $W_a^*(F_s, E)$ .*

*Proof.* For  $\alpha_i \otimes \phi_i \in \mathcal{A}_c^*(F_s, E)$ , where  $\alpha_i$  a leafwise differential form of compact support and  $\phi_i$  is a section of  $E$ , we set

$$(\alpha_1 \otimes \phi_1) \wedge (\alpha_2 \otimes \phi_2) = (\phi_1, \phi_2) \alpha_1 \wedge \alpha_2 \quad \text{and} \quad (\alpha_1 \otimes \phi_1) \wedge *(\alpha_2 \otimes \phi_2) = (\phi_1, \phi_2) \alpha_1 \wedge * \alpha_2,$$

where  $(\cdot, \cdot)$  is the Hermitian metric on  $E$ . Similarly for  $F'_s$  and  $E'$ .

Since  $p_f$  is leafwise uniformly proper,

$$C = \sup_{[\gamma'] \in \mathcal{G}'} \int_{p_f^{-1}([\gamma'])} d \text{vol}_{\text{vert}} < +\infty.$$

Thanks to 3.7, we then have for any  $\alpha \otimes \phi \in \mathcal{A}_c^\ell(\tilde{L}'_{f(x)}, E') = C_c^\infty(\tilde{L}'; \wedge^\ell T^* \tilde{L}'_{f(x)} \otimes E')$ ,

$$\begin{aligned} \|p_f^*((\alpha \otimes \phi)_{f(x)})\|_0^2 &= \int_{\tilde{L}_x \times B^k} (p_f^* \phi, p_f^* \phi) p_f^* \alpha \wedge * p_f^* \alpha = \int_{\tilde{L}_x \times B^k} (p_f^* \phi, p_f^* \phi) d \text{vol}_{\text{vert}} \wedge p_f^*(\alpha \wedge * \alpha) \\ &= \int_{\tilde{L}'_{f(x)}} \left[ \int_{p_f^{-1}([\gamma'])} d \text{vol}_{\text{vert}} \right] (\phi, \phi) \alpha \wedge * \alpha \leq C \int_{\tilde{L}'_{f(x)}} (\phi, \phi) \alpha \wedge * \alpha = C \|\alpha \otimes \phi\|_0^2. \end{aligned}$$

This inequality extends to all  $\xi \in \mathcal{A}_{(2)}^\ell(\tilde{L}'_{f(x)}, E') = W_0^\ell(\tilde{L}'_{f(x)}, E')$ , so  $p_f^*$  extends to a uniformly bounded (i.e. independent of  $x$ ) operator from  $W_0^\ell(\tilde{L}'_{f(x)}, E')$  to  $W_0^\ell(\tilde{L}_x \times B^k, p_f^* E')$ , that is  $p_f^*$  defines a bounded operator from  $W_0^\ell(F'_s, E')$  to  $W_0^\ell(F_s \times B^k, p_f^* E')$ .

Choose a sub-bundle  $\hat{H} \subset TF \oplus TB^k$  so that for each  $L_x$ , it is a horizontal distribution for the submersion  $p_f : L_x \times B^k \rightarrow L'_{f(x)}$ . The map  $(r \times \text{id})_* : TF_s \oplus TB^k \rightarrow TF \oplus TB^k$  is an isomorphism on each fiber, so  $\hat{H}$  determines a sub-bundle  $H$  of  $TF_s \oplus TB^k$ , and  $H | \tilde{L}_x \times B^k$  is a horizontal distribution for the submersion  $p_f : \tilde{L}_x \times B^k \rightarrow \tilde{L}'_{f(x)}$ . Choose a finite collection of leafwise vector fields  $\hat{Y}_1, \dots, \hat{Y}_N$  on  $M'$  which generate  $C^\infty(TF')$  over  $C^\infty(M')$ . Lift these to leafwise (for  $F'_s$ ) vector fields  $Y_1, \dots, Y_N$  on  $\mathcal{G}'$ , and lift these latter to sections of  $H$ , denoted  $X_1, \dots, X_N$ . If  $X^{\text{vert}}$  is a vertical vector field on  $\tilde{L} \times B^k$  with respect to  $p_f$ , then  $i_{X^{\text{vert}}} \circ p_f^* = 0$ . Modulo such vector fields, the  $X_i$  generate  $T\tilde{L} \oplus TB^k$  over  $C^\infty(\tilde{L} \times B^k)$ . In addition  $i_{X_j} \circ p_f^* = p_f^* \circ i_{Y_j}$ . Thus for any  $\xi \in \mathcal{A}_c^\ell(\tilde{L}'_{f(x)}, E')$ , any  $Y_K = Y_{k_1} \wedge \dots \wedge Y_{k_\ell}$ , and any  $j_1, \dots, j_m$ , with  $j_i \in \{1, \dots, N\}$ ,

