Coarse Cohomology for Families

James L. Heitsch [*]and Steven Hurder [†]
Department of Mathematics (m/c 249)
University of Illinois at Chicago
851 S. Morgan Street
Chicago Illinois 60607-7045 USA
heitsch@math.uic.edu, hurder@uic.edu

November 4, 1998
(final revision, May 9, 2000)

Abstract

We introduce coarse cohomology for families of metric spaces and develop the properties of this cohomology theory. Foliations provide the primary examples of such families and for these there are de Rham, Čech and Alexander-Spanier versions of this theory, which are all isomorphic. We compute the coarse cohomology for a number of important examples.

[*]Partially supported by NSF Grant DMS 9400676
[†]Partially supported by NSF Grant DMS-9704768

1 Introduction

Coarse cohomology for metric spaces, introduced by Roe [R1, R2, R3] provides a new approach to the topological study of non-compact Riemannian manifolds. Coarse cohomology theory evolved from the considerations of the index theory of geometric (Dirac-type) operators on open complete manifolds, and has yielded surprising new "index" invariants beyond the usual index. In this paper, we extend coarse cohomology theory to parametrized families of metric spaces.

The leaves of a foliation $\mathcal{F}$ of a compact manifold M have a complete metric whose coarse isometry class is an invariant of the transverse pseudo-group [P, HK], and the family of metric spaces given by the holonomy covering of the leaves viewed as a family over M provides a natural application of this work. The coarse cohomology of a foliation $\mathcal{F}$ of a compact manifold M depends only on the coarse geometry of the leaves and the topological properties of the graph $\mathcal{G}_{\mathcal{F}}$ of the foliation viewed as a generalized coarse fibration. The coarse cohomology theory for foliations has applications to defining new differential invariants of foliations; for example, the secondary de Rham cohomology classes of a foliation [HK] admit a natural pairing with its coarse cohomology, yielding new families of invariants. Coarse cohomology of foliations also appears as a natural setting for the cohomological invariants of leafwise-elliptic differential operators associated to the spectrum in a neighborhood of 0, both of index and spectral flow invariants for families of geometric operators. These applications of the theory will be developed in subsequent papers.

There are several aspects of the coarse cohomology theory for parametrized families of metric spaces that are novel. First is the discovery that while Roe's theory associates coarse cohomology invariants to the subspace of continuous functions on the manifold with a condition on their discrete gradients, coarse cohomology for families is more naturally defined using the full de Rham complex of forms along the fibers (or leaves) with a gradient decay condition. A second discovery is the bivariant nature of the theory, involving cohomology of the parameter space, and uniformly bounded homology along the fibers. This is illustrated in numerous examples calculated in this paper.

We begin with a general definition of the coarse cohomology of a continuous family of metric spaces, and develop some of its basic properties. We show that coarse cohomology is a fiberwise coarse invariant, i.e. it depends only on the coarse type of the elements of the family. This fact implies that the coarse cohomology of a foliation is invariant under leafwise surgery, and that it is the same as the coarse cohomology of the associated transverse groupoid.

As our main interest and the applications we have in mind are in the field of foliations, we then restrict our attention to them. For foliations, we can define three more coarse theories, the de Rham, the Čech and the Alexander-Spanier theories. They are all isomorphic to our original theory. When the graph of the foliation is a fiber bundle, there is a spectral sequence which converges to the coarse cohomology. Its E_2 term is the cohomology of the ambient manifold with coefficients in the Roe coarse cohomology of the holonomy cover of a leaf. (This is a special case of a more general result which calculates coarse cohomology using sheaves [He3]). We use this spectral sequence to compute the coarse cohomology of several important classes of foliations. We then show that coarse cohomology for foliations is a leafwise homotopy invariant. Finally, we construct a canonical map from the coarse cohomology to the usual cohomology of the graph and we give conditions under which this map is an isomorphism.

We note that our construction of coarse cohomology for a foliation makes essential use of the Hausdorff property of the holonomy groupoid. This appears immediately in the semi-continuity assumption in section 2, and subsequently in the properties of the coarsening map for basic open sets. It is possible that coarse cohomology for foliations can be defined for foliations whose graphs are non-Hausdorff, as the authors do not know of non-technical obstacles to such an extension of this work. But this will likely require yet another approach to the theory, using for example, a dual definition of coarse homology in terms of foliation currents.

It is a pleasure to thank A. K. Bousfield for helpful conversations.

2 Parametrized families of metric spaces

A parametrized family of metric spaces or *metric family* is a common generalization to the setting of coarse geometry of both the "balanced product" from topology and the *metric holonomy groupoid* of a foliation.

A *parametrized family of metric spaces*, or more briefly a *metric family*, is a 4-tuple $\mathcal{F} = \{\mathcal{G}, d, \pi, M\}$ consisting of a paracompact Hausdorff space $\mathcal{G}$, a metric space M, a continuous map $\pi\colon \mathcal{G} \to M$, and a family of fiberwise metrics $\{d_x \mid x \in M\}$ satisfying certain continuity and properness conditions. For each $x \in M$, the subspace $\mathcal{G}_x = \pi^{-1}(x)$ is endowed with is a metric d_x so that the metric topology on $\mathcal{G}_x$ coincides with the induced topology from $\mathcal{G}$, and $\mathcal{G}_x$ is a proper metric space. By proper we mean that the closure of any bounded subset of $\mathcal{G}_x$ is compact. In particular, this implies that each $\mathcal{G}_x$ is a complete metric space.

The map $\pi\colon \mathcal{G} \to M$ need not be a fibration, and without an additional continuity hypothesis on the family of metrics $\{d_x \mid x \in M\}$, there can exists a variety of pathologies in the relation between the topology of $\mathcal{G}$ and the fiberwise metrics $\{d_x \mid x \in M\}$. The condition we impose is a "semi-fibration" property on π, motivated by examples arising from foliations: Given a point $x \in M$ and a bounded set $K_x \subset \mathcal{G}_x$ then there exists an open neighborhood $U_x \subset M$ of x and a continuous map $\varphi\colon K_x \times U_x \to \mathcal{G}$ satisfying:

- φ is a homeomorphism onto its image.
- $\varphi(K_x \times \{x\}) = K_x$ and $\pi(\varphi(K_x \times \{y\})) = y$ for all $y \in U_x$ (i.e., φ is a fiberwise map.)
- For each $y \in U_x$ let $\varphi_y\colon K_x \to \mathcal{G}$ be the restriction of φ, and $\varphi_y^* d$ the pull-back metric induced on K_x. Then the the family of metrics $\{\varphi_y^* d \mid y \in U_x\}$ on K_x is continuous; that is, the induced map $\varphi^* d\colon K_x \times K_x \times U_x \to [0, \infty)$ is continuous.

If $\pi\colon \mathcal{G} \to M$ is actually a smooth fibration with compact fibers, and the metrics $\{d_x\}$ are obtained from the restrictions of a Riemannian metric on $\mathcal{G}$, then we can choose $K_x = \mathcal{G}_x$ which will satisfy these conditions. If $\pi\colon \mathcal{G} \to M$ is a smooth fibration but with non-compact fibers, then the choice of any complete Riemannian metric on $\mathcal{G}$ will again result in fiberwise metrics satisfying these conditions. Given a bounded set $K_x \subset \mathcal{G}_x$ the open set $U_x \subset M$ and fiberwise map $\varphi\colon K_x \times U_x \to \mathcal{G}$ exist by the fibration assumption, and the continuity property above is automatically satisfied.

More delicate examples arise from considering the metric holonomy groupoids of foliations. Recall, the holonomy groupoid of a foliation $\mathcal{F}$ of a manifold M is a topological space $\mathcal{G}_{\mathcal{F}}$ equipped with source and range maps to $s, r: \mathcal{G}_{\mathcal{F}} \to M$. [Hae1, Hae2]), The fibers $s^{-1}(x)$ and $r^{-1}(x)$ over a point $x \in M$ are diffeomorphic to the holonomy covering $\widetilde{L_x}$ of the leaf L_x through the point. Thus, the holonomy groupoid is not in general a fibration over M, as the holonomy covers $\widetilde{L_x}$ and $\widetilde{L_y}$ of distinct leaves $L_x \neq L_y$ are typically not even homeomorphic. Nonetheless, there is a local product structure for foliations: given a compact set $K \subset L_x$ in a leaf L_x which is leafwise homotopic to a point, there is an open neighborhood V_K of K so that the restriction of $\mathcal{F}$ to V_K is a product foliation with each leaf of V_K diffeomorphic to K. This implies the semi-fibration property for the map $s: \mathcal{G}_{\mathcal{F}} \to M$, where the metrics on the fibers of $\mathcal{G}_{\mathcal{F}}$ are induced by the choice of a complete Riemannian metric on M.

The metric equivalence relation obtained from the holonomy groupoid of a foliation (described in [HK]) yields the most general form of a parametrized family of metric spaces arising in a geometric context. The semi-fibration property of the holonomy groupoid is fundamental to many studies of the relationship between the geometry and analysis of foliations (cf. [He2, Hu1, Hu2, HK]).

In the study of dynamical systems and representation theory, there is a basic concept of a Borel equivalence relation, which yields a Borel measure space $\mathcal{R}$ and a Borel map $\pi: \mathcal{R} \to M$ where M is the topological space of units for $\mathcal{R}$ (cf. [FM]). Depending on the context, the fibers $\pi^{-1}(x)$ may be manifolds, and thus have well-defined cohomology groups associated to each fiber. This was first described by Mackey in [M] (see also the discussion of leafwise cohomology in Moore-Schochet [MS].) However, many pathologies can arise from this construction, as the dependence of the fiberwise metrics on M is only assumed to be Borel. This suffices for the original motivation for these constructions, which were introduced to abstract a class of constructions of *von Neumann algebras*. The best that can then be asserted is that the correspondence $x \mapsto H^*(\pi^{-1}(x))$ is a Borel function of $x \in M$, but there is no *a priori* relation between the cohomology groups of the fibers, even at the cochain level. The semi-fibration property we impose in our definition of a parametrized family of metric spaces implies the existence of a spectral sequence relating the (coarse) cohomologies of the fibers and that of the base M and total space $\mathcal{G}$.

3 Coarse Cohomology for Families

In this section we define the coarse cohomology of a parametrized family of metric spaces, and show that the coarse cohomology is a fiberwise coarse invariant, i.e. it depends only on the coarse type of the elements of the family.

Roe's elegant approach to defining coarse cohomology for metric spaces is to introduce *anti-Čech* systems of coverings, where successive coverings are "coarser" rather than "finer", and then consider the limiting cohomology associated to such a system. We also adopt this approach, and begin by introducing the coarsening map on fibers.

Let $\mathcal{F} = \{\mathcal{G}, d, \pi, M\}$ be a given parametrized family of metric spaces. Given any subset $A \subset \mathcal{G}$, denote by $Pen(A, r)$ the set of points $y \in \mathcal{G}$ so that $x = \pi(y) \in \pi(A)$, and the distance in $\mathcal{G}_x$ from y to $A \cap \mathcal{G}_x$ is less than r. This has the effect of replacing a given set $A \subset \mathcal{G}$ with a fiberwise expansion to include all points a distance less than r along fibers from A.

Lemma 1 *If $U \subset \mathcal{G}$ is open, then for all $r > 0$, $Pen(U, r)$ is open.*

Proof: Fix $r > 0$. It suffices to show that each $x \in U$ has an open neighborhood $V \subset U$ such that $Pen(V, r)$ is open. Let $W_x \subset \mathcal{G}_x$ be a bounded open neighborhood of x contained in U. As $\mathcal{G}_x$ is complete, the closure $K_x = \overline{Pen(W_x, r+1)}$ is compact. By the semi-fibration property, there exists an open neighborhood $U_x \subset M$ of $\pi(x)$ and a homeomorphism onto its image, $\varphi: K_x \times U_x \to \mathcal{G}$, such that $\varphi(K_x \times \{x\}) = K_x$ and $\pi(\varphi(K_x \times \{y\})) = y$ for all $y \in U_x$. Moreover, the family of metrics $\{\varphi_x^* d \mid x \in U_x\}$ on K_x is continuous. It follows that for U_x sufficiently small, $Pen(\varphi(W_x \times U_x), r) \subset \varphi(K_x \times U_x)$. The image $V = \varphi(W_x \times U_x) \subset \mathcal{G}$ is an open neighborhood of x, and $Pen(V, r)$ will be open as the induced family of metrics on $K_x \times U_x$ is continuous. $\square$

Let $\widetilde{\mathcal{U}}$ be a locally finite open cover of $\mathcal{G}$. For $U \in \widetilde{\mathcal{U}}$, set $U(n) = Pen(U, n)$. Now consider the system of open covers of $\mathcal{G}$ given by $\widetilde{\mathcal{U}}(n) = \{U(n) \mid U \in \widetilde{\mathcal{U}}\}$.

Lemma 2 *For each $n > 0$ the open cover $\widetilde{\mathcal{U}}(n)$ of $\mathcal{G}$ is locally finite.*

Proof: Let $K \subset \mathcal{G}$ be a compact subset, and $K(n) = \overline{Pen(K, n)}$ be the closure of its penumbra. Then $K(n)$ is also compact, as $\pi(K) \subset M$ is compact and by the proper hypothesis on the fiberwise metrics $\{d_x\}$ each fiber $K(n)_x = K(n) \cap \pi^{-1}(x)$ is compact. Let $\{U_\alpha(n) \mid \alpha \in \mathcal{A}\}$ be the collection of open sets in $\widetilde{\mathcal{U}}(n)$ with non-empty intersection with K. Then the collection $\{U_\alpha \mid \alpha \in \mathcal{A}\}$ in $\widetilde{\mathcal{U}}$ also all have non-empty intersection with the compact set $K(n)$, hence $\mathcal{A}$ must be a finite set. $\square$

For each open cover $\widetilde{\mathcal{U}}(n)$ let $\check{\mathrm{C}}_{\mathrm{c}}^*(\widetilde{\mathcal{U}}(\mathrm{n}), \boldsymbol{R})$ denote its Čech cochain complex with compact support. A cochain $\omega \in \check{\mathrm{C}}_{\mathrm{c}}^{\mathrm{q}}(\widetilde{\mathcal{U}}(\mathrm{n}), \boldsymbol{R})$ assigns to an ordered $(q+1)$-tuple $(U_0, \ldots, U_q) \in \widetilde{\mathcal{U}}(n)^{q+1}$ a section $\omega(U_0, \ldots, U_q) \in \Gamma(U_0 \cap \cdots \cap U_q)$. Compact support means that ω is non-zero for at most finitely many $(q+1)$-tuples in $\widetilde{\mathcal{U}}(n)^{q+1}$. The cochains in $\check{\mathrm{C}}_{\mathrm{c}}^*(\widetilde{\mathcal{U}}(\mathrm{n}), \boldsymbol{R})$ naturally define linear maps on the locally-finite chains of $\widetilde{\mathcal{U}}(n)^{q+1}$. If the condition of compact support is omitted, then the resulting cochain complex $\check{\mathrm{C}}^*(\widetilde{\mathcal{U}}(\mathrm{n}), \boldsymbol{R})$ yields the usual Čech cohomology group of the cover $\widetilde{\mathcal{U}}(n)$. (For more discussion on this point, see Chapter 3, [R3].)

We use the system of open covers $\widetilde{\mathcal{U}}(n)$ to define the inverse system of Čech cochain complexes with compact support $\check{\mathrm{C}}_{\mathrm{c}}^*(\widetilde{\mathcal{U}}(\mathrm{n}), \boldsymbol{R})$ whose inverse limit $\check{\mathrm{C}}\mathrm{X}_{\mathrm{c}}^*(\widetilde{\mathcal{U}}, \boldsymbol{R})$ we call the coarse cochain complex of the cover $\widetilde{\mathcal{U}}$. We denote its cohomology by $\mathcal{H}X^*(\widetilde{\mathcal{U}})$.

If $\widetilde{\mathcal{V}}$ is a cover of $\mathcal{G}$ which refines $\widetilde{\mathcal{U}}$ and $\lambda: \widetilde{\mathcal{V}} \to \widetilde{\mathcal{U}}$ is a refining map, then λ induces refining maps $\lambda: \widetilde{\mathcal{V}}(n) \to \widetilde{\mathcal{U}}(n)$ and so a cochain map $\lambda^*: \check{\mathrm{C}}\mathrm{X}_{\mathrm{c}}^*(\widetilde{\mathcal{U}}, \boldsymbol{R}) \to \check{\mathrm{C}}\mathrm{X}_{\mathrm{c}}^*(\widetilde{\mathcal{V}}, \boldsymbol{R})$. The usual proof shows that the induced map $\lambda^*: \mathcal{H}X^*(\widetilde{\mathcal{U}}) \to \mathcal{H}X^*(\widetilde{\mathcal{V}})$ is independent of the choice of refining map λ.

Definition 3 The Coarse Cohomology *of the metric family $\mathcal{F}$ is the direct limit,*

$$\mathcal{H}X^*(\mathcal{F}) = \varinjlim_{\widetilde{\mathcal{U}}} \mathcal{H}X^*(\widetilde{\mathcal{U}})$$

over all locally finite open covers $\widetilde{\mathcal{U}}$ of $\mathcal{G}$.

As cohomology commutes with direct limits, $\mathcal{H}X^*(\mathcal{F}) \cong H^*(\varinjlim_{\widetilde{\mathcal{U}}} \check{\mathrm{C}}\mathrm{X}_{\mathrm{c}}^{\mathrm{k}}(\widetilde{\mathcal{U}}, \boldsymbol{R}))$.

Recall that a map $f: X \to Y$ of topological spaces is *proper* if for every compact set $K \subset Y$ the preimage $f^{-1}(K)$ has compact closure in X. We say that f is a *Borel* map if for every Borel subset $Z \subset Y$ the preimage $f^{-1}(Z)$ is Borel.