$$\begin{aligned} \|i_{X_{j_1}} d \cdots i_{X_{j_m}} d(p_f^*(\xi))(Y_K)\|_0 &= \|p_f^*(i_{Y_{j_1}} d \cdots i_{Y_{j_m}} d(\xi)(Y_K))\|_0 \\ &\leq \sqrt{C} \|i_{Y_{j_1}} d \cdots i_{Y_{j_m}} d(\xi)(Y_K)\|_0. \end{aligned}$$

A classical argument then shows that for any  $a \geq 1$ ,  $p_f^*$  extends to a uniformly bounded operator from  $W_a^\ell(\tilde{L}'_{f(x)}, E')$  to  $W_a^\ell(\tilde{L}_x \times B^k, p_f^* E')$ , that is a bounded operator from  $W_a^\ell(F'_s, E')$  to  $W_a^\ell(F_s \times B^k, p_f^* E')$ .

The operator  $e_\omega$  maps  $W_a^\ell(\tilde{L}_x \times B^k, p_f^* E')$  to  $W_a^{k+\ell}(\tilde{L}_x \times B^k, p_f^* E')$  and is uniformly bounded, since  $\omega$  and all its derivatives are bounded. Thus for  $a \geq 0$ ,  $e_\omega \circ p_f^*$  is a bounded operator from  $W_a^\ell(F'_s, E')$  to  $W_a^{k+\ell}(F_s \times B^k, p_f^* E')$ .

For the case of  $s < 0$ , we dualize the argument above. Denote by  $p_{f,*}$  integration of fiber compactly supported forms along the fibers of the submersion  $p_f$ . We claim that for any  $\alpha \in \mathcal{A}_c^{k+\ell}(\tilde{L}_x \times B^k)$ ,

$$\mathbf{3.10.} \quad p_{f,*}\alpha \wedge *'p_{f,*}\alpha \leq C p_{f,*}(\alpha \wedge *\alpha),$$

where as above we identify the oriented volume elements of  $\tilde{L}'_{f(x)}$  at a point with  $\mathbb{R}_+^*$ . Any such  $\alpha$  may be written as  $\alpha = \alpha_1 + \alpha_2$ , where  $p_{f,*}(\alpha_2) = 0$  and  $\alpha_1 = d \text{vol}_{\text{vert}} \wedge \alpha_3$ , with  $\alpha_3 \in C_c^\infty(p_f^*(\wedge^\ell T^* \tilde{L}'_{f(x)}))$ . Then

$$p_{f,*}(\alpha \wedge *\alpha) = p_{f,*}(\alpha_1 \wedge *\alpha_1) + p_{f,*}(\alpha_2 \wedge *\alpha_2) + p_{f,*}(\alpha_1 \wedge *\alpha_2) + p_{f,*}(\alpha_2 \wedge *\alpha_1).$$

The last two terms are zero, since  $\alpha_1 \wedge *\alpha_2 = 0$  as  $d \text{vol}_{\text{vert}} \wedge *\alpha_2 = 0$ , and  $p_{f,*}(\alpha_2 \wedge *\alpha_1) = 0$  since  $\alpha_2 \wedge *\alpha_1$  does not contain  $d \text{vol}_{\text{vert}}$ . Thus

$$p_{f,*}(\alpha \wedge *\alpha) = p_{f,*}(\alpha_1 \wedge *\alpha_1) + p_{f,*}(\alpha_2 \wedge *\alpha_2) \geq p_{f,*}(\alpha_1 \wedge *\alpha_1).$$

But,

$$p_{f,*}\alpha_1 \wedge *'p_{f,*}\alpha_1 = p_{f,*}\alpha \wedge *'p_{f,*}\alpha,$$

so we need only prove 3.10 for  $\alpha = d \text{vol}_{\text{vert}} \wedge \alpha_3$ , with  $\alpha_3 \in C_c^\infty(p_f^*(\wedge^\ell T^* \tilde{L}'_{f(x)}))$ .

Choose a finite collection of sections  $\beta_1, \dots, \beta_r$  of  $\wedge^\ell T^* F'$ , so that  $\beta_i \wedge *' \beta_j = 0$  if  $i \neq j$ , and the  $\beta_i$  generate  $C^\infty(\wedge^\ell T^* F')$  over  $C^\infty(M')$ . Denote also by  $\beta_i$  the lift of these sections to sections of  $\wedge^\ell T^* F'_s$ . Then  $\alpha = d \text{vol}_{\text{vert}} \wedge \alpha_3$  may be written as

$$\alpha = \sum_i g_i d \text{vol}_{\text{vert}} \wedge p_f^* \beta_i,$$

where the  $g_i$  are smooth compactly supported functions on  $\tilde{L}_x \times B^k$ . Now,

$$p_{f,*}\alpha \wedge *'p_{f,*}\alpha = \sum_i p_{f,*}(g_i d \text{vol}_{\text{vert}}) \beta_i \wedge *' \sum_j p_{f,*}(g_j d \text{vol}_{\text{vert}}) \beta_j = \sum_i [p_{f,*}(g_i d \text{vol}_{\text{vert}})]^2 \beta_i \wedge *' \beta_i.$$

Thanks to 3.8,

$$\begin{aligned} p_{f,*}(\alpha \wedge *\alpha) &= p_{f,*}\left(\sum_i (g_i d \text{vol}_{\text{vert}} \wedge p_f^* \beta_i) \wedge * \sum_j (g_j d \text{vol}_{\text{vert}} \wedge p_f^* \beta_j)\right) = \\ &= p_{f,*}\left(\sum_{i,j} g_i g_j d \text{vol}_{\text{vert}} \wedge p_f^* (\beta_i \wedge *' \beta_j)\right) = \sum_i p_{f,*}(g_i^2 d \text{vol}_{\text{vert}}) \beta_i \wedge *' \beta_i \geq \\ &= \sum_i \frac{[p_{f,*}(g_i \cdot 1 d \text{vol}_{\text{vert}})]^2}{p_{f,*}(1 d \text{vol}_{\text{vert}})} \beta_i \wedge *' \beta_i \geq \frac{1}{C} \sum_i [p_{f,*}(g_i d \text{vol}_{\text{vert}})]^2 \beta_i \wedge *' \beta_i = \frac{1}{C} p_{f,*}\alpha \wedge *' p_{f,*}\alpha, \end{aligned}$$

proving 3.10. Note that the second to last inequality is just Cauchy-Schwartz.

Thus for all  $\alpha \in \mathcal{A}_c^{k+\ell}(\tilde{L}_x \times B^k)$ ,

$$\|p_{f,*}\alpha\|_0^2 = \int_{\tilde{L}'_{f(x)}} p_{f,*}\alpha \wedge *'p_{f,*}\alpha \leq C \int_{\tilde{L}'_{f(x)}} p_{f,*}(\alpha \wedge *\alpha) = C \int_{\tilde{L}_x \times B^k} \alpha \wedge *\alpha = C \|\alpha\|_0^2.$$