Definition 4 *Let $\mathcal{F}_1 = (\mathcal{G}_1, d_1, \pi_1, M_1)$ and $\mathcal{F}_2 = (\mathcal{G}_2, d_2, \pi_2, M_2)$ be two metric families. A leafwise map $\Phi: \mathcal{F}_1 \to \mathcal{F}_2$ consists of a continuous map $\phi: M_1 \to M_2$ and a proper Borel map $\widetilde{\phi}: \mathcal{G}_1 \to \mathcal{G}_2$ so that $\phi \cdot \pi_1 = \pi_2 \cdot \widetilde{\phi}$. In addition,*

1. *for all $R > 0$ there is $S_R > 0$ so that if $x_1, y_1 \in \mathcal{G}_1$ with $\pi_1(x_1) = \pi_1(y_1) = x$ and $d_{1,x}(x_1, y_1) < R$, then $d_{2,\phi(x)}(\widetilde{\phi}(x_1), \widetilde{\phi}(y_1)) < S_R$*
2. *there is $N > 0$ so that for any $y \in \mathcal{G}_1$ and any neighborhood V of $\widetilde{\phi}(y)$, there is a neighborhood U of y so that $\widetilde{\phi}(U) \subset Pen(V, N)$.*

Proposition 5 *If $\Phi: \mathcal{F}_1 \to \mathcal{F}_2$ is a leafwise map, then it induces a well defined map*

$$\Phi^*: \mathcal{H}X^*(\mathcal{F}_2) \to \mathcal{H}X^*(\mathcal{F}_1).$$

Proof: Let $\widetilde{\mathcal{U}}_2$ be a cover of $\mathcal{G}_2$ as above. For $y \in \mathcal{G}_1$ with $\widetilde{\phi}(y) \in V \in \widetilde{\mathcal{U}}_2$, let $U(V, y)$ be a neighborhood of y with $\widetilde{\phi}(U(V, y)) \subset V(N)$. Let $\widetilde{\mathcal{U}}_1$ be an open locally finite refinement of $\{U(V, y)\}$. There is a map $\lambda: \widetilde{\mathcal{U}}_1 \to \widetilde{\mathcal{U}}_2$ so that for all $U \in \widetilde{\mathcal{U}}_1, \widetilde{\phi}(U) \subset \lambda(U)(N)$. Now suppose that $y \in U(n)$. Then there is a $y_1 \in U$ so that $d_{1,x}(y, y_1) < n$ so $d_{2,\phi(x)}(\widetilde{\phi}(y), \widetilde{\phi}(y_1)) < S_n$, and as $\widetilde{\phi}(y_1) \in \lambda(U)(N)$, $\widetilde{\phi}(y) \in \lambda(U)(N + S_n)$. Now since $\widetilde{\phi}$ is a proper map, we have a well defined induced map for each n

$$\Phi^*: \check{\mathrm{C}}^*_{\mathrm{c}}(\widetilde{\mathcal{U}}_2(\mathrm{N}+\mathrm{S_n}), \boldsymbol{R}) \to \check{\mathrm{C}}^*_{\mathrm{c}}(\widetilde{\mathcal{U}}_1(\mathrm{n}), \boldsymbol{R}).$$

We may assume that $n \mapsto S_n$ is a non-decreasing map and taking the inverse limit over n we obtain the map

$$\Phi^*: \check{\mathrm{C}}^*_{\mathrm{c}}(\widetilde{\mathcal{U}}_2, \boldsymbol{R}) \to \check{\mathrm{C}}^*_{\mathrm{C}}(\widetilde{\mathcal{U}}_1, \boldsymbol{R}).$$

As above we have that the map $\Phi^*: \mathcal{H}X^*(\widetilde{\mathcal{U}}_2) \to \mathcal{H}X^*(\widetilde{\mathcal{U}}_1)$ is independent of the choice of the map λ. Standard techniques on direct limits now finish the proof of the proposition. $\square$

Definition 6 *We say that two leafwise maps $\Phi_1, \Phi_2: \mathcal{F}_1 \to \mathcal{F}_2$ are close if $\phi_1 = \phi_2$ and there is $S > 0$ so that for all $y \in \mathcal{G}_1$, $d_{2,\phi_1(x)}(\widetilde{\phi}_1(y), \widetilde{\phi}_2(y)) < S$.*

Definition 7 *We say that two metric families $\mathcal{F}_1$ and $\mathcal{F}_2$ are leafwise coarsely equivalent if there are leafwise maps $\Phi_1 : \mathcal{F}_1 \to \mathcal{F}_2$ and $\Phi_2: \mathcal{F}_2 \to \mathcal{F}_1$ so that both $\Phi_1 \cdot \Phi_2$ and $\Phi_2 \cdot \Phi_1$ are close to the identity.*

Theorem 8 *Close leafwise maps of metric families induce the same map in coarse cohomology. Leafwise coarsely equivalent metric families have isomorphic coarse cohomology.*

Proof: The second statement follows immediately from the first.

Lemma 9 (See [R3] section 3.5) *Let $\mathcal{F} = (\mathcal{G}, d, \pi, M)$ be a metric family. Let (K, d_K) be a compact metric space. Set $\mathcal{F} \times K = (\mathcal{G} \times K, \max\{d, d_K\}, \pi \cdot \rho, M)$, where $\rho: \mathcal{G} \times K \to \mathcal{G}$ is the projection. Then the natural projection $\Pi: \mathcal{F} \times K \to \mathcal{F}$ induces an isomorphism on coarse cohomology. Thus, the inclusion $\mathcal{F} \to \mathcal{F} \times K$ corresponding to any point in K induces the inverse isomorphism.*

Proof: Let $\widetilde{\mathcal{U}}$ be a locally finite open cover of $\mathcal{G}$. Then $\widetilde{\mathcal{U}} \times K = \{U \times K \mid U \in \widetilde{\mathcal{U}}\}\}$ is a locally finite open cover of $\mathcal{G} \times K$. In addition, the incidence data for the two covers $\widetilde{\mathcal{U}}$ and $\widetilde{\mathcal{U}} \times K$ are the same. It follows that for all n, $\mathcal{H}_c^k(\widetilde{\mathcal{U}}(n), \boldsymbol{R}) = \mathcal{H}_c^k((\widetilde{\mathcal{U}} \times K)(n), \boldsymbol{R})$. Thus $\varprojlim_n \mathcal{H}_c^k(\widetilde{\mathcal{U}}(n), \boldsymbol{R}) = \varprojlim_n \mathcal{H}_c^k((\widetilde{\mathcal{U}} \times K)(n), \boldsymbol{R})$ and $\varprojlim_n{}^1 \mathcal{H}_c^k(\widetilde{\mathcal{U}}(n), \boldsymbol{R}) = \varprojlim_n{}^1 \mathcal{H}_c^k((\widetilde{\mathcal{U}} \times K)(n), \boldsymbol{R})$. The Five Lemma and Proposition 5 give that $\mathcal{H}X^k(\widetilde{\mathcal{U}}) = \mathcal{H}X^k(\widetilde{\mathcal{U}} \times K)$. Taking direct limits we have that

$$\mathcal{H}X^*(\mathcal{F}) = \varinjlim_{\widetilde{\mathcal{U}}} \mathcal{H}X^*(\widetilde{\mathcal{U}} \times K).$$

Let $\widetilde{\mathcal{V}}$ be any locally finite open cover of $\mathcal{G} \times K$. As $\mathcal{G} \times K$ has the product topology, ρ is an open map and hence $\widetilde{\mathcal{U}} = \{\rho(V) \mid V \in \widetilde{\mathcal{V}}\}$ is a locally finite open cover of $\mathcal{G}$. For any $N > \text{diameter}(K)$, $\widetilde{\mathcal{V}}(N) = (\widetilde{\mathcal{U}} \times K)(N)$. Thus $\varprojlim_n \mathcal{H}_c^k(\widetilde{\mathcal{V}}(n), \boldsymbol{R}) = \varprojlim_n \mathcal{H}_c^k((\widetilde{\mathcal{U}} \times K)(n), \boldsymbol{R})$, and we have the Lemma. □

The proof of Theorem 8 now follows by an application of Lemma 9. Let $\Phi_1, \Phi_2: \mathcal{F}_1 \to \mathcal{F}_2$ be close. There is a leafwise map $\Psi: \mathcal{F}_1 \times \{1, 2\} \to \mathcal{F}_2$ with $\Psi \cdot i_1 = \Phi_1$, and $\Psi \cdot i_2 = \Phi_2$ where $i_1, i_2: \mathcal{F}_1 \to \mathcal{F}_1 \times \{1, 2\}$ are the obvious inclusions. By Lemma 9, i_1 and i_2 induce the same map in coarse cohomology. □

Finally, note that we have immediately from the definition of $\mathcal{H}X^*(\widetilde{\mathcal{U}})$

Proposition 10 *Let $\widetilde{\mathcal{U}}$ be a locally finite open cover of $\mathcal{G}$. Then there is a short exact sequence*

$$0 \mapsto \varprojlim_n{}^1 \mathcal{H}_c^{k-1}(\widetilde{\mathcal{U}}(n), \boldsymbol{R}) \to \mathcal{H}X^k(\widetilde{\mathcal{U}}) \to \varprojlim_n \mathcal{H}_c^k(\widetilde{\mathcal{U}}(n), \boldsymbol{R}) \to 0.$$

As direct limits preserve exact sequences, we have

Proposition 11 *There is a short exact sequence*

$$0 \mapsto \varinjlim_{\widetilde{\mathcal{U}}}(\varprojlim_n{}^1 \mathcal{H}_c^{k-1}(\widetilde{\mathcal{U}}(n), \boldsymbol{R})) \to \mathcal{H}X^k(\mathcal{F}) \to \varinjlim_{\widetilde{\mathcal{U}}}(\varprojlim_n \mathcal{H}_c^k(\widetilde{\mathcal{U}}(n), \boldsymbol{R})) \to 0.$$

4 Some Examples

Before developing alternate definitions of the coarse cohomology for foliations and their properties, we discuss a series of examples which illustrate some methods of calculation and relations between the coarse cohomology and geometric properties of foliations.

Example 1. Let (L, d) be a proper metric space and $\pi: L \to pt$ the map to a point, then $\mathcal{F} = (L, d, \pi, pt)$ is a metric family and its coarse cohomology is just the coarse cohomology of L as defined by Roe [R3].

An open covering $\mathcal{V}$ of a metric space (L, d) is *bounded* if there exists $R > 0$ so that every open set $U \in \mathcal{V}$ has diameter at most R. Given a bounded covering $\mathcal{V}$ of (L, d), then Roe's definition of the coarse cohomology $HX(L, d)$ of (L, d) is the cohomology of the inverse limit cochain complex $\check{\mathrm{C}}\mathrm{X}^*_{\mathrm{c}}(\mathcal{V}, \boldsymbol{R})$. This cohomology is independent of the choice of initial bounded covering $\mathcal{V}$.

The definition of the coarse cohomology of (L, d) considered as a parametrized family is the cohomology of the direct limit of inverse limit cochain complexes, $\mathcal{H}X^*(\mathcal{F}) \cong H^*(\varinjlim_{\mathcal{U}} \check{\mathrm{C}}\mathrm{X}^{\mathrm{k}}_{\mathrm{c}}(\mathcal{U}, \boldsymbol{R}))$.

Given an arbitrary open covering $\mathcal{U}$ of (L, d) and a bounded covering $\mathcal{V}$ of (L, d), then the intersection $\mathcal{U} \cap \mathcal{V}$ is a bounded refinement of $\mathcal{U}$. Thus, the bounded open covers of (L, d) are cofinal among all open covers, so we see that $\mathcal{H}X^*(\mathcal{F}) \cong HX(L, d)$.

Example 2. Let $\pi: P \to M$ be a smooth fiber bundle, and assume that each fiber $L_x = \pi^{-1}(x)$ over $x \in M$ is compact. Choose a complete Riemannian metric d on P, and let the metric d_x on L_x be that obtained from the restriction of d to L_x. Then each fiber is (L_x, d_x) is a proper metric space. If $K \subset M$ is a compact set, the diameters of the spaces $\{(L_x, d_x) \mid x \in K\}$ are bounded from above by a constant λ_K. Then the coarse cohomology of the metric family $\mathcal{F}$ is just $\check{H}^*_c(M)$, the usual Čech cohomology of M with compact supports.

To see this, first note P is a metric space for the metric obtained from the Riemannian metric d, and the bounded open covers of P are cofinal among all open covers. Thus, it suffices to consider cochains in $\check{\mathrm{C}}\mathrm{X}^{\mathrm{p}}_{\mathrm{c}}(\widetilde{\mathcal{U}}, \boldsymbol{R})$ where $\widetilde{\mathcal{U}}$ is a locally-finite bounded open cover $\widetilde{\mathcal{U}}$ of P. Let $\omega \in \check{\mathrm{C}}\mathrm{X}^{\mathrm{p}}_{\mathrm{c}}(\widetilde{\mathcal{U}}, \boldsymbol{R})$, then define the support of ω to be the union of all open sets U_i such that ω is non-zero on some $(p+1)$-tuple with U_i as an element. As ω has compact support, there are only finitely many such, and each has bounded diameter, hence there is a compact set $K_\omega \subset M$ such that the support of ω is contained in $\pi^{-1}(K_\omega)$. For $n > \lambda_{K_\omega}$ each open set $U_i(n) \in \widetilde{\mathcal{U}}(n)$ is of the form $U_i(n) = \pi^{-1}(\pi(U_i))$. Define an open covering $\mathcal{V}$ of M whose elements have the form $\pi(U)$ for $U \in \widetilde{\mathcal{U}}$. It follows that $\check{\mathrm{C}}\mathrm{X}^*_{\mathrm{c}}(\widetilde{\mathcal{U}}, \boldsymbol{R}) \cong \check{\mathrm{C}}^*_{\mathrm{c}}(\mathcal{V}, \boldsymbol{R})$. Taking cohomology and passing to the direct limit over covers $\widetilde{\mathcal{U}}$ yields the result.

Note that the hypothesis that P is a fibration was not essentially used in the above argument; we required only that $\pi: P \to M$ is an open map between smooth manifolds, and that the diameter of the fibers L_x is a continuous function of the basepoint $x \in M$. Here is an interesting example suggested by the referee which illustrates this more general case. For the plane $\boldsymbol{R}^2$ let $\pi: \boldsymbol{R}^2 \to [0, \infty)$ be the polar coordinate fibration, associating to $(x, y) \in \boldsymbol{R}^2$ its magnitude $\pi(x, y) = \sqrt{x^2 + y^2}$. For $r > 0$ the fiber $\pi^{-1}(r)$ is a circle of circumference $2\pi r$, while $\pi^{-1}(0)$ is a singleton. The above arguments show that $\mathcal{H}X^*(\mathcal{F}) = \check{H}^*_c([0, \infty))$ which is trivial.

Example 3. If F is a foliation of a compact manifold M whose graph $\mathcal{G}_F$ is Hausdorff, then $\mathcal{F} = (\mathcal{G}_F, d, s, M)$ is a metric family where $s: \mathcal{G}_F \to M$ is the source map and the metric d_x is the metric on the holonomy cover $\mathcal{G}_x = \widetilde{L}_x$ of the leaf through $x \in M$ induced from any choice of metric on M. We refer to this metric family as the metric family associated to the foliation F. For more on this see Section 5 below. As noted there, compactness is not essential and can be replaced by: let F be a foliation of a Riemannian manifold so that all its leaves with the induced metric are proper metric spaces.

A foliation of a compact manifold whose graph is compact is called a generalized Seifert fibration. Equivalently, F has a compact graph if all leaves of F are compact, and the leaf space M/F with the quotient topology is a Hausdorff space [EMS]. The same argument as in Example 2 above shows that for a generalized Seifert fibration F, $\mathcal{H}X^*(\mathcal{F}) = \check{H}^*_c(M)$. Note that there are examples of foliations of compact manifolds by compact leaves whose graphs are not compact; see [S, Vo1, Vo2, Vo3]. It is a very interesting problem to calculate $\mathcal{H}X^*(\mathcal{F})$ for such foliations.

Example 4. Leafwise surgery on a foliation does not change its coarse cohomology. In particular, let M be a compact m dimensional manifold with foliation F of codimension q. Let $N^{m-q} \times D^k \times S^{q-k}$ be an embedded submanifold of M such that for each $x \in N$, the image of $\{x\} \times D^k \times S^{q-k}$ is contained in a leaf. If $k = q$ we require that there is a uniform bound on the leafwise distance between the two copies of D^q. If we perform leafwise surgery on the images of the $D^k \times S^{q-k}$, we obtain a new manifold M_1 and a new foliation F_1. These two foliations are leafwise coarsely equivalent, and thus $\mathcal{H}X^*(\mathcal{F}) \cong \mathcal{H}X^*(\mathcal{F}_1)$.

Example 5. (The transversal groupoid) Let F be a codimension–q foliation of a compact manifold M with Hausdorff holonomy groupoid $\mathcal{G}_F$. Let $\mathcal{U}$ be a finite cover of M for F which is "good" in the sense of [HecH, CC]. For each $U_j \in \mathcal{U}$ with foliation chart $\phi_j: U_j \to (-1,1)^p \times (-1,1)^q$, let $T_j = \phi^{-1}(0 \times (-1,1)^q)$ be the transversal in U_j. The "good" hypotheses implies that the closure $\overline{T}_j$ of each T_j is an embedded copy of the closed disc $[-1,1]^q$, and moreover the transversals $\{\overline{T_j} \mid U_j \in \mathcal{U}\}$ are pairwise disjoint. Denote by $\mathcal{T} = \cup_i \overline{T}_j$ the complete closed transversal associated to $\mathcal{U}$. We may assume that the $\overline{T}_j$ are pairwise disjoint. The *closed transversal groupoid* $\mathcal{T}_F \subset \mathcal{G}_F$ is the preimage of $\mathcal{T} \times \mathcal{T}$ under the map $s \times r: \mathcal{G}_F \to M \times M$. Denote by $s: \mathcal{T}_F \to M$ the map induced from $s: \mathcal{G}_F \to M$. Then, $\mathcal{T}_F$ consists of all the holonomy equivalence classes of paths in $\mathcal{G}_F$ which start and end in the complete closed transversal $\mathcal{T}$. Give $\mathcal{T}_F$ the fiberwise metric d inherited from $\mathcal{G}_F$. That is, for $y_1, y_2 \in \mathcal{G}_F$ with $s(y_1) = s(y_2) = x$ the distance $d_x(y_1, y_2)$ is the length of the shortest geodesic from y_1 to y_2 in the holonomy covering $\widetilde{L_x}$. Then $\mathcal{F}_T = (\mathcal{T}_F, d, s, M)$ is a metric family and it is leafwise coarsely equivalent to the metric family $\mathcal{F} = (\mathcal{G}_F, d, s, M)$, so we have

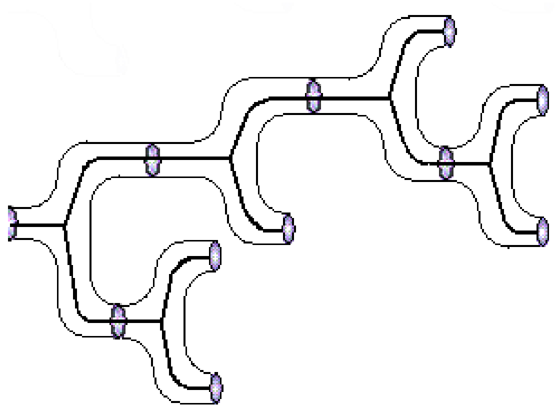
Proposition 12 *If F is a foliation of a compact manifold with Hausdorff graph, then*
$$\mathcal{H}X^*(\mathcal{F}) \cong \mathcal{H}X^*(\mathcal{F}_T).$$

Example 6. (The Hirsch foliation) We give a brief description of the construction, sufficient to sketch the calculation of its coarse cohomology. The Hirsch foliation is described in detail on pages 371–373 of [CC], with illustrations. This example was introduced by Morris Hirsch [Hir1] to show there exist real analytic foliations with only exceptional minimal sets. The example is also well-known for its geometric properties, as it is transversally affine and every leaf has a Cantor set of ends.

Choose an analytic embedding of S^1 in the solid torus $D^2 \times S^1$ so that its image is twice a generator of the fundamental group of the solid torus. Remove an open tubular neighborhood of the embedded S^1. What remains is a three dimensional manifold M_1 whose boundary is two disjoint copies of T^2. $D^2 \times S^1$ fibers over S^1 with fibers the 2-disc. This fibration restricted to M_1 foliates M_1 with leaves consisting of 2-disks with two open subdisks removed.

Now identify the two components of the boundary of M_1 by a diffeomorphism which covers the map $z \mapsto z^2$ of S^1 to obtain the manifold M. Endow M with a Riemannian metric; then the punctured 2-disks foliating M_1 can now be viewed as pairs of pants.

As the foliation of M_1 is transverse to the boundary, the punctured 2-disks assemble to yield a foliation of foliation $\mathcal{F}$ on M, where the leaves without holonomy (corresponding to irrational points for the chosen doubling map of S^1) are infinitely branching surfaces, decomposable into pairs-of-pants which correspond to the punctured disks in M_1.



The leaves with holonomy, corresponding to periodic orbits for the doubling map of S^1, are as above but with a genus one handle attached (see [CC] for details.) In all cases, the holonomy covers of all leaves are diffeomorphic (and in fact are quasi-isometric). As the foliations is analytic, it follows that its graph $\mathcal{G}_F$ is Hausdorff.

We next sketch of the proof that the source map $\pi: \mathcal{G}_F \to M$ is a fibration. Consider the pull-back of $b^*: \widetilde{\mathcal{G}_F} \to M_1$, where $b: M_1 \to M$ is the identification map on the boundary. It is clear that $b^*: \widetilde{\mathcal{G}_F} \to M_1$ is a trivial fibration, where the fiber $\mathcal{G}_{F,x}$ over $x \in M_1$ is obtained by freely attaching pairs-of-pants to the punctured disk containing x. If we call the circle corresponding to the outer boundary of the punctured disk the "waist", and each circle corresponding to boundary of a puncture a "legging", then the identification b must map the waist to a legging. However, as the basepoint $x \in M_1$ is moved along the circumference of M_1, the two punctures follow the embedding of S^1 and so implements an isotopy of the fibers of $b^*: \widetilde{\mathcal{G}_F} \to M_1$ from one legging to another. Thus, the identification of the fibers $\mathcal{G}_{F,x}$ as x transverses the outer boundary of M_1 with the fibers along the inner boundary is a "shift" of the surface $\mathcal{G}_{F,x}$ followed by an isotopy which starts at the identity, rotates through half a turn halfway around the outer boundary of M_1 (at which points the two leggings have been switched) then returns to the identity as we complete the circumnavigation. This defines a global identification of the fibers of $\widetilde{\mathcal{G}_F}$ along the outer boundary of M_1 with the fibers along the inner boundary.

We claim that the fibers of $s: \mathcal{G}_F \to M$ are all coarsely equivalent to a tripartite tree. Let Δ denote the "3-spoked complex" on the pair of pants (as indicated on the picture above.) The gluing map of the boundary of M_1 also identifies the vertices at the end of the branches of Δ, yielding a fibration $s: \mathcal{G}_\delta \to M$ whose fiber $\Gamma_x = s^{-1}(x)$ over $x \in M$ is a tripartite tree. Let $\mathcal{F}_\delta$ be the metric family $\mathcal{F}_\delta = \{\mathcal{G}_\delta, d, s, M\}$ where d is the induced fiberwise metric from the one on $\mathcal{G}_F$. The fiberwise inclusions $\Gamma_x \subset \widetilde{L}_x = s^{-1}(x)$ induce an inclusion $\Phi_i: \mathcal{G}_\delta \to \mathcal{G}_F$. There is also a fiberwise "collapsing map" $\Phi_C: \mathcal{G}_F \to \mathcal{G}_\delta$, induced on each pair of pants by the map which collapses the pair of pants to Δ, and extended to all of the fiber. The composition $\Phi_C \circ \Phi_i$ is the identity, while the composition $\Phi_i \circ \Phi_C$ is close to the identity. Thus both $\mathcal{F}$ and $\mathcal{F}_\delta$ have the same coarse cohomology.

The fibers Γ_x are convex, so given any bounded subset $U \subset \Gamma_x$ there is an $n > 0$ such that its penumbra $U(n)$ is also convex. Moreover, since all the fibers Γ_x are coarsely isometric, there is a function $C(R)$ so that if U has diameter less than R then for $n > C(R)$ then $U(n)$ is convex. It follows that if $\widetilde{\mathcal{U}}$ is a locally finite cover of G_δ with all open sets bounded in diameter by R, and whose projections to M are contractible, then for integers $n > C(r)$, the collection of covers $\{\widetilde{\mathcal{U}}_j(n) \mid U_j \in \widetilde{\mathcal{U}}\}_n$ is a cofinal sequence of Leray covers (i.e. any non-empty intersection of elements of the cover is contractible.) Thus, the inverse limit stabilizes for $n > C(R)$,

$$\lim_{\overleftarrow{m}} \mathcal{H}^k_c(\widetilde{\mathcal{U}}(m), \boldsymbol{R}) \cong \check{H}^k_c(\widetilde{\mathcal{U}}(n), \boldsymbol{R})$$

Taking the direct limit over the open covers $\widetilde{\mathcal{U}}$ we have

$$\mathcal{H}X^*(\mathcal{F}) \cong \mathcal{H}X^*(\mathcal{F}_\delta) \cong \check{H}^*_c(\mathcal{G}_\delta, \boldsymbol{R}).$$

The cohomology group $\check{H}^*_c(\mathcal{G}_\delta, \boldsymbol{R})$ can be calculated using the spectral sequence for the fibration $s: \mathcal{G}_\delta \to M$, where we note that the $E_2^{p,q}$ term is isomorphic to $H^p_c(M; H^q_c(\Gamma_x))$ where the coefficients $H^q_c(\Gamma_x)$ are a module over $\pi_1(M)$. The cohomology group $H^q_c(\Gamma_x)$ is trivial for $q = 0$ or $q > 1$, while $H^1_c(\Gamma_x) \cong C(\epsilon(\Gamma_x))$, the space of continuous functions on the ends of Γ_x. The action of $\pi_1(M)$ on Γ_x on a typical fiber $\mathcal{G}_x$ induces an action on the space of ends $\epsilon(\Gamma_x)$, which is surprisingly complicated. One generator of $\pi_1(M)$ corresponds to a longitude in $\pi_1(M_1)$, and its action on the ends is trivial, as can be seen from the fibration structure described above. On the other hand, $\pi_1(M)$ is an HNN-extension derived from the doubling map on the meridinal generator of $\pi_1(M_1)$, and this group acts on the ends of $\mathcal{G}_x$ leaving only the constant functions invariant.

Example 7. (The double Reeb foliation) Let F be the foliation of $S^1 \times S^2$ which is obtained by gluing together two copies of $S^1 \times D^2$, each with a Reeb foliation. There is one compact leaf which is T^2, and we choose the gluing so that the meridian generates the holonomy on both sides, hence its holonomy cover is $S^1 \times \boldsymbol{R}$ which is coarsely equivalent to $\boldsymbol{R}$. All the other leaves are copies of $\boldsymbol{R}^2$ embedded so that they (and their holonomy covers) are coarsely equivalent to the closed half line $\boldsymbol{R}^+$.

There are obvious collapsing maps (in the sense of coarse geometry) from the holonomy covers to their coarsely equivalent spaces. Denote by $\mathcal{G}_\delta$, the space obtained from $\mathcal{G}_F$ by collapsing the holonomy covers to their coarsely equivalent spaces. Let $\mathcal{F}_\delta$ be the metric family $\mathcal{F}_\delta = \{\mathcal{G}_\delta, d, \pi, M\}$ where d is coarsely equivalent to the metric on $\mathcal{G}_F$ and $\pi: \mathcal{G}_\delta \to M$ is the induced projection. Denote by $\mathcal{F}$ the metric family $\mathcal{F} = \{\mathcal{G}_F, d, \pi, M\}$ corresponding to F. The collapsing map $C: \mathcal{G}_F \to \mathcal{F}_\delta$

defines a leafwise map $\Phi_C = \{C, Id\}$. There is a canonical leafwise map $\Phi_i = \{i, Id\}$, where $i: \mathcal{G}_\delta \to G_F$ is an injection. The composition $\Phi_C \cdot \Phi_i$ is the identity, while the composition $\Phi_i \cdot \Phi_C$ is close to the identity. Thus both $\mathcal{F}$ and $\mathcal{F}_\delta$ have the same coarse cohomology. As above, it is not difficult to see that $\mathcal{G}_\delta$ has a cofinal set $\widetilde{\mathcal{U}}$ of locally finite covers for which the cover $\widetilde{\mathcal{U}}(n)$ is a good cover in the sense of Leray whenever $n > C(R)$ for $\widetilde{\mathcal{U}}$ a bounded (by R) locally-finite covering whose open sets project to contractible open sets in M. Thus, by arguments similar to above,

Proposition 13 *For the double Reeb foliation F, $\mathcal{H}X^*(\mathcal{F}) \cong \check{H}^*_c(\mathcal{G}_\delta, \boldsymbol{R})$.*

The coarse cohomology groups $\mathcal{H}X^*(\mathcal{F})$ of the double Reeb foliation are calculated in [He3].

5 Coarse de Rham Theory

We now restrict our attention to foliations. We shall introduce three more "coarse theories" and show that they are all isomorphic to our original theory. We will use these theories to compute more examples and to further develop the properties of coarse cohomology. We begin with the coarse de Rham theory of a foliation.

Let F be a codimension q foliation of a compact[1] n dimensional manifold M without boundary. Let $\mathcal{U}$ be a fixed finite cover of M by foliation charts. We assume that $\mathcal{U}$ is a so-called *good cover* as in [HecH].

We now present and discuss in greater detail the construction the *holonomy groupoid* $\mathcal{G}_F$ of F as needed to define the coarse de Rham theory of F. (More discussion and other applications can be found in Winkelnkemper [Wi].) A point $y = [\gamma] \in \mathcal{G}_F$ is the equivalence class of a path $\gamma: [0,1] \to M$ whose image is contained in a single leaf L. Such a γ is called a *leafwise path.* Two leafwise paths γ_1 and γ_2 are equivalent provided $\gamma_1(0) = \gamma_2(0)$, $\gamma_1(1) = \gamma_2(1)$ and the holonomy along the two paths is the same on some transversal containing $\gamma_1(0)$. There are natural maps $s, r: \mathcal{G}_F \to M$ defined by $s(y) = \gamma(0)$, and $r(y) = \gamma(1)$. $\mathcal{G}_F$ is a (generally non-Hausdorff) $2n - q$ dimensional manifold with the topology generated by the following sets. Let $y \in \mathcal{G}_F$ and U and V be foliation charts of $s(y)$ and $r(y)$ respectively. We require that U and V each be contained in some element of our fixed finite cover $\mathcal{U}$. Let $\gamma \in y$. Then the set $W = (U, \gamma, V)$ consists of all equivalence classes of leafwise paths which start in U, end in V and which are homotopic to γ through a homotopy of leafwise paths whose end points remain in U and V respectively. We may write

$$W \cong \bigcup_{x \in T} P_x \times P'_{\gamma(x)}$$

where T is a transversal in U, P_x is the plaque in U containing x, $P'_{\gamma(x)}$ is the plaque in V containing $\gamma(x)$, $\gamma: T \to T'$ is the holonomy along γ and T' is a transversal in V. Note that the domain of the holonomy map γ is in general a proper subset of T which is not necessarily contractible. If U and V are elements of $\mathcal{U}$, we say W is a *basic neighborhood* for $\mathcal{G}_F$. Denote by $\widetilde{\mathcal{U}}$ the collection of all such

[1] Compactness is not essential and can be replaced by: let F be a foliation of a Riemannian manifold so that all its leaves with the induced metric are proper metric spaces, and the open covering by good foliation charts has bounded diameters.

basic neighborhoods. The maps s and r are continuous with respect to this topology. M embeds in $\mathcal{G}_F$ by associating to each $x \in M$ the constant path $*x$ at x. For each $x \in M$, $s^{-1}(x) = \widetilde{L}_x$ is the holonomy cover of the leaf L_x of F containing x. These submanifolds form a foliation F_s of $\mathcal{G}_F$.

Note that $\mathcal{G}_F$ is not necessarily Hausdorff, [Wi, Prop. 2.1], in which case the fiberwise metrics do not satisfy our continuity condition. To insure that the associated family $\mathcal{F} = \{\mathcal{G}_F, d, s, M\}$ is a metric family, we henceforth assume that $\mathcal{G}_F$ is Hausdorff.

The Hausdorff property for $\mathcal{G}_F$ has the following consequences. First, any basic neighborhood is relatively compact. Second, the proposition above implies the set of basic neighborhoods $\widetilde{\mathcal{U}}$ forms a locally finite cover for the foliation F_s of $\mathcal{G}_F$. We caution the reader that even if the cover $\mathcal{U}$ is a good cover in the sense of Leray, i.e. any non-empty intersection of elements of the cover is contractible, the cover $\widetilde{\mathcal{U}}$ is not in general a good cover in this sense. See the example in Section 7, where this question is addressed.

A basic neighborhood for $\mathcal{G}_F^\ell = \times_\ell \mathcal{G}_F$ is a product of ℓ basic neighborhoods for $\mathcal{G}_F$. The collection of all these basic neighborhoods is denoted by $\widetilde{\mathcal{U}}^\ell$. This cover is locally finite and each element is relatively compact.

We denote by $\mathcal{G}_\ell$ the submanifold of $\mathcal{G}_F^\ell$ consisting of those points $(y_1, \ldots, y_\ell)$ with $s(y_1) = s(y_j)$ for $j = 2, \ldots, \ell$. $\mathcal{G}_\ell$ has dimension $n + \ell(n-q)$, and we have $s\colon \mathcal{G}_\ell \to M$ given by $s(y_1, \ldots, y_\ell) = s(y_1)$. We also have $M \to \mathcal{G}_\ell$ given by $x \mapsto (*x, \ldots, *x)$. Note that $s^{-1}(x) \cong \widetilde{L}_x \times \ldots \times \widetilde{L}_x$. The topology on $\mathcal{G}_\ell$ is that induced from $\mathcal{G}_F^\ell$ and has the following description. Let $(y_1, \ldots, y_\ell) \in \mathcal{G}_\ell$ and let $W_1, \ldots, W_\ell$ be neighborhoods for $\mathcal{G}_F$ with $y_i \in W_i$. We may assume that $W_i = (U, \gamma_i, V_i)$, i.e. the same U for all i. Let T be a transversal in U, T_i a transversal in V_i, and $\gamma_i\colon T \to T_i$ the holonomy along the path γ_i. Then $W = (W_1 \times \ldots \times W_\ell) \cap \mathcal{G}_\ell$ may be written as

$$W \cong \bigcup_{x \in T} P_x \times P'_{\gamma_1(x)} \times \ldots \times P'_{\gamma_\ell(x)}.$$

We write $W = (U, \gamma_1, V_1, \ldots, \gamma_\ell, V_\ell)$. As above, if U and the $V_i \in \mathcal{U}$, we call W a *basic neighborhood* of $\mathcal{G}_\ell$. Denote the collection of all such basic neighborhoods by $\widetilde{\mathcal{U}}_\ell$, and note that this is a locally finite cover for the foliation whose leaves are $s^{-1}(x), x \in M$. In particular, each element of this cover is relatively compact.