Using the facts that  $p_{f,*}$  commutes with the de Rham differentials,  $p_{f,*} \circ i_{X_{\text{vert}}} = 0$  and  $i_{Y_j} \circ p_{f,*} = p_{f,*} \circ i_{X_j}$ , it is easy to deduce, just as for  $p_f^*$ , that for any  $a \geq 0$ ,  $p_{f,*} \circ e_\omega$  extends to a uniformly bounded operator (say with bound  $C_a$ ) from  $W_a^\ell(\tilde{L}_x \times B^k, p_f^* E')$  to  $W_a^\ell(\tilde{L}'_{f(x)}, E')$ . Now suppose that  $\xi' \in W_a^\ell(\tilde{L}'_{f(x)}, E')$  for some  $a < 0$ , and recall that  $\|(e_\omega \circ p_f^*)(\xi')\|_a$  is given by

$$\|(e_\omega \circ p_f^*)(\xi')\|_a = \sup_\xi \frac{|\langle \xi', (p_{f,*} \circ e_\omega)(\xi) \rangle|}{\|\xi\|_{-a}} \leq \sup_\xi \frac{\|\xi'\|_a \|(p_{f,*} \circ e_\omega)(\xi)\|_{-a}}{\|\xi\|_{-a}} \leq C_a \|\xi'\|_a,$$

where the supremums are taken over all  $\xi \in W_{-a}^\ell(\tilde{L}_x \times B^k, p_f^* E')$ . Thus for any  $a < 0$  (and so for all  $a \in \mathbb{Z}$ ),  $e_\omega \circ p_f^*$  is a uniformly bounded operator from  $W_a^\ell(\tilde{L}'_{f(x)}, E')$  to  $W_a^{k+\ell}(\tilde{L}_x \times B^k, p_f^* E')$ , so  $e_\omega \circ p_f^*$  is a bounded operator from  $W_a^\ell(F'_s, E')$  to  $W_a^\ell(F_s \times B^k, p_f^* E')$ .

For all  $a \in \mathbb{Z}$ , the image of  $e_\omega \circ p_f^*$  consists of  $\pi_1$ -fiber compactly supported distributional forms. The argument above for  $p_{f,*}$  applied to  $\pi_{1,*}$  shows that it is uniformly bounded as a map from  $\text{Im}(e_\omega \circ p_f^*) \subset W_a^{k+\ell}(\tilde{L}_x \times B^k, p_f^* E')$  to  $W_a^\ell(\tilde{L}_x, E)$ . Thus for all  $a \in \mathbb{Z}$ ,  $f^{(i,\omega)}$  extends to a bounded operator from  $W_a^\ell(F'_s, E')$  to  $W_a^\ell(F_s, E)$ .  $\square$

We now consider the action of  $f^{(i,\omega)}$  on the leafwise reduced  $L^2$  cohomology of the foliations  $F_s$  and  $F'_s$ , denoted  $H_{(2)}^*(F_s, E)$ , and  $H_{(2)}^*(F'_s, E')$ .  $H_{(2)}^*(F_s, E)$  is the field of Hilbert spaces over  $M$ , where for  $x \in M$ ,  $H_{(2)}^*(F_s, E)_x = H_{(2)}^*(\tilde{L}_x, E)$ , the reduced  $L^2$  cohomology of  $\tilde{L}_x$  with coefficients in the leafwise flat bundle  $E$ . Recall that  $\mathcal{A}_c^*(\tilde{L}_x, E)$  is the space of differential forms on  $\tilde{L}_x$  with compact support and with coefficients in  $E|_{\tilde{L}_x}$ . Because of the flatness of  $E|_{\tilde{L}_x}$ , the usual exterior derivative extends to  $d_k : \mathcal{A}^k(\tilde{L}_x, E) \rightarrow \mathcal{A}^{k+1}(\tilde{L}_x, E)$ , which further extends to  $d_{k,(2)} : W_0^k(\tilde{L}_x, E) \rightarrow W_0^{k+1}(\tilde{L}_x, E)$  on the  $L^2$  completions. Then  $H_{(2)}^k(\tilde{L}_x, E)$  is the kernel of  $d_{k,(2)}$  modulo the closure of the image of  $d_{k-1,(2)}$ . Similarly for  $F'_s$  and  $E'$ . As  $\omega$  is closed,  $e_\omega$  commutes with de Rham differentials. The image of  $e_\omega \circ p_f^*$  is contained in the  $\pi_1$ -fiber compactly supported forms, so  $f^{(i,\omega)} = \pi_{1,*} \circ e_\omega \circ p_f^*$  commutes with de Rham differentials. It follows immediately that the extension of  $f^{(i,\omega)}$  to the  $L^2$  forms also commutes with the closures of the de Rham differentials, so  $f^{(i,\omega)}$  induces a well defined map  $\tilde{f}^* : H_{(2)}^*(F'_s, E') \rightarrow H_{(2)}^*(F_s, E)$ . As remarked above, the properties of this map (using this definition) are not immediately obvious, e.g. its independence of  $i$  and  $\omega$ . To deal with this problem, we now switch our point of view to that in [HL91], and give another construction of the map  $\tilde{f}^*$ .