The plaques of our fixed cover $\mathcal{U}$ which lie in a given leaf L_x cover it and their inverse images under r cover $\widetilde{L}_x$. A connected component of the inverse image of a plaque P in M is also called a plaque and it is diffeomorphic to P under r. Define the plaque distance function $D(\cdot,\cdot)$ on $\widetilde{L}_x$ by

$$D(y, y') = \inf_\sigma \{N_F(\sigma)\}$$

Here σ is a path in $\widetilde{L}_x$ with $\sigma(0) = y$, and $\sigma(1) = y'$. $N_F(\sigma)$ is the minimum number of plaques needed to cover the image of σ. Note that D is not a distance function, but it is sufficiently like one to serve our purposes. See [P],[G], [Hu1].

Definition 14 *Given $A \subseteq \mathcal{G}_F^\ell$ and $r > 0$,*

$$Pen(A, r) = \{(y'_1, \ldots, y'_\ell) \mid \text{there is } (y_1, \ldots, y_\ell) \in A \text{ with } s(y_i) = s(y'_i)$$
$$\text{and } D(y_i, y'_i) < r \text{ for } i = 1, \ldots, \ell\ \}$$

Note that if $A \subset \mathcal{G}_\ell$, then for all $r > 0$, $Pen(A, r)$ is actually a subset of $\mathcal{G}_\ell$. The set A we will be interested in is the diagonal Δ_ℓ ($\cong \mathcal{G}_F$) of $\mathcal{G}_F^\ell$.

Proposition 15 *If $A \subset \mathcal{G}_F^\ell$ is relatively compact, than for any $r > 0$, $Pen(A, r)$ is relatively compact.*

Proof: We may assume that $A \subset W$ a basic neighborhood. Then, $Pen(A, r) \subset Pen(W, r)$. As $\widetilde{\mathcal{U}}^\ell$ is locally finite and W is relatively compact, it is easy to see that $Pen(W, r)$ is a union of a finite number of basic neighborhoods and so is relatively compact. □

We next define a bicomplex of forms for F. Denote by $A^{k,\ell}(F)$ the set of all smooth k-forms on $\mathcal{G}_{\ell+1}$. Define $\delta_v: A^{k,\ell}(F) \to A^{k,\ell+1}(F)$ by

$$\delta_v \omega = \sum_{j=0}^{\ell+1} (-1)^j \pi_j^* \omega$$

where the $\pi_j: \mathcal{G}_{\ell+2} \to \mathcal{G}_{\ell+1}$ are the obvious projection maps. Denote by $d: A^{k,\ell}(F) \to A^{k+1,\ell}(F)$ the usual exterior derivative. The cohomology of the bicomplex $\{A^{*,*}(F), \delta_v, d\}$ is just the usual cohomology of M. To see this, let $S: \mathcal{G}_\ell \to \mathcal{G}_{\ell+1}$ be given by $S(y_1, \ldots, y_\ell) = (*s(y_1), y_1, \ldots, y_\ell)$. It is easy to check that for fixed k, S induces a contracting homotopy $S^*: A^{k,\ell}(F) \to A^{k,\ell-1}(F)$ for the complex $\{A^{k,*}(F), \delta_v\}$. Thus the cohomology of the columns of this bicomplex is trivial in positive dimensions. The kernel of $\delta_v: A^{k,0}(F) \to A^{k,1}(F)$ is easily seen to be $A^k(M)$ the differential k-forms on M. The E_1 term of one of the spectral sequences associated to the double complex is thus $E_1^{k,0} = A^k(M)$ and all other terms are zero. The differential $d_1: E_1^{k,0} \to E_1^{k+1,0}$ is just the usual exterior derivative. Thus the E_2 term satisfies $E_2^{k,0} = H^k(M)$ and all other terms are zero. The result is then immediate. To obtain a new theory we must introduce a restriction on the support of the forms.

Denote by $A_c^{k,\ell}(F)$ the subspace of $A^{k,\ell}(F)$ consisting of k-forms ω such that for all $r > 0$, $sup(\omega) \cap Pen(\Delta_{\ell+1}, r)$ is relatively compact. Note that this condition is independent of the finite cover used to define $Pen(\Delta_{\ell+1}, r)$.

Proposition 16 *δ_v and d preserve the subspace $A_c^{*,*}(F)$. In addition, $d^2 = \delta_v^2 = 0$, and $d\delta_v = \delta_v d$.*

Proof: As d does not increase supports, it is clear that it preserves the bicomplex. The facts $d^2 = \delta_v^2 = 0$, and $d\delta_v = \delta_v d$ are easy to check. Thus we need only show that δ_v preserves the bicomplex.

Let $\omega \in A_c^{k,\ell}(F)$. We may assume that the $sup(\omega) \cap Pen(\Delta_{\ell+1}, r)$ is contained in a single basic neighborhood $W = (U, \gamma_0, V_0, \ldots, \gamma_\ell, V_\ell)$. Now consider $sup(\pi_0^*\omega) \cap Pen(\Delta_{\ell+2}, r)$. This is certainly contained in the union of all basic neighborhoods of the form $W' = (U, \gamma, V, \gamma_0, V_0, \ldots, \gamma_\ell, V_\ell)$, where (U, γ, V) is a basic neighborhood such that for some $y \in (U, \gamma, V)$ and some i and $y_i \in (U, \gamma_i, V_i)$, $D(y, y_i) < 2r$. But there are only a finite number of such (U, γ, V) since the cover $\widetilde{\mathcal{U}}$ of $\mathcal{G}_{\mathcal{F}}$ is locally finite. It follows immediately that $\delta_v \omega \in A_c^{k,\ell+1}(F)$. □

We call $\{A_c^{*,*}(F), \delta_v, d\}$ the *coarse de Rham bicomplex of* F, and we call an element $\omega \in A_c^{k,\ell}(F)$ a *coarse de Rham k, ℓ cochain for F.*

Definition 17 *For $p > 0$ set*

$$AX^p(F) = \sum_{k+\ell=p} A_c^{k,\ell}(F)$$

and define $\delta: AX^p(F) \to AX^{p+1}(F)$ by

$$\delta \mid A_c^{k,\ell}(F) = d + (-1)^k \delta_v.$$

The Coarse de Rham Cohomology of F, *denoted $HX^*(F)$, is the cohomology of the complex $\{AX^*(F), \delta\}$.*

We shall prove below, see Theorem 28, that $HX^*(F)$ is isomorphic to $\mathcal{H}X^*(\mathcal{F})$, the coarse cohomology of the metric family associated to F.

We now define the doubly alternating subcomplex of the coarse de Rham bicomplex and show that the inclusion of this subcomplex induces an isomorphism in cohomology. The terminology *doubly alternating* will be explained in Section 7.

Let $\beta \in \mathcal{S}_{\ell+1}$, the symmetric group on $\ell + 1$ elements. Then β acts on $\mathcal{G}_{\ell+1}$ as follows. For each $(y_0, \ldots, y_\ell) \in \mathcal{G}_{\ell+1}$,

$$\beta(y_0, \ldots, y_\ell) = (y_{\beta(0)}, \ldots, y_{\beta(\ell)}).$$

The subspace of *doubly alternating* coarse de Rham cochains is

$$A_{c,a}^{k,\ell}(F) = \{\omega \in A_c^{k,\ell}(F) \mid \beta^*(\omega) = (-1)^\beta \omega\}.$$

Both differentials preserve this subspace of the coarse de Rham cochains so it forms a subcomplex.

Theorem 18 *The inclusion map $i: A_{c,a}^{*,*}(F) \to A_c^{*,*}(F)$ induces an isomorphism in cohomology.*

Proof: We will prove that i induces an isomorphism on the cohomology of the columns of the two bicomplexes, i.e. an isomorphism on the E_1 terms of one of the two spectral sequences associated to the bicomplexes. This implies the theorem. Define an inverse system of covers $\widetilde{\mathcal{U}}(n)$ of $\mathcal{G}_F$ as follows. Let $\mathcal{U}$ be a finite cover of M for F. For $n > 0$ and $W \in \widetilde{\mathcal{U}}$, set $W(n) = Pen(W, 3^n)$, and

$$\widetilde{\mathcal{U}}(n) = \{W(n) \mid W \in \widetilde{\mathcal{U}}\}.$$

Given any open set $U \subset \mathcal{G}_F$, set $U_q = U^q \cap \mathcal{G}_q$. Fix an integer $k \geq 0$, and consider the k^{th} columns of the two bicomplexes. Define the presheaves Γ^q and $\Gamma^{a,q}$ as follows

$$\Gamma^q(U) = A^k(\mathcal{G}_{q+1}) \mid U_{q+1}$$

$$\Gamma^{a,q}(U) = A_a^k(\mathcal{G}_{q+1}) \mid U_{q+1}.$$

Here A^k denotes smooth k-forms and A_a^k the doubly alternating forms. The Alexander-Spanier coboundary δ induces maps of presheaves $\delta: \Gamma^q \to \Gamma^{q+1}$ and $\delta: \Gamma^{a,q} \to \Gamma^{a,q+1}$, and i induces the map of presheaves $i: \Gamma^{a,q} \to \Gamma^q$. Note that $\Gamma^{a,0} = \Gamma^0$, and that for each open set $U \in \mathcal{G}_F$, $\Gamma^0(U) = A^k(\mathcal{G}_F) \mid U$. Define the presheaf $\Gamma = ker(\delta: \Gamma^0 \to \Gamma^1) = ker(\delta: \Gamma^{a,0} \to \Gamma^{a,1})$. Note that for any open set $U \subset \mathcal{G}_F, \Gamma(U) = s^*(A^k(M)) \mid U$. The one inclusion is obvious. The other follows because

$\delta\omega = 0$ on U implies that $\delta\omega = 0$ on $\overline{U}$ which implies that $\omega = s^*(\omega_1)$ for some ω_1 on $s(\overline{U}) = \overline{s(U)}$. But any such ω_1 extends to all of M. In particular, note that for $k > \dim M$, $\Gamma = \mathbf{0}$, the zero presheaf.

We first consider the presheaves Γ^*. Following the proof of Theorem 3.23 of [R3], form the double complex

$$B^{p,q} = \lim_{\underset{n}{\longleftarrow}} C_c^p(\widetilde{\mathcal{U}}(n), \Gamma^q).$$

Roe's proof that the q^{th} row of this bicomplex forms an acyclic resolution of $\lim\limits_{\underset{n}{\longleftarrow}} H_c^0(\widetilde{\mathcal{U}}(n), \Gamma^q)$ carries over to our case word for word. In addition, it is easy to see that

$$\lim_{\underset{n}{\longleftarrow}} H_c^0(\widetilde{\mathcal{U}}(n), \Gamma^q) = A_c^{k,q}(F).$$

We now show that the p^{th} column of $B^{p,q}$ gives an acyclic resolution of $\lim\limits_{\underset{n}{\longleftarrow}} C_c^p(\widetilde{\mathcal{U}}(n), \Gamma)$. It then follows immediately that the cohomology of $\{A_c^{k,*}(F), \delta_v\}$ (i.e. the cohomology of the k^{th} column of $A_c^{*,*}(F)$), is the same as the cohomology of $\lim\limits_{\underset{n}{\longleftarrow}} C_c^*(\widetilde{\mathcal{U}}(n), \Gamma)$. To finish the proof of the theorem, repeat the argument with $\Gamma^{a,*}$ in place of Γ^*. We then have that the cohomology of $\{A_{a,c}^{k,*}(F), \delta_v\}$ is also isomorphic to the cohomology of $\lim\limits_{\underset{n}{\longleftarrow}} C_c^*(\widetilde{\mathcal{U}}(n), \Gamma)$. Because of the naturalness of the construction of these two isomorphisms, it is obvious that i induces an isomorphism on the cohomology of the k^{th} column of the doubly alternating subcomplex to the cohomology of the k^{th} column of the de Rham bicomplex.

We may not use Roe's proof for the columns of $B^{p,q}$ since the presheaves Γ^* do not give a uniform resolution of the presheaf Γ. Instead, we give a direct proof that the p^{th} column gives an acyclic resolution of $\lim\limits_{\underset{n}{\longleftarrow}} C_c^p(\widetilde{\mathcal{U}}(n), \Gamma)$.

Fix p and consider $f = (f_0, f_1, \ldots) \in B^{p,q}$. Denote by $i_n: C_c^p(\widetilde{\mathcal{U}}(n), \Gamma^q) \to C_c^p(\widetilde{\mathcal{U}}(n-1), \Gamma^q)$, the map induced by the natural refinement map $\widetilde{\mathcal{U}}(n-1) \to \widetilde{\mathcal{U}}(n)$. Then each $f_n \in C_c^p(\widetilde{\mathcal{U}}(n), \Gamma^q)$, and satisfies $i_n(f_n) = f_{n-1}$. Define $D: B^{p,q} \to B^{p,q-1}$ as follows. Let $W_0, \ldots, W_p \in \widetilde{\mathcal{U}}$ be such that $s(W_0) \cap \ldots \cap s(W_p) \neq \emptyset$. Choose the smallest positive integer $m = m(W_0, \ldots, W_p)$ such that $W_0(m) \cap \ldots \cap W_p(m) \supset *(s(W_0) \cap \ldots \cap s(W_p))$. Let $s_q: \mathcal{G}_q \to \mathcal{G}_{q+1}$ be given by $s_q(y_1, \ldots, y_q) = (*s(y_1), y_1, \ldots, y_q)$. For any open set $U \subset \mathcal{G}_F$, with $U \supset *s(U)$, this map induces a map also denoted $s_q: U_q \to U_{q+1}$. Now for f as above, set $Bf = (Bf_0, Bf_1, \ldots)$ where if $W_0(n) \cap \ldots \cap W_p(n) \neq \emptyset$ and $n \geq m(W_0, \ldots, W_p)$

$$Bf_n(W_0(n) \cap \ldots \cap W_p(n)) = s_q^*(f_n(W_0(n) \cap \ldots \cap W_p(n))).$$

For $n < m$,

$$Bf_n(W_0(n) \cap \ldots \cap W_p(n)) = s_q^*(f_m(W_0(m) \cap \ldots \cap W_p(m))) \mid [(W_0(n) \cap \ldots \cap W_p(n))]_q.$$

It is straight forward (if somewhat tedious) to show that $Bf \in B^{p,q-1}$ and that $\delta \cdot B + B \cdot \delta = I$. Thus B is a contracting homotopy and we have that $H^q(B^{p,*}) = 0$ for $q > 0$. To finish the proof we note that $H^0(B^{p,*}) = \lim\limits_{\underset{n}{\longleftarrow}} C_c^p(\widetilde{\mathcal{U}}(n), \Gamma)$. □

The proof has as a corollary that the cohomology of the k^{th} columns of both $A_c^{*,*}(F)$ and $A_{a,c}^{*,*}(F)$ are zero provided that $k > \dim\, M$. This is an immediate consequence of the fact that $\Gamma = \mathbf{0}$ in that case. Thus we have the following result.

Proposition 19 *Set $\widehat{A}_c^{k,\ell}(F) = A_c^{k,\ell}(F)$ for $k \leq \dim M$ and $\widehat{A}_c^{k,\ell}(F) = 0$ for $k > \dim M$. Then the natural cochain map $\pi\colon A_c^{*,*}(F) \to \widehat{A}_c^{*,*}(F)$ induces an isomorphism on cohomology. A similar result holds for $A_{a.c}^{*,*}(F)$.*

Proof: π induces an isomorphism on the cohomology of the columns of the two bicomplexes. □

As usual, the complete anti-symmetrization map $\mathcal{A}\colon A_c^{k,\ell}(F) \to A_{c,a}^{k,\ell}(F)$ given by

$$\mathcal{A}(\omega) = \frac{1}{(\ell+1)!}\sum_{\beta}(-1)^{\beta}\beta^{*}\omega$$

is a cochain map and induces the inverse of i in cohomology. We leave the proof of this to the reader.

An alternate definition of the coarse de Rham cohomology of F is given as follows. Choose a metric on M. This induces a metric on each leaf L and so also on $\widetilde{L}$ and $\times_{\ell}\widetilde{L}$ which makes them complete Riemannian manifolds. In addition, their quasi-isometry types are independent of the choice of metric. This follows from the fact that any two metrics on M are quasi-isometric since M is compact. Use these metrics to define $Pen(A, r)$, and so also $A_c^{*,*}(F)$. This is the same bicomplex constructed above from the finite cover $\mathcal{U}$. In particular, it is independent of the metric used to define it.

As an application of this idea, we now extend Roe's definition of coarse cohomology for complete Riemannian manifolds.

Let N be a complete Riemannian manifold. Denote by $A_c^{k,\ell}(N)$ the set of all smooth k-forms ω on $N^{\ell+1}$ such that for all $r > 0$, $sup(\omega) \cap Pen(\Delta_{\ell+1}, r)$ is relatively compact. Here $\Delta_{\ell+1} \cong N$ is the diagonal of $N^{\ell+1}$. As before, we have two differentials d and δ_v, which preserve the bicomplex $A_c^{*,*}(N)$. The proof that d preserves it is the same as before. For the proof that δ_v preserves it, see [R3], or proceed as follows. Choose a discrete subset $Y \subset N$ so that the distance between points of Y is at least 1 and so that the open balls of radius 2 about points of Y cover N. This open cover of N is locally finite, [R3], and we may use it to define $Pen(\Delta_{\ell+1}, r)$. We may now repeat the proof given above for foliated manifolds to show that δ_v preserves the complex. The *extended coarse cohomology of* N is the cohomology of this bicomplex. We denote this cohomology by $HX_e^{*}(N)$. Note that the subcomplex $\{A_c^{0,\ell}(N), \delta_v\}$ is the complex used by Roe to define the coarse cohomology of N, denoted $HX^{*}(N)$.

Theorem 20 *The extended coarse cohomology of N is naturally isomorphic to the coarse cohomology of N.*

Proof: Consider the k^{th} column $A_c^{k,*}(N)$ of the bicomplex. Repeat the proof of Theorem 3.23 of [R3], but for $k > 0$, replace the constant presheaf $\boldsymbol{R}$ by the constant presheaf $\mathbf{0}$, i.e. the presheaf of k-forms on a point. Set Γ^q to be the presheaf

$$\Gamma^q(U) = \{\text{smooth } k\text{-forms on } N^{q+1}\} \mid U^{q+1}.$$

Then

$$\mathbf{0} \mapsto \mathbf{0} \to \Gamma^0 \to \Gamma^1 \to \dots$$

is a uniform resolution of the presheaf $\mathbf{0}$ by fine, flabby ω-sheaves. Roe's theorem then says that for $k > 0$, the cohomology of the k^{th} column of the bicomplex is identically zero. As the cohomology of the zero th column is just $HX^*(N)$, we are done. □

Now assume that $\mathcal{G}_F$ is a fiber bundle over M, and the identifications of fibers under local trivializations can be chosen to be quasi-isometries. Riemannian foliations and foliations constructed from the suspension of locally-free group actions are two examples of foliations which satisfy this condition. Using the fact that $HX^*_e(\widetilde{L}) = HX^*(\widetilde{L})$, the proof for the usual cohomology of fiber bundles carries over to prove the following.

Theorem 21 *Suppose that $\mathcal{G}_F$ is an orientable fiber bundle over M with fiber $\widetilde{L}$. Then there is a spectral sequence which converges to the coarse cohomology of F whose E_2 term is*

$$E_2^{p,q} = H^p(M, HX^q(\widetilde{L})),$$

where the cohomology $H^p(M, HX^q(\widetilde{L}))$ is with twisted coefficients.

We finish this section by noting that the coarse cohomology for foliations has a Mayer-Vietoris sequence with respect to the base. Let U be an open subset of M. We define the coarse cohomology of F over U, denoted $HX^*(F,U)$ to be the cohomology of the bicomplex $A_c^{*,*}(F,U)$ where $A_c^{k,\ell}(F,U)$ consists of all smooth k-forms on the subspace $s_{\ell+1}^{-1}(U) \subset \mathcal{G}_{\ell+1}$ satisfying the coarse support condition. The usual proof (see [BT]) of the exactness of the Mayer-Vietoris sequence also proves the following.

Theorem 22 *Suppose that $M = U \cup V$ where U and V are open. Then there is a long exact sequence*

$$\dots \to HX^q(F) \to HX^q(F,U) \oplus HX^q(F,V) \to HX^q(F, U\cap V) \to HX^{q+1}(F) \to \cdots.$$

6 More Examples

Example 8. The foliations constructed in [He1], which showed that the secondary characteristic classes for foliations are highly non-trivial, have graphs which are orientable fiber bundles over M. The holonomy covers $\widetilde{L}$ have non-positive curvature and are homeomorphic to Euclidean space of a fixed dimension, say ℓ (depending on the particular example). Thus $HX^\ell(\widetilde{L}) \cong \boldsymbol{R}$ and is zero otherwise. See [R3], Theorem 3.42. The spectral sequence of Theorem 21 collapses. The action of $\pi_1(M)$ on the fiber is a translation, but this induces the identity map on the coarse cohomology $HX^\ell(\widetilde{L})$, so there is no twisting. Thus we have $HX^*(F) \cong H^{*-\ell}(M) \cong H_c^*(\mathcal{G}_F)$ the usual cohomology of $\mathcal{G}_F$ with compact supports.

Example 9. The natural foliations of flat bundles M over a compact Riemannian manifold N constructed from *faithful* orientation preserving representations of $\pi_1(N)$ have graphs which are orientable fiber bundles over M. The holonomy covers $\widetilde{L}$ are all isometric to the universal cover $\widetilde{N}$ of N as the holonomy representation is injective. The E_2 term of the spectral sequence in thus $E_2^{p,q} = H^p(M, HX^q(\widetilde{N}))$, and one may use the spectral sequence to compute in specific cases. In particular, if $\widetilde{N}$ is rescaleable, or uniformly contractible, or globally of non-positive curvature then $HX^*(\widetilde{N})$ is naturally isomorphic to $H_c^*(\widetilde{N})$, see [R3], and if there is no twisting, it follows that $HX^*(F)$ is then isomorphic to $H_c^*(\mathcal{G}_F)$.

Example 10. Let Σ_g be a closed orientable surface of genus g. Its fundamental group $\pi_1(\Sigma_g)$ has a set of generators $\alpha_1, \beta_1, \ldots, \alpha_g, \beta_g$ where the elements $\alpha_1, \ldots, \alpha_g$ generate a free subgroup of rank g. Choose rationally independent irrational angles $\theta_1, \ldots, \theta_g$ with $0 < \theta_i < 2\pi$ and define $\phi\colon \pi_1(\Sigma_g) \to SO_2$ by

$$\phi(\beta_i) = I \text{ and } \phi(\alpha_i) = \text{ rotation by } \theta_i.$$

The image of ϕ is a free abelian group of rank g. Form the associated flat bundle

$$M_g = \widetilde{\Sigma_g} \times_\phi S^1$$

and let F be the natural flat foliation. This foliation has no leafwise holonomy and all the leaves, and their holonomy covers, are diffeomorphic and are $\boldsymbol{Z}^g$ covers of Σ_g. They are coarsely equivalent to $\boldsymbol{R}^g$ so $HX^g(\widetilde{L}) = \boldsymbol{R}$ and all other $HX^k(\widetilde{L}) = 0$. This is a Riemannian foliation so $\mathcal{G}_F$ is an oriented fiber bundle over M. As above, the spectral sequence collapses and there is no twisting, so $HX^*(F) = H^*(M_g) \otimes HX^*(\widetilde{L})$. In particular, $HX^{g+3}(F) \neq 0$. This shows that given any positive integer k, there is a codimension one foliation F of a three manifold such that $HX^k(F) \neq 0$.

Example 11. If $\mathcal{G}_F$ is a product bundle, then $HX^*(F) \cong H_c^*(M) \otimes HX^*(\widetilde{L})$. To see this use the spectral sequence coming from the filtration of the bicomplex by the columns. The E_1 term of this spectral sequence is the cohomology of the rows of the bicomplex. The usual proof of the Künneth formula gives that

$$H^*(A_c^{*,\ell}(F), d) \cong H_c^*(M) \otimes H^*(A_c^{*,\ell}(\widetilde{L}), d).$$

One need only check that the maps used preserve the support conditions, but this is easy. The E_1 term is thus $H_c^*(M) \otimes H^*(A_c^{*,\ell}(\widetilde{L}), d)$, and the differential d^1 acts only on the second term of the tensor product and it is the differential of the E_1 term of the spectral sequence which computes $HX_e^*(\widetilde{L})$. That is, the spectral sequence which computes $HX^*(F)$ is just $H_c^*(M)$ tensored with the spectral sequence which computes $HX_e^*(\widetilde{L})$.

7 Coarse Čech and Alexander-Spanier Cohomologies

In this section we define the coarse Čech cohomology of F and show that it is isomorphic to the coarse de Rham cohomology of F. This is essentially an extension and application of Weil's beautiful proof of the de Rham theorem [W]. We then prove the main result of this section, Theorem 28, that the coarse Čech cohomology of a foliation is isomorphic to the coarse cohomology of its associated metric family. Finally, we give the definition of the coarse Alexander-Spanier cohomology of F, which of course is isomorphic to all the other coarse theories of F. We leave the proof of this fact to the reader (another application of [W]).

Recall the fixed finite cover $\mathcal{U}$ of M from section 5, and the associated locally finite cover $\widetilde{\mathcal{U}}_{\ell+1}$ of $\mathcal{G}_{\ell+1}$. As noted above, even if the cover $\mathcal{U}$ is good in the sense of Leray, (i.e. all non empty intersections are contractible), the covers $\widetilde{\mathcal{U}}_{\ell+1}$ are not necessarily good in this sense. A simple example shows this. Let $M = S^1 \times S^2$ where $S^1 \subset \boldsymbol{R}^2$ and $S^2 \subset \boldsymbol{R}^3$ are the standard embeddings as unit vectors. Then M is a foliated S^2 bundle over S^1 which is the suspension of the time-one flow of the vector field on S^2 which is the projection of the vector field $\partial/\partial z$ on R^3. (Since the diffeomorphism of S^2 defined by the time-one flow is isotopic to the identity, the suspension is a product bundle.) Now take any foliation neighborhood U of the point $((1,0),(0,0,-1))$ and any foliation neighborhood V of the point $((1,0),(0,0,1))$. Then it is not hard to see that for any leafwise path γ from U to V which is sufficiently long, the basic neighborhood (U,γ,V) has the homotopy type of S^1.

As it will be essential in this section that the covers $\widetilde{\mathcal{U}}_{\ell+1}$ are good in the sense of Leray, we proceed as follows. Endow $\mathcal{G}_F$ with the Riemannian metric pull-pack via the immersion $s \times r: \mathcal{G}_F \to M \times M$. We replace the cover $\widetilde{\mathcal{U}}$ by a locally finite refinement, also denoted $\widetilde{\mathcal{U}}$, so that every open set $W \in \widetilde{\mathcal{U}}$ is geodesically convex. In particular, we may view the basic neighborhood

$$W = (U,\gamma,V) \cong \bigcup_{x \in T} P_x \times P'_{\gamma(x)},$$

as a product fiber bundle over the open subset $U' = \bigcup_{x \in \mathrm{dom}\gamma} P_x$ of U and we take the metric on W which is the pull back of the metric on U' tensored with the metric on the fibers $P'_{\gamma(x)}$. As each $W \in \widetilde{\mathcal{U}}$ is geodesically convex, it follows that $\widetilde{\mathcal{U}}$ is a good cover in the sense of Leray. In addition, the associated covers $\widetilde{\mathcal{U}}_{\ell+1}$ constructed from the new $\widetilde{\mathcal{U}}$ also consist of sets which are geodesically convex and so these covers are also good in the sense of Leray. By abuse of notation, we will continue to write $W = (U,\gamma,V)$ for W in the new $\widetilde{\mathcal{U}}$, and $W = (U,\gamma_0,V_0,\ldots,\gamma_{\ell+1},V_{\ell+1})$ for W in the new $\widetilde{\mathcal{U}}_{\ell+1}$. We will call any such cover $\widetilde{\mathcal{U}}$ a *convex* cover of $\mathcal{G}_F$.

Recall that $Pen(\Delta_{\ell+1},r)$ is defined using the finite cover $\mathcal{U}$ of M and not $\widetilde{\mathcal{U}}_{\ell+1}$ of $\mathcal{G}_{\ell+1}$. A *coarse Čech k,ℓ cochain f* assigns a real number $f(W_0,\ldots,W_k)$ to each $(W_0,\ldots,W_k) \in \widetilde{\mathcal{U}}_{\ell+1}^{k+1}$ with $W_0 \cap \ldots \cap W_k \neq \emptyset$ so that for each $r > 0$, there are only a finite number of elements $(W_0,\ldots,W_k)$ with $W_0 \cap \ldots \cap W_k \cap Pen(\Delta_{\ell+1},r) \neq \emptyset$, and $f(W_0,\ldots,W_k) \neq 0$.

The vector space of coarse Čech k,ℓ cochains associated to $\widetilde{\mathcal{U}}$ is denoted by $\check{\mathrm{C}}_c^{k,\ell}(\widetilde{\mathcal{U}})$.

We define two differentials $\delta_h: \check{\mathrm{C}}_{\mathrm{c}}^{\mathrm{k},\ell}(\widetilde{\mathcal{U}}) \to \check{\mathrm{C}}_{\mathrm{c}}^{\mathrm{k}+1,\ell}(\widetilde{\mathcal{U}})$ and $\delta_v: \check{\mathrm{C}}_{\mathrm{c}}^{\mathrm{k},\ell}(\widetilde{\mathcal{U}}) \to \check{\mathrm{C}}_{\mathrm{c}}^{\mathrm{k},\ell+1}(\widetilde{\mathcal{U}})$ as follows. For $(W_0,\ldots,W_{k+1}) \in \widetilde{\mathcal{U}}_{\ell+1}^{k+2}$ with $W_0 \cap \ldots \cap W_{k+1} \neq \emptyset$,

$$\begin{aligned} \delta_h f(W_0,\ldots,W_{k+1}) &= \sum_{j=0}^{k+1} (-1)^j f(W_0,\ldots,W_{j-1},W_{j+1},\ldots,W_{k+1}) \\ &= \sum_{j=0}^{k+1} (-1)^j f \cdot \pi_j(W_0,\ldots,W_{k+1}) \end{aligned}$$

where $\pi_j: \widetilde{\mathcal{U}}_{\ell+1}^{k+2} \to \widetilde{\mathcal{U}}_{\ell+1}^{k+1}$ is the map which deletes the j^{th} entry.

To define δ_v, let $\rho_i: \widetilde{\mathcal{U}}_{\ell+2} \to \widetilde{\mathcal{U}}_{\ell+1}$ be the map given by

$$\rho_i(U,\gamma_0,V_0,\ldots,\gamma_{\ell+1},V_{\ell+1}) =$$

$$(U, \gamma_0, V_0, \dots, \gamma_{i-1}, V_{i-1}, \gamma_{i+1}, V_{i+1}, \dots, \gamma_{\ell+1}, V_{\ell+1}).$$

Then

$$\delta_v f(W_0, \dots, W_k) = \sum_{i=0}^{\ell+1} (-1)^i f(\rho_i W_0, \dots, \rho_i W_k).$$

As in Section 5, the fact that the covers $\widetilde{\mathcal{U}}_\ell$ are locally finite implies that δ_v, and δ_h do preserve the complex $\check{\mathrm{C}}_c^{*,*}(\widetilde{\mathcal{U}})$. It is also easy to see that $\delta_v^2 = \delta_h^2 = 0$ and $\delta_v \delta_h = \delta_h \delta_v$.

Definition 23 *For $p = 0, 1, \dots$ set*

$$\check{C}X_D^p(\widetilde{\mathcal{U}}) = \sum_{k+\ell=p} \check{C}_c^{k,\ell}(\widetilde{\mathcal{U}})$$

and define $\delta\colon \check{C}X_D^p(\widetilde{\mathcal{U}}) \to \check{C}X_D^{p+1}(\widetilde{\mathcal{U}})$ by

$$\delta \mid \check{C}_c^{k,\ell}(\widetilde{\mathcal{U}}) = \delta_h + (-1)^k \delta_v.$$

Denote the cohomology of the complex $\{\check{C}X_D^*(\widetilde{\mathcal{U}}), \delta\}$ by $\check{H}X_D^*(\widetilde{\mathcal{U}})$.

If $\widetilde{\mathcal{V}}$ is a convex cover of $\mathcal{G}_F$ which refines $\widetilde{\mathcal{U}}$, then $\widetilde{\mathcal{V}}_\ell$ refines $\widetilde{\mathcal{U}}_\ell$, and given a refining map $\lambda\colon \widetilde{\mathcal{V}} \to \widetilde{\mathcal{U}}$, it defines a refining map $\lambda\colon \widetilde{\mathcal{V}}_\ell \to \widetilde{\mathcal{U}}_\ell$. To see that λ induces a cochain map

$$\lambda^*\colon \check{C}X_D^*(\widetilde{\mathcal{U}}) \to \check{C}X_D^*(\widetilde{\mathcal{V}}),$$

note that any $W \in \widetilde{\mathcal{U}}_\ell$ has compact closure so it has non-trivial intersection with at most a finite number of elements of the locally finite cover $\widetilde{\mathcal{V}}_\ell$. This implies immediately that λ^* preserves the relative compactness condition.

Lemma 24 *The map induced by λ^* on cohomology is independent of λ.*

Proof: Let λ and μ be two refining maps. For fixed ℓ, $K\colon \check{C}_c^{k,\ell}(\widetilde{\mathcal{U}}) \to \check{C}_c^{k-1,\ell}(\widetilde{\mathcal{V}})$, where

$$Kf(W_0, \dots, W_{k-1}) = \sum_{j=0}^{k-1} f(\lambda(W_0), \dots, \lambda(W_j), \mu(W_j), \dots, \mu(W_{k-1})),$$

is a cochain homotopy between the maps $\lambda^*, \mu^*\colon \{\check{C}_c^{*,\ell}(\widetilde{\mathcal{U}}), \delta_h\} \to \{\check{C}_c^{*,\ell}(\widetilde{\mathcal{V}}), \delta_h\}$. The relative compactness of the elements of $\widetilde{\mathcal{U}}_\ell$ again implies that K does preserve the relative compactness condition. Thus, λ and μ induce the same map on the E_1 term of one of the two spectral sequences associated to the bicomplex. It follows immediately (since it is a first quadrant bicomplex) that λ and μ induce the same map from $\check{H}X_D^*(\widetilde{\mathcal{U}})$ to $\check{H}X_D^*(\widetilde{\mathcal{V}})$. □

Definition 25 The Coarse Čech Cohomology *of F is the direct limit,*

$$\check{H}X^*(F) = \varinjlim_{\widetilde{\mathcal{U}}} \check{H}X_D^*(\widetilde{\mathcal{U}}).$$

Theorem 26 *$\check{H}X^*(F)$ is isomorphic to $HX^*(F)$. In particular, for every convex cover $\widetilde{\mathcal{U}}$ of $\mathcal{G}_F$, $\check{H}X^*_D(\widetilde{\mathcal{U}})$ is isomorphic to $HX^*(F)$.*

Proof: For fixed ℓ, note that $\{\check{C}^{*,\ell}_c(\widetilde{\mathcal{U}}), \delta_h\}$ is just the usual Čech complex, and that $\{A^{*,\ell}_c(F), d\}$ is the usual de Rham complex for $\mathcal{G}_{\ell+1}$, (with the addition of the support conditions). Let $\{\phi_{\widetilde{U}} \mid \widetilde{U} \in \widetilde{\mathcal{U}}\}$ be a partition of unity subordinate to $\widetilde{\mathcal{U}}$. Given $W = (\times_{j=0}^{\ell}\widetilde{U}_j) \cap \mathcal{G}_{\ell+1} \in \widetilde{\mathcal{U}}_{\ell+1}$, and $y = (y_0, \ldots, y_\ell) \in W$, set

$$\phi_W(y) = \prod_{j=0}^{\ell} \phi_{\widetilde{U}_j}(y_j).$$

Extend ϕ_W to all of $\mathcal{G}_{\ell+1}$ by making it zero off W. Then $\{\phi_W \mid W \in \widetilde{\mathcal{U}}_{\ell+1}\}$ is a partition of unity subordinate to $\widetilde{\mathcal{U}}_{\ell+1}$. Define $A\colon \check{C}^{k,\ell}_c(\widetilde{\mathcal{U}}) \to A^{k,\ell}_c(F)$ by

$$Af = \sum_{\widetilde{\mathcal{U}}^{k+1}_{\ell+1}} f(W_0, \ldots, W_k)\phi_{W_0}\, d\phi_{W_1} \wedge \ldots \wedge d\phi_{W_k}.$$

It is easy to check that Af is indeed in $A^{k,\ell}_c(F)$ and that $A\delta_h = dA$.