Let  $K = \bigcup_{\tilde{L}} K_{\tilde{L}}$  be a bounded leafwise triangulation of  $F_s$  (see [HL91]) induced from a bounded leafwise

triangulation to  $F$ . Then  $K_{\tilde{L}}$  is a triangulation of the leaf  $\tilde{L}$ , so that the volumes and diameters of the simplices of dimension  $\geq 1$  are uniformly bounded away from zero. These triangulations vary measurably transversely. A simplicial  $k$ -cochain  $\varphi$  on  $K_{\tilde{L}}$  with coefficients in  $E$  assigns to each  $k$ -simplex  $\sigma$  of  $K_{\tilde{L}}$  an element  $\varphi(\sigma) \in E_\sigma$ , the fiber of  $E$  over the barycenter of  $\sigma$ . To define the co-boundary map  $\delta$ , we identify  $E_\sigma$  with the fibers of  $E$  over the barycenters of the simplices in the boundary of  $\sigma$  using the flat structure of  $E$ . This is well defined since  $\sigma$  is contractible. Denote by  $\mathcal{C}_{(p)}^k(K_{\tilde{L}}, E)$  the space of simplicial  $k$ -cochains  $\varphi$  on  $K_{\tilde{L}}$  with coefficients in  $E$  such that

$$\sum_{\sigma \text{ } k\text{-simplex of } K_{\tilde{L}}} (\varphi(\sigma), \varphi(\sigma))^{p/2} < +\infty.$$

The homology of the complex  $(\mathcal{C}_{(p)}^*(K_{\tilde{L}}, E), \delta)$  is the  $\ell^p$  cohomology of the simplicial complex  $K_{\tilde{L}}$  with coefficients in  $E$ . It is denoted  $H_{\Delta,p}^*(\tilde{L}, E)$ . Denote by  $\mathcal{A}_{(p)}^*(\tilde{L}, E)$  the  $L^p$  forms on  $\tilde{L}$  with coefficients in  $E$ . The classical Whitney and de Rham maps extend to well defined chain morphisms

$$W : \mathcal{C}_{(p)}^*(K_{\tilde{L}}, E) \rightarrow \mathcal{A}_{(p)}^*(\tilde{L}, E) \text{ and } \int : \mathcal{A}_{(p)}^*(\tilde{L}, E) \rightarrow \mathcal{C}_{(p)}^*(K_{\tilde{L}}, E),$$

which induce bounded isomorphisms in cohomology (which are inverses of each other), with bounds independent of  $\tilde{L}$ , for  $p = 1, 2$ . See [HL91] for  $p = 2$ , and [GKS88] for  $p = 1$ . As above, to define these maps, we use the classical definitions coupled with the fact that for any point  $x \in \sigma$ , the flat structure of  $E|_\sigma$  gives a natural isomorphism between  $E_x$  and  $E_\sigma$ .