Lemma 27

$$A\delta_v = \delta_v A$$

Proof:

$$\begin{aligned}
A(\delta_v f) &= \sum_{\widetilde{\mathcal{U}}^{k+1}_{\ell+2}} (\delta_v f)(W_0, \ldots, W_k)\phi_{W_0}\, d\phi_{W_1} \wedge \ldots \wedge d\phi_{W_k} \\
&= \sum_{\widetilde{\mathcal{U}}^{k+1}_{\ell+2}} \sum_i (-1)^i f(\rho_i W_0, \ldots, \rho_i W_k)\phi_{W_0}\, d\phi_{W_1} \wedge \ldots \wedge d\phi_{W_k} \\
&= \sum_{\widetilde{\mathcal{U}}^{k+1}_{\ell+1}} \sum_i (-1)^i f(\widehat{W}_0, \ldots, \widehat{W}_k)\pi_i^*[\phi_{\widehat{W}_0}\, d\phi_{\widehat{W}_1} \wedge \ldots \wedge d\phi_{\widehat{W}_k} \sum_{\mathcal{A}} \phi_{a_0}\phi_{a_1}\ldots\phi_{a_k}] \\
&+ \sum_{\widetilde{\mathcal{U}}^{k+1}_{\ell+1}} \sum_i (-1)^i f(\widehat{W}_0, \ldots, \widehat{W}_k)\pi_i^*[\phi_{\widehat{W}_0}\phi_{\widehat{W}_1}\, d\phi_{\widehat{W}_2} \wedge \ldots \wedge d\phi_{\widehat{W}_k} \sum_{\mathcal{A}} \phi_{a_0}\, d\phi_{a_1}\phi_{a_2}\ldots\phi_{a_k}] \\
&+ \\
&\vdots \\
&+ \sum_{\widetilde{\mathcal{U}}^{k+1}_{\ell+1}} \sum_i (-1)^i f(\widehat{W}_0, \ldots, \widehat{W}_k)\pi_i^*[\phi_{\widehat{W}_0}\phi_{\widehat{W}_1} \wedge \ldots \wedge \phi_{\widehat{W}_k} \sum_{\mathcal{A}} \phi_{a_0}\, d\phi_{a_1}\ldots d\phi_{a_k}]
\end{aligned}$$

where $\widehat{W}_i = (\widehat{U}_i, \ldots) \in \widetilde{\mathcal{U}}_{\ell+1}$ and given $(\widehat{W}_0, \ldots, \widehat{W}_k) \in \widetilde{\mathcal{U}}^{k+1}_{\ell+1}$, $\mathcal{A} = \prod_{i=0}^{k} \mathcal{A}_i$ where

$$\mathcal{A}_i = \{\widetilde{U} \in \widetilde{\mathcal{U}} \mid \widetilde{U} = (U, \gamma, V) \text{ and } U \cap \widehat{U}_i \neq \emptyset\}.$$

Note that at any point in $\mathcal{G}_{\ell+2}$, where $\pi_i^*(\phi_{\widehat{W}_0}\ldots\phi_{\widehat{W}_k}) \neq 0$, $\pi_i^*(\sum_{\mathcal{A}_j} \phi_{a_j}) = 1$. Thus the first sum above is just $\delta_v(Af)$, all the succeeding sums are zero, and we have $A\delta_v = \delta_v A$.

Thus A is a map of first quadrant bicomplexes and to show that it induces an isomorphism of the total cohomology, we need only show that A induces isomorphisms on the cohomology of the rows, i.e. on the E_1 term of one of the associated spectral sequences. For that we need only repeat Weil's proof of the de Rham theorem ([W] [BT]) for each row, and the proof goes through because the covers $\widetilde{\mathcal{U}}_{\ell+1}$ are good covers in the sense of Leray. For the ℓ^{th} row, the bicomplex we use is $C^p_c(\widetilde{\mathcal{U}}_{\ell+1}, \Omega^q)$ of p cochains f on the cover $\widetilde{\mathcal{U}}_{\ell+1}$ with values in the q-forms such that for any $r > 0$, there are only finitely many elements $(W_0, \ldots, W_p) \in \widetilde{\mathcal{U}}^{p+1}_{\ell+1}$ so that $W_0 \cap \ldots \cap W_p \cap Pen(\Delta_{\ell+1}, r) \neq \emptyset$, and $f(W_0, \ldots, W_p) \neq 0$. As the map which A induces in the cohomology of a given row is the isomorphism constructed by Weil (at least up to sign), we are done. $\square$

We say that $f \in \check{\mathrm{C}}^{\mathrm{k},\ell}_{\mathrm{c}}(\widetilde{\mathcal{U}})$ is *alternating* if for all $(W_0, \ldots, W_k) \in \widetilde{\mathcal{U}}^{k+1}_{\ell+1}, \alpha \in \mathcal{S}_{k+1}$,

$$f(W_{\alpha(0)}, \ldots, W_{\alpha(k)}) = (-1)^\alpha f(W_0, \ldots, W_k).$$

The alternating cochains form a subcomplex of the Čech cochains and A restricted to this subcomplex has the same image (modulo exact forms) as A. Thus Theorem 26 remains true if we replace the Čech bicomplex by its alternating subcomplex.

For each $\beta \in \mathcal{S}_{\ell+1}$, define a map $\beta: \widetilde{\mathcal{U}}_{\ell+1} \to \widetilde{\mathcal{U}}_{\ell+1}$ as follows. Suppose that $W = (U, \gamma_0, V_0, \ldots, \gamma_\ell, V_\ell)$. Then ,

$$\beta(W) = (U, \gamma_{\beta(0)}, V_{\beta(0)}, \ldots, \gamma_{\beta(\ell)}, V_{\beta(\ell)}).$$

We say that $f \in \check{\mathrm{C}}^{\mathrm{k},\ell}_{\mathrm{c}}(\widetilde{\mathcal{U}})$ is *doubly alternating* if for all $(W_0, \ldots, W_k) \in \widetilde{\mathcal{U}}^{k+1}_{\ell+1}, \alpha \in \mathcal{S}_{k+1}, \beta \in \mathcal{S}_{\ell+1}$,

$$f(\beta(W_{\alpha(0)}), \ldots, \beta(W_{\alpha(k)})) = (-1)^\alpha (-1)^\beta f(W_0, \ldots, W_k).$$

As $\beta^* \phi_W = \phi_{\beta^{-1}(W)}$, it follows that $\beta^*(Af) = (-1)^\beta Af$ if f is doubly alternating, i.e. A maps doubly alternating Čech cochains to doubly alternating de Rham cochains. Essentially the same proof now shows that Theorem 26 still holds if we replace the de Rham and Čech bicomplexes by their subcomplexes of doubly alternating cochains. In particular, the inclusion of the doubly alternating Čech cochains into the complex of all Čech cochains induces an isomorphism in cohomology so each coarse Čech cohomology class has a representative which is doubly alternating.

We now prove that the coarse Čech cohomology of a foliation is isomorphic to the coarse cohomology of the associated metric family.

Theorem 28 *Let F be a foliation of a compact manifold with $\mathcal{G}_F$ Hausdorff, and let $\mathcal{F}$ denote the associated metric family. Then $\mathcal{H}X^*(F)$ is naturally isomorphic to $\mathcal{H}X^*(\mathcal{F})$.*

Proof: Let $\mathcal{U}$ be a finite cover of M for F and $\widetilde{\mathcal{U}}$ an associated convex cover of $\mathcal{G}_F$. We shall prove that $\mathcal{H}X^*(F)$ is naturally isomorphic to $\mathcal{H}X^*(\widetilde{\mathcal{U}})$. The fact that convex covers are cofinal in the set of all locally finite open covers of $\mathcal{G}_F$ then implies the Theorem.

We begin by giving an alternate description of the coarse Čech k, ℓ cochains associated to $\widetilde{\mathcal{U}}$. For $W_j \in \widetilde{\mathcal{U}}^{\ell+1}$, we write $W_j = (U^j_0, \ldots, U^j_\ell)$, and for each $n \geq 1$ and $k, \ell \geq 0$ set

$$\mathcal{W}^{k+1}_{\ell+1}(n) = \{(W_0, \ldots, W_k) \in \times_{k+1} \widetilde{\mathcal{U}}^{\ell+1} \mid \bigcap_{i=0}^{\ell} Pen(U^0_i \cap \ldots \cap U^k_i, n) \neq \emptyset\}.$$

Denote by $\check{\mathrm{C}}_{\mathrm{c}}^{\mathrm{k},\ell}(\mathcal{W}(\mathrm{n}))$ the set of all finitely non-zero maps $g\colon \mathcal{W}_{\ell+1}^{k+1}(n) \to \boldsymbol{R}$. Then the coarse Čech k,ℓ cochains $\check{\mathrm{C}}_c^{k,\ell}(\widetilde{\mathcal{U}})$ are naturally identified with $\varprojlim_n \check{\mathrm{C}}_{\mathrm{c}}^{\mathrm{k},\ell}(\mathcal{W}(\mathrm{n}))$. To see this, note that a cochain $g \in \check{\mathrm{C}}_c^{k,\ell}(\widetilde{\mathcal{U}})$ assigns a real number to each $(W_0,\ldots,W_k) \in \times_{k+1}\widetilde{\mathcal{U}}^{\ell+1}$ with $\bigcap_{i=0}^{\ell} Pen(U_i^0 \cap \ldots \cap U_i^k, n) \neq \emptyset$ for some n. For a given *fixed* n, there are only a finite number of $(W_0,\ldots,W_k) \in \times_{k+1}\widetilde{\mathcal{U}}^{\ell+1}$ with $\bigcap_{i=0}^{\ell} Pen(U_i^0 \cap \ldots \cap U_i^k, n) \neq \emptyset$ for which $g(W_0,\ldots,W_k) \neq 0$.

There are obvious forgetful maps, $\pi_j\colon \mathcal{W}_{\ell+1}^{k+1}(n) \to \mathcal{W}_{\ell+1}^{k}(n)$ and $\rho_i\colon \mathcal{W}_{\ell+1}^{k+1}(n) \to \mathcal{W}_{\ell}^{k+1}(n)$. As above, these may be used to define differentials $\delta_h\colon \check{\mathrm{C}}_{\mathrm{c}}^{\mathrm{k}-1,\ell}(\mathcal{W}(\mathrm{n})) \to \check{\mathrm{C}}_{\mathrm{c}}^{\mathrm{k},\ell}(\mathcal{W}(\mathrm{n}))$ and $\delta_v\colon \check{\mathrm{C}}_{\mathrm{c}}^{\mathrm{k},\ell-1}(\mathcal{W}(\mathrm{n})) \to \check{\mathrm{C}}_{\mathrm{c}}^{\mathrm{k},\ell}(\mathcal{W}(\mathrm{n}))$. These differentials induce the differentials $\delta_h\colon \check{\mathrm{C}}_{\mathrm{c}}^{\mathrm{k}-1,\ell}(\widetilde{\mathcal{U}}) \to \check{\mathrm{C}}_{\mathrm{c}}^{\mathrm{k},\ell}(\widetilde{\mathcal{U}})$ and $\delta_v\colon \check{\mathrm{C}}_{\mathrm{c}}^{\mathrm{k},\ell-1}(\widetilde{\mathcal{U}}) \to \check{\mathrm{C}}_{\mathrm{c}}^{\mathrm{k},\ell}(\widetilde{\mathcal{U}})$. Note that under this description of the coarse Čech bicomplex, the subcomplex $\{\check{\mathrm{C}}_c^{0,\ell}(\widetilde{\mathcal{U}}), \delta_v\}$ is just the coarse Čech cochain complex $\check{\mathrm{C}}\mathrm{X}_{\mathrm{c}}^*(\widetilde{\mathcal{U}}, \boldsymbol{R})$.

We now compute the E_2 term of the spectral sequence associated to our bicomplex whose E_1 term is the cohomology of its columns. For $s = 0,\ldots,k$, denote by $i_s\colon \mathcal{W}_{\ell+1}^{k+1}(n) \to \mathcal{W}_{\ell+1}^{k+2}(n)$ the map given by $i_s(W_0,\ldots,W_k) = (W_0,\ldots,W_s,W_s,\ldots,W_k)$. This map induces $i_s^*\colon \check{\mathrm{C}}_c^{k+1,\ell}(\widetilde{\mathcal{U}}) \to \check{\mathrm{C}}_c^{k,\ell}(\widetilde{\mathcal{U}})$. Now for each fixed k, and all ℓ, define $K_{\ell+1}\colon \check{\mathrm{C}}_c^{k,\ell+1}(\widetilde{\mathcal{U}}) \to \check{\mathrm{C}}_c^{k,\ell}(\widetilde{\mathcal{U}})$ to be

$$K_{\ell+1} g(W_0,\ldots,W_k) = \sum_{j=0}^{\ell} (-1)^j g(W_0^j,\ldots,W_k^j),$$

where $W_i = (U_0^i,\ldots,U_\ell^i)$ and we set $W_0^j = (U_0^0,\ldots,U_j^0,U_j^1,\ldots,U_\ell^1)$ and for $i > 0$, $W_i^j = (U_0^i,\ldots,U_j^i,U_j^i,\ldots,U_\ell^i)$. We leave it to the reader to check that for each fixed k, $K_{\ell+1}\cdot\delta_v + \delta_v\cdot K_\ell = \pi_0^*\cdot i_0^* - I$. As $\pi_0 \cdot i_0 = I$, we have that π_0^* and i_0^* induce isomorphisms between the columns of the E_1 term, and that $i_0^* = \pi_0^{*-1}$. Thus each column is isomorphic to the first column. It is easy now to show that for all s, $i_s^*\cdot\pi_s^* = I$ on the columns of the E_1 term. In addition, as $\pi_s \cdot i_s = \pi_{s+1}\cdot i_s$, we have that $\pi_s^* = \pi_{s+1}^*$, and the differential $d_1^{k,\ell} = \sum_{i=0}^{k+1}(-1)^i\pi_i^* = \sum_{i=0}^{k+1}(-1)^i\pi_0^* = 0$ or π_0^* depending on whether k is even or odd. In particular, the E_2 term is non zero only in the first column and that column is just $\mathcal{H}X^*(\widetilde{\mathcal{U}})$. □

We have immediately

Corollary 29 *If $\widetilde{\mathcal{U}}$ is a convex cover of $\mathcal{G}_F$ such that each cover $\widetilde{\mathcal{U}}(n)$ is a good cover of $\mathcal{G}_F$ in the sense of Leray, then $\mathcal{H}X^*(F) \cong H_c^*(\mathcal{G}_F)$.*

Proof: The inverse system $\mathcal{H}_c^*(\widetilde{\mathcal{U}}(n), \boldsymbol{R})$ is degenerate: all the groups $\mathcal{H}_c^*(\widetilde{\mathcal{U}}(n), \boldsymbol{R})$ are isomorphic to $H_c^*(\mathcal{G}_F)$, and all the maps are the identity. Thus $\varprojlim_n{}^1 \mathcal{H}_c^*(\widetilde{\mathcal{U}}(n), \boldsymbol{R}) = 0$ and Proposition 5 gives the result. □

Note that examples 8 and 9 of Section 6 satisfy the hypotheses of Corollary 29.