Let  $f_{K,K'} : K_{\tilde{L}} \rightarrow K'_{\tilde{L}}$  be an oriented leafwise simplicial approximation of  $\tilde{f}$  as in [HL91]. It is uniformly proper, so it defines a pull-back map  $f_{\Delta}^*$  on  $\ell^p$  cochains with coefficients in  $E'$ , which commutes with the coboundaries. The induced map on cohomology is also denoted  $f_{\Delta}^*$ . Set  $f_D^* = W \circ f_{\Delta}^* \circ \int$

**Proposition 3.11.**  $\tilde{f}^* = f_D^* : H_{(2)}^*(F'_s, E') \rightarrow H_{(2)}^*(F_s, E)$ .

*Proof.* As  $B^k$  is a finite CW-complex, the map  $p_f$  induces the well defined map

$$p_{f,\Delta}^* : H_{\Delta,2}^*(\tilde{L}', E') \rightarrow H_{\Delta,2}^*(\tilde{L} \times B^k, p_f^* E').$$

Denote by  $\beta$  the simplicial  $k$  cocycle  $\int \omega$  on  $B^k$ , and by  $\pi_2 : \tilde{L} \times B^k \rightarrow B^k$  a simplicial approximation (after suitable subdivisions) of the projection. We choose the subdivision fine enough so that the cup product by the bounded  $k$  cocycle  $\pi_2^* \beta$  induces the well defined map

$$[\beta] \cup : \mathbf{H}_{\Delta,2}^*(\tilde{L} \times B^k, p_f^* E') \rightarrow \mathbf{H}_{\Delta,2,c}^{*+k}(\tilde{L} \times B^k, p_f^* E'),$$

where  $\mathbf{H}_{\Delta,2,c}^*(\tilde{L} \times B^k, p_f^* E')$  denotes the  $\ell^2$  simplicial cohomology of cochains which are zero on any simplex that intersects the boundary of  $\tilde{L} \times B^k$ , that is ‘‘fiber compactly supported’’ cocycles. Cap product with the fundamental cycle  $[B^k]$  of  $B^k$  gives the map

$$\cap[B^k] : \mathbf{H}_{\Delta,2,c}^{*+k}(\tilde{L} \times B^k, p_f^* E') \rightarrow \mathbf{H}_{\Delta,2}^*(\tilde{L}, E).$$

Denote by  $\mathbf{H}_{(2),c}^*(\tilde{L} \times B^k, p_f^* E')$  the cohomology of  $L^2$  forms which are zero on some neighborhood of the boundary  $\tilde{L} \times B^k$ . Note that  $\mathbf{H}_{\Delta,2}^*(\tilde{L} \times B^k, p_f^* E')$  is a module over  $\mathbf{H}_{\Delta,2}^*(\tilde{L} \times B^k)$ ,  $\mathbf{H}_{(2)}^*(\tilde{L} \times B^k, p_f^* E')$  is a module over  $\mathbf{H}_{(2)}^*(\tilde{L} \times B^k)$ , and  $\cap[B^k] : \mathbf{H}_{\Delta,2,c}^{*+k}(\tilde{L} \times B^k, p_f^* E') \rightarrow \mathbf{H}_{\Delta,2}^*(\tilde{L}, E)$  is defined. Then the following diagram commutes.

$$\begin{array}{ccccccc} \mathbf{H}_{\Delta,2}^*(\tilde{L}', E') & \xrightarrow{p_{f,\Delta}^*} & \mathbf{H}_{\Delta,2}^*(\tilde{L} \times B^k, p_f^* E') & \xrightarrow{[\beta] \cup} & \mathbf{H}_{\Delta,2,c}^{*+k}(\tilde{L} \times B^k, p_f^* E') & \xrightarrow{\cap[B^k]} & \mathbf{H}_{\Delta,2}^*(\tilde{L}, E) \\ \downarrow W & & \downarrow W & & \downarrow W & & \downarrow W \\ \mathbf{H}_{(2)}^*(\tilde{L}', E') & \xrightarrow{p_f^*} & \mathbf{H}_{(2)}^*(\tilde{L} \times B^k, p_f^* E') & \xrightarrow{[\omega] \wedge} & \mathbf{H}_{(2),c}^{*+k}(\tilde{L} \times B^k, p_f^* E') & \xrightarrow{\pi_{1,*}} & \mathbf{H}_{(2)}^*(\tilde{L}, E). \end{array}$$

Since  $p_f$  is a smooth submersion, it defines the bounded operator  $p_f^* : \mathbf{H}_{(2)}^*(\tilde{L}', E') \rightarrow \mathbf{H}_{(2)}^*(\tilde{L} \times B^k, p_f^* E')$ , and  $W \circ p_{f,\Delta}^* = p_f^* \circ W$  by the naturality of the Whitney map. The square in the middle commutes because  $W$  is compatible with cup and wedge products in cohomology and  $W[\beta] = [\omega]$ . Finally the RHS square is commutative because  $W$  is compatible with cap products, and integration over the fibers of  $\pi_1$  is exactly cap product by the fundamental class in homology of  $B^k$ .

The bottom line of this diagram is  $\tilde{f}^*$ , so we need only show that

$$W \circ \cap[B^k] \circ [\beta] \cup \circ p_{f,\Delta}^* \circ W^{-1} = f_D^* = W \circ f_{\Delta}^* \circ \int.$$

As  $W^{-1} = \int$ , this reduces to showing that

$$\cap[B^k] \circ [\beta] \cup \circ p_{f,\Delta}^* = f_{\Delta}^*.$$

The zero section  $i : \tilde{L} \hookrightarrow \tilde{L} \times B^k$  induces

$$i_{\Delta}^* : \mathbf{H}_{\Delta,2}^*(\tilde{L} \times B^k, p_f^* E') \rightarrow \mathbf{H}_{\Delta,2}^*(\tilde{L}, E),$$

and the projection  $\pi_1 : \tilde{L} \times B^k \rightarrow \tilde{L}$  induces

$$\pi_{1,\Delta}^* : \mathbf{H}_{\Delta,2}^*(\tilde{L}, E) \rightarrow \mathbf{H}_{\Delta,2}^*(\tilde{L} \times B^k, p_f^* E').$$

These maps satisfy

$$\pi_{1,\Delta}^* \circ i_{\Delta}^* = \text{id}_{\mathbf{H}_{\Delta,2}^*(\tilde{L} \times B^k, p_f^* E')}.$$

Thus we have

$$([\beta] \cup) \circ p_{f,\Delta}^* = ([\beta] \cup) \circ \pi_{1,\Delta}^* \circ i_{\Delta}^* \circ p_{f,\Delta}^* = ([\beta] \cup) \circ \pi_{1,\Delta}^* \circ f_{\Delta}^*.$$

By the Thom Isomorphism Theorem,  $([\beta] \cup) \circ \pi_{1,\Delta}^* : \mathbf{H}_{\Delta,2}^*(\tilde{L}, E) \rightarrow \mathbf{H}_{\Delta,2,c}^{*+k}(\tilde{L} \times B^k, p_f^* E')$  is an isomorphism whose inverse is precisely  $\cap[B^k]$ .  $\square$

**Theorem 3.12.** *The map  $\tilde{f}^* : \mathbf{H}_{(2)}^*(F'_s, E') \rightarrow \mathbf{H}_{(2)}^*(F_s, E)$  on leafwise reduced  $L^2$  cohomology induced by  $f^{(i,\omega)}$  does not depend on the choices of  $i$  and  $\omega$ . If  $f_1$  and  $f_2$  are leafwise homotopy equivalent, then  $\tilde{f}_1^* = \tilde{f}_2^*$ . If  $g : (M', F') \rightarrow (M, F)$  is a leafwise homotopy inverse for  $f$ , then  $\tilde{g}^* \circ \tilde{f}^* = \text{id}$  and  $\tilde{f}^* \circ \tilde{g}^* = \text{id}$ , so  $\tilde{f}^*$  is an isomorphism, with inverse  $\tilde{g}^*$ .*