We finish by defining the coarse Alexander-Spanier cohomology of F. Denote by $\Delta_{\ell+1}^{k+1}$ the diagonal of $\mathcal{G}_{\ell+1}$ in $\mathcal{G}_{\ell+1}^{k+1}$, i.e. $\Delta_{\ell+1}^{k+1} = \{(z,\ldots,z) \mid z \in \mathcal{G}_{\ell+1}\}$. Denote by $\Delta_{k+1}(Pen(\Delta_{\ell+1}, r)) \subset \mathcal{G}_{\ell+1}^{k+1}$ the set $\{(z,\ldots,z) \mid z \in Pen(\Delta_{\ell+1}, r)\}$. The set $C_c^{k,\ell}(F)$ consists of all locally bounded functions $\phi\colon \mathcal{G}_{\ell+1}^{k+1} \to$

$\boldsymbol{R}$ such that for all $r > 0$, there is an open neighborhood $\mathcal{W}$ of $\Delta_{k+1}(Pen(\Delta_{\ell+1}, r))$ in $\mathcal{G}_{\ell+1}^{k+1}$ with $\sup \phi \cap \mathcal{W}$ relatively compact. Define $\delta_h: C_c^{k,\ell}(F) \to C_c^{k+1,\ell}(F)$ and $\delta_v: C_c^{k,\ell}(F) \to C_c^{k,\ell+1}(F)$ by

$$\delta_h \phi(z_0, \ldots, z_{k+1}) = \sum_{j=0}^{k+1} (-1)^j \phi \cdot \pi_j(z_0, \ldots, z_{k+1})$$

$$\delta_v \phi(z_0, \ldots, z_k) = \sum_{i=0}^{\ell+1} (-1)^i \phi(\rho_i(z_0), \ldots \rho_i(z_k)),$$

where $\pi_j: \mathcal{G}_{\ell+1}^{k+2} \to \mathcal{G}_{\ell+1}^{k+1}$ deletes the j^{th} entry, and $\rho_i: \mathcal{G}_{\ell+2} \to \mathcal{G}_{\ell+1}$ deletes the i^{th} entry. $C_0^{k,\ell}(F)$ consists of those $\phi \in C_c^{k,\ell}(F)$ such that there is a neighborhood $\mathcal{W}$ of $\Delta_{\ell+1}^{k+1}$ in $\mathcal{G}_{\ell+1}^{k+1}$ with $\phi \mid \mathcal{W} \equiv 0$. Then δ_h and δ_v induce differentials on the Alexander-Spanier bicomplex $CX^{k,\ell}(F) = C_c^{k,\ell}(F)/C_0^{k,\ell}(F)$. The resulting cohomology, $HX_{AS}^*(F)$, is the coarse Alexander-Spanier cohomology of F. It is isomorphic to the coarse de Rham and Čech cohomologies by arguments similar to those above. In addition, there are alternating, doubly alternating, and smooth forms of this theory, each of which is isomorphic to $HX^*(\mathcal{F})$.

8 Functoriality of Coarse Cohomology

The functorial properties of coarse cohomology require that a given map between foliated manifolds induce a "nice" map on the corresponding holonomy groupoids. In order to guarantee this, it is necessary to impose geometric hypotheses on the foliations and the map between them. We first formulate these conditions, then we will use the coarse de Rham theory, in particular the covers $\widetilde{\mathcal{U}}$ and $\widetilde{\mathcal{U}}_\ell$ associated to the finite cover $\mathcal{U}$ of M as in Section 5, to show that the coarse cohomology of a foliation is a leafwise homotopy invariant.

Let $\mathcal{F}$ and $\mathcal{F}'$ be foliations of compact manifolds M and M' respectively. A continuous map $f: M \to M'$ which takes each leaf of $\mathcal{F}$ to a leaf of $\mathcal{F}'$ is called a leafwise map. To insure that f induces a map on the graphs of the foliations, we must assume that f maps leafwise paths with holonomy the identity to leafwise paths with holonomy the identity. (Leafwise homotopy equivalences, see below, satisfy this property.) Let f be such a map.

Lemma 30 *Given a finite cover $\mathcal{U}'$ for $\mathcal{F}'$, there is a finite cover $\mathcal{U}$ for $\mathcal{F}$ such that for all $r > 0$, $f(Pen(\Delta_\ell, r)) \subseteq Pen(\Delta'_\ell, r)$.*

Proof: The collection $\{f^{-1}(U') \mid U' \in \mathcal{U}'\}$ is an open cover of M. Choose $\mathcal{U}$ to be subordinate to this cover. Then for any plaque P of $\mathcal{U}$, there is a plaque P' of $\mathcal{U}'$ with $f(P) \subseteq P'$. It follows that the pull-back of the plaque distance function f^*D', where D' is defined using the cover $\mathcal{U}'$, dominates the plaque distance function D defined using the cover $\mathcal{U}$. The lemma follows immediately. $\square$

More is true. If $\mathcal{U}$ and $\mathcal{U}'$ are <u>any</u> finite covers for F and F', then there is a fixed $N > 0$ so that for any $W \in \widetilde{\mathcal{U}}_\ell$, $f(W)$ is covered by at most N elements of $\widetilde{\mathcal{U}}'_\ell$. In particular, let $\mathcal{V}$ be a finite cover of M subordinate to $f^{-1}(\mathcal{U}')$. Then there is $N_1 > 0$ so that any element of $\mathcal{U}$ is covered by N_1

elements of $\mathcal{V}$. It follows that any element of $\widetilde{\mathcal{U}}_\ell$ is covered by at most $N = N_1^{\ell+1}$ elements of $\mathcal{V}$. As f maps each element of $\widetilde{\mathcal{V}}_\ell$ into an element of $\widetilde{\mathcal{U}}'_\ell$, this N works.

Note that a leafwise map does *not* in general induce a map on coarse cochain complexes, since it will not necessarily preserve the compact support condition on cochains, as the following example shows. Let F be an irrational slope line foliation of the torus T^2, and let F' be the one dimensional foliation of the circle S^1. Then the projection $\rho: T^2 \to S^1$ is a leafwise map, but the inverse image of any non-empty relatively compact set in $\mathcal{G}'_\ell$ is never a relatively compact set in $\mathcal{G}_\ell$.

Definition 31 *Let $f: M \to M'$ be a leafwise map. We say that $f: \mathcal{G}_F \to \mathcal{G}_{F'}$ is *-proper if for all ℓ, the induced map $f: \mathcal{G}_\ell \to \mathcal{G}'_\ell$ is proper is the sense that the inverse image of any relatively compact is relatively compact.*

Lemma 32 *If $f: \mathcal{G}_F \to \mathcal{G}_{F'}$ is a proper map, then f is *-proper.*

Proof: Let W be a basic neighborhood of $\mathcal{G}'_\ell$. There are basic neighborhoods $W_1, \ldots, W_\ell$ of $\mathcal{G}_{F'}$ so that $W = (W_1 \times \ldots \times W_\ell) \cap \mathcal{G}'_\ell$. But then $f^{-1}(W) = (f^{-1}(W_1) \times \ldots \times f^{-1}(W_\ell)) \cap \mathcal{G}_\ell$, and the result follows. □

Definition 33 *Two leafwise maps $f, g: M \to M'$ are leafwise homotopic is there is a continuous map $H: M \times I \to M'$ so that for all $x \in M$, $H(x,0) = f(x)$, $H(x,1) = g(x)$, and $H(L_x \times I) \subset L_{f(x)}$. The map H is a leafwise homotopy from f to g.*

Lemma 34 *If $f, g: M \to M'$ are leafwise homotopic, and f is *-proper, then g is *-proper.*

Proof: Let $H: M \times I \to M$ be a leafwise homotopy from f to g. Let $F \times I$ be the foliation of $M \times I$ whose leaves are $L \times I$. Let $\mathcal{U} \times I = \{U \times I \mid U \in \mathcal{U}\}$, which is a finite cover for $F \times I$. Note that $\mathcal{G}_{F\times I} = \mathcal{G}_F \times I^2$, and that $\widetilde{\mathcal{U}}_{F\times I} = \{W \times I^2 \mid W \in \widetilde{\mathcal{U}}\}$. H is a continuous leafwise map so there is $N > 0$ so that for any element $W \in \widetilde{\mathcal{U}}, H(W \times I^2)$ is covered by at most N elements of $\widetilde{\mathcal{U}}'$. To show that H is *-proper (and thus that g is also proper), we need only show that $H^{-1}(W')$ is relatively compact for any $W' \in \widetilde{\mathcal{U}}'$. If $H(W \times I^2) \cap W' \neq \emptyset$, then $H(W \times I^2) \subset Pen(W', N)$. As $f(W) = H(W \times \{(0,0)\})$, we have that $f(W) \cap Pen(W', N) \neq \emptyset$, so $W \subset Pen(f^{-1}(Pen(W', N), 1)$, a relatively compact set. But it then follows immediately that $W \times I^2 \subset Pen(f^{-1}(Pen(W', N), 1) \times I^2$, which is also a relatively compact set. Thus H is *-proper. □

Proposition 35 *If $f: M \to M'$ is a *-proper map, then f induces a well defined map*

$$f^*: HX^*(F') \to HX^*(F)$$

which depends only on the leafwise homotopy class of f.

Proof: First assume that f is smooth. Clearly f induces a map f^* from the space of *all* differential forms on $\mathcal{G}'_\ell$ to the space of all differential forms on $\mathcal{G}_\ell$, and $f^*d = df^*$ and $\delta_v f^* = f^*\delta_v$. Lemma 30 says that for all $\omega \in A_c^{k,\ell}(F')$,

$$sup(f^*\omega) \cap Pen(\Delta_{\ell+1}, r) \subseteq f^{-1}(sup(\omega) \cap Pen(\Delta'_{\ell+1}, r)).$$

This and the fact that f is *-proper immediately imply that f^* preserves the support condition, i.e. f induces a map of bicomplexes

$$f^*: A_c^{*,*}(F') \to A_c^{*,*}(F).$$

The usual proof of homotopy invariance for de Rham cohomology shows that f^* induces a map on the cohomology of each row of the bicomplexes which depends only on the smooth leafwise homotopy class of f. Thus $f^*: HX^*(F') \to HX^*(F)$ depends only on the smooth leafwise homotopy class of f.

An arbitrary *-proper map f is leafwise homotopic to a smooth *-proper map, say g, and we define f^* on coarse de Rham cohomology to be g^*. It is not difficult to show that two smooth leafwise maps are leafwise homotopic through a smooth leafwise homotopy if and only if they are leafwise homotopic through a continuous leafwise homotopy. Thus f^* is well defined on coarse de Rham cohomology, and depends only on the leafwise homotopy class of f. $\square$

A leafwise map $f: M \to M'$ is a leafwise homotopy equivalence if there is a leafwise map $f': M' \to M$ so that $f' \circ f$ and $f \circ f'$ are leafwise homotopic to the identity .

Theorem 36 *Suppose that F and F' are foliations of compact manifolds M and M' respectively. If $f: M \to M'$ is a leafwise homotopy equivalence, then f induces an isomorphism from the coarse cohomology of F' to that of F.*

Proof: We need only check that f is *proper, i.e. for any element W' of $\widetilde{\mathcal{U}}', S = f^{-1}(W')$ is relatively compact. Let $f': M \to M$ be a leafwise homotopy inverse of f. Then $f' \circ f(S) = f'(W')$ which is covered by finite number of elements of $\widetilde{\mathcal{U}}$, so relatively compact. To finish, we only need the following lemma.

Lemma 37 *There is $r > 0$ so that $S \subseteq Pen(f' \circ f(S), r)$.*

Proof: Let $H: M \times I \to M$ be a leafwise homotopy from the identity to $f' \circ f$. Let $F \times I$ and $\mathcal{U} \times I$ be as above. H is a continuous leafwise map so there is $N > 0$ so that for any element $W \in \widetilde{\mathcal{U}}, H(W \times I^2)$ is covered by at most N elements of $\widetilde{\mathcal{U}}$. Now $f' \circ f(S)$ is contained in some finite union $\cup W_i$, where the $W_i \in \widetilde{\mathcal{U}}$. If $y \in W \cap S$, then $y = H(y, 0)$ and $H(y, 1) = f' \circ f(y) \in W_i$ for some W_i. So $H(W \times I^2) \cap W_i \neq \emptyset$ and $H(W \times I^2) \cap W \neq \emptyset$. As $H(W \times I^2)$ can be covered by at most N elements of $\widetilde{\mathcal{U}}$, $W \subseteq Pen(W_i, N+1)$ and $S \subseteq Pen(f' \circ f(S), r)$ for any $r > N + 1$. $\square$

We now indicate how to show that a *-proper map f induces a well defined map in coarse Čech cohomology, and that the obvious diagram relating the maps f induces in coarse Čech and de Rham cohomology commutes. Analogous results hold for coarse Alexander-Spanier cohomology.

Let $\widetilde{\mathcal{U}}$ and $\widetilde{\mathcal{U}}'$ be good covers for $\mathcal{G}_F$ and $\mathcal{G}_{F'}$ respectively. We may assume $\widetilde{\mathcal{U}}$ is subordinate to the cover $f^{-1}\widetilde{\mathcal{U}}'$. For each $\widetilde{U} \in \widetilde{\mathcal{U}}$ choose $\lambda(\widetilde{U}) \in \widetilde{\mathcal{U}}'$ such that $f(\widetilde{U}) \subset \lambda(\widetilde{U})$. Then λ induces canonical maps $\lambda_*: \widetilde{\mathcal{U}}_\ell \to \widetilde{\mathcal{U}}'_\ell$ for all ℓ, and also a cochain map $\lambda^*: \check{\mathrm{C}}_c^{*,*}(\widetilde{\mathcal{U}}') \to \check{\mathrm{C}}_c^{*,*}(\widetilde{\mathcal{U}})$. We leave it to the reader to check that the support condition is preserved. To see that the map induced in cohomology is independent of the choice of λ repeat the proof of Lemma 24. The usual proof that passing to subordinate covers leads to the obvious commuting diagram in cohomology goes through and it follows immediately that f induces a well defined map $f^*: \mathcal{H}X(F') \to \mathcal{H}X(F)$.

To see that this f^* is the same one as in coarse de Rham cohomology, proceed as in the proof of Theorem 26, (i.e. fix ℓ), and then repeat the usual proof of this fact for the space $\mathcal{G}_\ell$. This gives a commutative diagram for the maps induced by f on the E_1 terms of one of the associated spectral sequences and the result follows immediately.

9 The Relationship with Usual Cohomology

In this section we show that there is a natural map from the coarse cohomology of a foliation to the usual cohomology with compact supports of its graph. We also give conditions on the foliation which guarantee that this map is an isomorphism.

Set

$$A_0^{k,\ell}(F) = \{\omega \in A_c^{k,\ell}(F) \mid \omega \equiv 0 \text{ on a neighborhood of } \Delta_{\ell+1}\}.$$

This is a subcomplex of $A_c^{k,\ell}(F)$ and we define *the standard de Rham bicomplex for* F to be

$$AS_c^{k,\ell}(F) = A_c^{k,\ell}(F)/A_0^{k,\ell}(F)$$

with the induced differentials d and δ_v. Denote its cohomology by $H_R^*(\mathcal{G}_F)$. Let c be the natural map $c: A_c^{k,\ell}(F) \to AS_c^{k,\ell}(F)$, and denote the induced map in cohomology by c also.

Theorem 38 *$H_R^*(\mathcal{G}_F)$ is isomorphic to $H_c^*(\mathcal{G}_F)$, the usual cohomology of $\mathcal{G}_F$ with compact supports.*

Proof: Let $\widetilde{\mathcal{U}}$ be a convex cover of $\mathcal{G}_F$ and set $\mathcal{W}_1 = \mathcal{G}_F = \mathcal{G}_1$. For $\ell > 1$ set

$$\mathcal{W}_\ell = [\bigcup_{W \in \widetilde{\mathcal{U}}} (\times_\ell W)] \cap \mathcal{G}_\ell.$$

Then $\mathcal{W}_\ell$ is a neighborhood of Δ_ℓ in $\mathcal{G}_\ell$ and for each j, $\pi_j: \mathcal{W}_{\ell+1} \to \mathcal{W}_\ell$ is onto. Set

$$A_c^{k,\ell}(\mathcal{W}) = A_c^k(\mathcal{W}_{\ell+1}),$$

the differential k-forms on $\mathcal{W}_{\ell+1}$ with relatively compact support (i.e. the closure in $\mathcal{G}_{\ell+1}$ of the support is compact). Equip $A_c^{k,\ell}(\mathcal{W})$ with the differentials obtained by restriction of the differentials on $A_c^{k,\ell}(F)$.

If $\widetilde{\mathcal{U}}'$ is locally finite refinement of $\widetilde{\mathcal{U}}$, then we have the restriction map, $A_c^{k,\ell}(\mathcal{W}) \to A_c^{k,\ell}(\mathcal{W}')$. Note that by definition,

$$\{AS_c^{*,*}(F), \delta\} = \varinjlim_{\mathcal{W}} \{A_c^{*,*}(\mathcal{W}), \delta\}.$$

Thus

$$H_R^*(\mathcal{G}_F) = \varinjlim_{\mathcal{W}} H^*(\{A_c^{*,*}(\mathcal{W}), \delta\}).$$

Lemma 39 *Given any convex cover of $\mathcal{G}_F$, there is a locally finite refinement $\widetilde{\mathcal{U}}$ such that Δ_ℓ is a deformation retraction of $\mathcal{W}_\ell$ by a canonical retraction.*

Proof: Choose the refinement $\widetilde{\mathcal{U}}$ so that each plaque is geodesically convex in its leaf. Now suppose that $(y_1, \ldots, y_\ell) \in \mathcal{W}_\ell$. Then there is $W \in \widetilde{\mathcal{U}}$ with $y_1, \ldots, y_\ell \in W$, and we may retract along the geodesics connecting $r(y_1)$ to $r(y_2), \ldots, r(y_\ell)$ to retract $(y_1, \ldots, y_\ell)$ to $(y_1, \ldots, y_1) \in \Delta_\ell$. This retraction is canonical and lies in $\mathcal{W}_\ell$. □

Now the usual proof shows that for such a refinement, and for fixed ℓ, $H^*(\{A_c^{*,\ell}(\mathcal{W}), d\})$ is isomorphic to $H_c^*(\Delta_{\ell+1})$. Thus the E_1 term of one of the spectral sequences of this bicomplex is just $E_1^{p,q} = H_c^p(\Delta_{\ell+1})$. As π_i^* on $H_c^p(\Delta_{\ell+1})$ is the identity, we have that the differential $d_1^{p,q}: E_1^{p,q} \to E_1^{p,q+1}$ is 0 if q is even and the identity if q is odd. The E_2 term is then

$$E_2^{p,0} = H_c^p(\Delta_{\ell+1}) \text{ and } E_2^{p,q} = 0 \text{ if } q > 0.$$

As $\Delta_{\ell+1} \cong \mathcal{G}_F$, $H^*(\{A_c^{*,*}(\mathcal{W}), \delta\}) \cong H_c^*(\mathcal{G}_F)$. Since the retractions used are canonical and there is a cofinal subset of such convex covers, we are done. □

Note that $\{A_c^{k,0}(F), d\} = \{AS_c^{k,0}(F), d\}$, i.e. the bottom rows of the coarse and standard bicomplexes are the same and both are just the usual de Rham complex with compact supports of $\mathcal{G}_F$. There are natural maps of complexes $A_c^{*,*}(F) \to A_c^{*,0}(F)$ and $AS_c^{*,*}(F) \to AS_c^{*,0}(F)$ which make the following following diagram commute.

$$\begin{array}{ccc} HX^*(F) & & \\ & \searrow & \\ c\downarrow & & H_c^*(\mathcal{G}_F) \\ & \nearrow & \\ H_R^*(\mathcal{G}_F) & & \end{array}$$

In particular, the lower right hand map is the isomorphism in Theorem 38.