*Proof.* For any choice of  $i$  and  $\omega$ ,  $\tilde{f}^* = f_D^*$ , so they are all the same. The other properties of  $\tilde{f}^*$  follow from these same properties for  $f_D^*$  which are easy to prove using classical arguments.  $\square$

#### 4. AN APPLICATION

The results of this paper are crucial for the proof of the main theorem of [BH11], that the higher harmonic signature,  $\sigma(F, E)$ , of a  $2\ell$  dimensional oriented Riemannian foliation  $F$  of a compact Riemannian manifold  $M$ , twisted by a leafwise flat complex bundle  $E$  over  $M$ , is a leafwise homotopy invariant. This result has important consequences for the Novikov conjecture for groups and for the Baum–Connes Novikov conjecture for foliations. See [BH11] for the details.

Recall that  $\sigma(F, E)$  is defined as follows. Assume that  $E$  admits a non-degenerate *possibly indefinite* Hermitian metric which is preserved by the leafwise flat structure. Recall the generalized deRham operator  $d_{k,(2)} : W_0^k(\tilde{L}_x, E) \rightarrow W_0^{k+1}(\tilde{L}_x, E)$ . The metric on  $\tilde{L}_x$  and the leafwise flat bundle  $E$  determine leafwise adjoints for the  $d_{k,(2)}$ , so also leafwise Laplacians  $\Delta_x^E$ , and Hodge  $*$  operators on the  $W_0^*(\tilde{L}_x, E)$ . The Hodge operator determines an involution which commutes with  $\Delta_x^E$ , so it splits as a sum  $\Delta_x^E = \Delta_x^{E,+} + \Delta_x^{E,-}$ , in particular in dimension  $\ell$ ,  $\Delta_{x,\ell}^E = \Delta_{x,\ell}^{E,+} + \Delta_{x,\ell}^{E,-}$ . Consider the bundles  $\text{Ker}(\Delta_\ell^{E,\pm}) = \cup_{x \in M} \text{Ker}(\Delta_{x,\ell}^{E,\pm})$  over  $M$ , whose fibers are the (in general, infinite dimensional) Hilbert spaces  $\text{Ker}(\Delta_{x,\ell}^{E,+})$  and  $\text{Ker}(\Delta_{x,\ell}^{E,-})$ . We assume that these bundles are smooth, that is the Schwartz kernels of the projections of  $W_0^\ell(\tilde{L}_x, E)$  onto the  $\text{Ker}(\Delta_{x,\ell}^{E,\pm})$  vary smoothly in  $x$ . This is true in many cases: if the preserved metric on  $E$  is positive definite; if  $E$  is a bundle associated to the normal bundle of the foliation; if  $E$  is a trivial bundle, so for the untwisted leafwise signature operator. There is a Chern–Connes character  $\text{ch}_a$  for the  $\text{Ker}(\Delta_\ell^{E,\pm})$  which takes values in the Haefliger cohomology of  $F$ , [BH08]. The higher harmonic signature of  $F$  is defined as

$$\sigma(F, E) = \text{ch}_a(\text{Ker}(\Delta_\ell^{E,+})) - \text{ch}_a(\text{Ker}(\Delta_\ell^{E,-})).$$

The main theorem of [BH11] is the following.

**Theorem 4.1.** *Suppose that  $M$  is a compact Riemannian manifold, with an oriented Riemannian foliation  $F$  of dimension  $2\ell$ , and that  $E$  is a leafwise flat complex bundle over  $M$  with a (possibly indefinite) non-degenerate Hermitian metric which is preserved by the leafwise flat structure. Assume that the bundles  $\text{Ker}(\Delta_\ell^{E,\pm})$  are smooth. Then  $\sigma(F, E)$  is a leafwise homotopy invariant.*

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