For the Čech case proceed as follows. Let $\widetilde{\mathcal{U}}$ be a convex cover of $\mathcal{G}_F$ and set

$$\check{\mathrm{C}}_0^{\mathrm{k},\ell}(\widetilde{\mathcal{U}}) = \{\mathrm{f} \in \check{\mathrm{C}}_{\mathrm{c}}^{\mathrm{k},\ell}(\widetilde{\mathcal{U}}) \mid \mathrm{f}(\mathrm{W}_0, \ldots, \mathrm{W}_\mathrm{k}) = 0 \text{ if } \mathrm{W}_0 \cap \ldots \cap \mathrm{W}_\mathrm{k} \cap \Delta_{\ell+1} \neq \emptyset\}.$$

The differentials preserve $\check{\mathrm{C}}_0^{\mathrm{k},\ell}(\widetilde{\mathcal{U}})$. The *standard Čech bicomplex for F* is

$$\check{\mathrm{C}}\mathrm{S}_{\mathrm{c}}^{\mathrm{k},\ell}(\widetilde{\mathcal{U}}) = \check{\mathrm{C}}_{\mathrm{c}}^{\mathrm{k},\ell}(\widetilde{\mathcal{U}}) / \check{\mathrm{C}}_0^{\mathrm{k},\ell}(\widetilde{\mathcal{U}})$$

with the induced differentials δ_h and δ_v. It follows from Theorem 40 below that the cohomology of this bicomplex is independent of $\widetilde{\mathcal{U}}$. Denote its cohomology by $H_{\check{C}}^*(\mathcal{G}_F)$. As above, let c be the natural map

$$c: \check{\mathrm{C}}_{\mathrm{c}}^{\mathrm{k},\ell}(\widetilde{\mathcal{U}}) \to \check{\mathrm{C}}\mathrm{S}_{\mathrm{c}}^{\mathrm{k},\ell}(\widetilde{\mathcal{U}}),$$

and also denote by c the induced map in cohomology.

Note that the bicomplex $\check{\mathrm{C}}\mathrm{S}_{\mathrm{c}}^{\mathrm{k},\ell}(\widetilde{\mathcal{U}})$ is isomorphic to the bicomplex

$$\check{\mathrm{C}}\mathrm{S}_0^{\mathrm{k},\ell}(\widetilde{\mathcal{U}}) = \{\mathrm{f}: \widetilde{\mathcal{U}}_{\ell+1}^{\mathrm{k}+1} \to \boldsymbol{R} \mid \sup(f) \text{ is finite and}$$

$$f(W_0, \ldots, W_k) = 0 \text{ if } W_0 \cap \ldots \cap W_k \cap \Delta_{\ell+1} = \emptyset\}.$$

We may make similar constructions in the Alexander-Spanier case, and we denote the resulting cohomology by $HX_{AS}(\mathcal{G}_F)$.

Theorem 40 *The following diagram commutes.*

$$\begin{tikzcd}
\check{H}X^*(F) \arrow[rrr, "c"] \arrow[dr] \arrow[dd] & & & H^*_C(\mathcal{G}_F) \arrow[dl] \arrow[dd] \\
& HX^*_{AS}(F) \arrow[r, "c"] \arrow[dl] & H^*_{AS}(\mathcal{G}_F) \arrow[dr] & \\
HX^*(F) \arrow[rrr, "c"] & & & H^*_R(\mathcal{G}_F)
\end{tikzcd}$$

The maps in the left triangle are the isomorphisms given in Section 7. Those of the right triangle are their analogues and they are also isomorphisms.

Proof: We will show that $H^*_C(\mathcal{G}_F) \cong H^*_R(\mathcal{G}_F)$ and leave the rest of the proof to the reader. Let $\widetilde{\mathcal{U}}_1$ be a convex cover of $\mathcal{G}_F$ as in Lemma 39 and let $\mathcal{W}$ be as in the proof of Theorem 38 for $\widetilde{\mathcal{U}}_1$. Choose a cover $\mathcal{U}$ of M for F, and an associated convex cover $\widetilde{\mathcal{U}}$ of $\mathcal{G}_F$ which satisfies the condition that for any $U \in \widetilde{\mathcal{U}}$ there is $U_1 \in \widetilde{\mathcal{U}}_1$ so that $Pen(U, 2) \subset U_1$ where Pen is with respect to the cover $\mathcal{U}$. Then the map A: $\check{\mathrm{C}}^{\mathrm{k},\ell}_{\mathrm{c}}(\widetilde{\mathcal{U}}) \to \mathrm{A}^{\mathrm{k},\ell}_{\mathrm{c}}(\mathrm{F})$ defined in Section 7 when restricted to $\check{\mathrm{C}}\mathrm{S}^{\mathrm{k},\ell}_0(\widetilde{\mathcal{U}})$ takes values in $A^{k,\ell}_c(\mathcal{W})$. To show that this induces isomorphisms on the ℓ^{th} rows of the bicomplexes, we merely repeat Weil's proof again, using the bicomplex $C^p_{c,0}(\widetilde{\mathcal{U}}, \Omega^q)$, where $f \in C^p_{c,0}(\widetilde{\mathcal{U}}, \Omega^q)$ assigns to each $(W_0, \ldots, W_p) \in \widetilde{\mathcal{U}}^{p+1}_{\ell+1}$ a q-form on $W_0 \cap \ldots \cap W_p$ such that $f(W_0, \ldots, W_p) = 0$ if $W_0 \cap \ldots \cap W_p \cap \Delta_{\ell+1} = \emptyset$ and the support of f is finite. The cohomology of the columns of this bicomplex is zero since $\widetilde{\mathcal{U}}$ is good. The fact that the cohomology of the rows is zero is proven just as in [W], [BT]. □

As in the coarse case, we may replace the standard de Rham, Čech and Alexander-Spanier bicomplexes by their subcomplexes of alternating or doubly alternating cochains. The cohomology of these subcomplexes is the same as that of the full bicomplexes and the above theorems still hold.

To finish this section we give four general results on when the map c is an isomorphism.

The results of Section 5 give that if F is the natural foliation of a flat bundle over a compact Riemannian manifold N constructed from a generic orientation preserving representation of $\pi_1(N)$ and $\widetilde{N}$ is rescaleable, or uniformly contractible, or globally of non-positive curvature, and there is no twisting, then c is an isomorphism.

To state the second result, choose a smooth metric on M. As noted in Section 5 this metric induces a smooth *leafwise* metric d on each $\mathcal{G}_\ell$ and we may use these to define $Pen(A, r)$. We say that F is *rescaleable* [R3] if there is a one-parameter group ρ_t of automorphisms of $\mathcal{G}_F$ mapping each holonomy cover $\widetilde{L}$ to itself so that for $y_1, y_2 \in \mathcal{G}_F$ with $s(y_1) = s(y_2)$, $d(\rho_t(y_1), \rho_t(y_2)) = e^t d(y_1, y_2)$, for all $t \in \boldsymbol{R}$.

Proposition 41 *If F is rescaleable, then the map c: $HX^*(F) \to H^*_R(\mathcal{G}_F)$ is an isomorphism.*

Proof: Let $r > 0$ and for $\ell = 1, 2, \ldots$ let $\mathcal{W}_\ell(r) = Pen(\Delta_\ell, r)$. Set

$$A_c^{k,\ell}(\mathcal{W}(r)) = A_c^k(\mathcal{W}_{\ell+1}(r)),$$

the differential k-forms on $\mathcal{W}_{\ell+1}(r)$ with relatively compact support (i.e. the closure in $\mathcal{G}_{\ell+1}$ of the support is compact). Equip $A_c^{k,\ell}(\mathcal{W}(r))$ with the differentials obtained by restriction of the differentials on $A_c^{k,\ell}(F)$. For $r = r_2 - r_1$ we have the restriction map,

$$i_r^*: A_c^{k,\ell}(\mathcal{W}(r_2)) \to A_c^{k,\ell}(\mathcal{W}(r_1)).$$

By definition,

$$AS_c^{*,*}(F) = \varinjlim_r A_c^{*,*}(\mathcal{W}(r)),$$

so

$$H_R^*(\mathcal{G}_F) = \varinjlim_r HX^*(\mathcal{W}(r)),$$

where $HX^*(\mathcal{W}(r))$ is the cohomology of $A_c^{*,*}(\mathcal{W}(r))$.

Now

$$A_c^{*,*}(F) = \varprojlim_r A_c^{*,*}(\mathcal{W}(r)),$$

and the inverse system $\{A_c^{*,*}(\mathcal{W}(r)), i_r^*\}$ satisfies the Mittag-Leffler condition [DG], see also [A]. Set $t = \log(r)$. Then $\rho_t: A_c^{*,*}(\mathcal{W}(r_2)) \to A_c^{*,*}(\mathcal{W}(r_1))$ is an isomorphism of bicomplexes and ρ_t is properly homotopic to i_r. It follows that these two maps induce the same map in cohomology and so all the maps i_r^* are isomorphisms. Thus $\varprojlim_r HX^*(\mathcal{W}(r)) \cong \varinjlim_r HX^*(\mathcal{W}(r))$ and the inverse system $\{HX^*(\mathcal{W}(r)), i_r^*\}$ also satisfies the Mittag-Leffler condition. By [DG] we have $\varprojlim_r HX^*(\mathcal{W}(r)) = HX^*(F)$. As the composition

$$HX^*(F) = \varprojlim_r HX^*(\mathcal{W}(r)) \cong \varinjlim_r HX^*(\mathcal{W}(r)) = H_R^*(\mathcal{G}_F)$$

is just the map c, we are done. □

For the third result, let $\mathcal{U}$ be a finite cover of M for F and $\widetilde{\mathcal{U}}$ an associated convex cover of $\mathcal{G}_F$. Recall the inverse system $\widetilde{\mathcal{U}}(n)$ of covers of $\mathcal{G}_F$ used to define the coarse Čech cohomology of F.

Definition 42 *We say that $A \subseteq \mathcal{G}_F$ is leafwise contractible inside $B \subseteq \mathcal{G}_F$ if there is continuous map $\rho: A \times I \to B$ so that $\rho \mid A \times 0$ is the inclusion of A in B, $s(\rho(y,t)) = s(y)$ for all $y \in A$, and if $s(y_1) = s(y_2)$, then $\rho(y_1, 1) = \rho(y_2, 1)$.*

Definition 43 *We say that F is uniformly contractible if there is a convex cover $\widetilde{\mathcal{U}}$ of $\mathcal{G}_F$ so that for each $n_1 > 0$ there is $n_2 > 0$ so that for all $U \in \widetilde{\mathcal{U}}$, $U(n_1)$ is leafwise contractible inside $U(n_2)$.*

Proposition 44 *If F is uniformly contractible, then the map $c: \mathcal{H}X^*(F) \to H_C^*(\mathcal{G}_F)$ is an isomorphism.*

Proof: For each n, consider the bicomplex $A_n^{*,*} = \check{\mathrm{C}}_c^*(\widetilde{\mathcal{U}}(\mathrm{n}), \Omega^*)$. Here Ω^k is the sheaf of k-forms. This is the bicomplex used in Weil's proof of the de Rham theorem. Set $A^{*,*} = \varprojlim_n A_n^{*,*}$. Weil's proof of the de Rham theorem shows that the cohomology of the bicomplex $H^*(A_n^{*,*}) \cong H_c^*(\mathcal{G}_F)$ for all n, and it is immediate that the inclusions $\widetilde{\mathcal{U}}(n) \to \widetilde{\mathcal{U}}(n+1)$ induce the identity map on cohomology. This implies that $\lim^1 H^*(A_n^{*,*}) = 0$ and so

$$H^*(A^{*,*}) = \varprojlim_n H^*(A_n^{*,*}) \cong H_c^*(\mathcal{G}_F).$$

Now we compute the column cohomology of the bicomplex $A^{*,*}$. Each column cohomology group is just the direct sum of groups $H^q(\varprojlim_n \Omega^*(U_0(n) \cap \ldots \cap U_p(n)))$.

Let $n_1 > 0$ and choose n_2 so that for all $U \in \widetilde{\mathcal{U}}$, $U(n_1)$ is leafwise contractible inside $U(n_2)$. Then note that for any $U_0, \ldots, U_p \in \widetilde{\mathcal{U}}$, $U_0(n_1) \cap \ldots \cap U_p(n_1)$ is leafwise contractible inside $U_0(2n_1 + n_2 + 1) \cap \ldots \cap U_p(2n_1 + n_2 + 1)$. In addition, $s(U_0(n_1) \cap \ldots \cap U_p(n_1))$ is contractible, so $U_0(n_1) \cap \ldots \cap U_p(n_1)$ is in fact contractible inside $U_0(2n_1 + n_2 + 1) \cap \ldots \cap U_p(2n_1 + n_2 + 1)$. Thus we have the following

Lemma 45 *For all $q > 0$,*

$$H^q(\Omega^*(U_0(2n_1 + n_2 + 1) \cap \ldots \cap U_p(2n_1 + n_2 + 1))) \to H^q(\Omega^*(U_0(n_1) \cap \ldots \cap U_p(n_1)))$$

is the zero map.

We now have that for $q > 0$, $\lim^1 H^q(\Omega^*(U_0(n) \cap \ldots \cap U_k(n))) = 0$, and $\varprojlim_n H^q(\Omega^*(U_0(n) \cap \ldots \cap U_k(n))) = 0$. Thus $H^q(\varprojlim_n \Omega^*(U_0(n) \cap \ldots \cap U_k(n))) = 0$ also. Thus the E_1 term for one of the spectral sequences associated to the bicomplex $A^{*,*}$ satisfies $E_1^{p,q} = 0$ if $q > 0$, and $E_1^{p,0} = \varprojlim_n \check{\mathrm{C}}_c(\widetilde{\mathcal{U}}(\mathrm{n}), \boldsymbol{R})$. It follows that the E_2 is non zero only for $p = 0$ and these groups are just the simple coarse cohomology groups of F. The Proposition follows. □

Finally, we have

Theorem 46 *Let F be a foliation such that each holonomy cover $\widetilde{L}$ is globally of non-positive curvature. Then the map $c\colon \mathcal{H}X^*(F) \to H_c^*(\mathcal{G}_F)$ is an isomorphism.*

Proof: Use the isomorphism $\mathcal{H}X^*(F) \cong HX_{AS}^*(F)$ and adapt the proof of Theorem 3.42 of [R3] using Corollary 29. □

References

[A] Michael F. Atiyah. Characters and cohomology of finite groups. *Pub. Math. IHES*, 9:274–289, 1961.

[BT] Raoul Bott and Loring Tu. *Differential Forms in Algebraic Topology.* Springer-Verlag, Berlin-New York, 1982.

[CC] Alberto Candell and Lawrence Conlon. *Foliations 1.* Amer. Math. Soc., Providence, RI, 2000.

[DG] Jean Dieudonne and Alexander Grothendieck. Elements de Geometrie algebrique. *Pub. Math. IHES*, 11:349–511, 1961.

[EMS] R. Edwards, K. Millet, and D. Sullivan. Foliations with all leaves compact. *Topology*, 16:13–32, 1977.

[FM] J. Feldman and C.C. Moore. Ergodic equivalence relations, cohomology, and von Neumann algebras I. *Transactions AMS*, 234:289–324, 1977.

[G] Mikhael Gromov. *Structures Metriques pour les Varièté́s Riemanniennes.* Editions CEDIC, Paris, 1981.

[Hae1] André. Haefliger. Homotopy and integrability. *Lect. Notes in Math.*, 197:133–166, Springer-Verlag 1971.

[Hae2] André. Haefliger. Groupoïdes d'holonomie et classifiants. *Astérisque*, 116:70–97, Société Mathématique de France 1984.

[He1] James L. Heitsch. Independent variation of secondary characteristic classes. *Annals of Mathematics*, 108:421–460, 1978.

[He2] James L. Heitsch. Bismut super-connections and the Chern character for Dirac operators on foliated manifolds. *K-Theory*, 9:507–528, 1995.

[He3] James L. Heitsch. Coarse sheaf cohomology for foliations. Illinois J. Math., to appear.

[HecH] Gilbert Hector and Ulrich Hirsch. *Introduction to the Theory of Foliations, Part A.* Vieweg und Sohn, Braunschwieg/Wiesbaden, second edition, 1986.

[Hir1] Moe Hirsch. *A stable analytic foliation with only exceptional minimal sets.* in Dynamical Systems, Warwick, 1974, Lect. Notes in Math., 468:9–10, Springer-Verlag 1975.

[Hu1] Steven Hurder. Coarse geometry of foliations. *Geometric Study of Foliations, Tokyo 1993* (eds. Mizutani et al). World Scientific, 1995, 35-96.

[Hu2] Steven Hurder. Exotic index theory and the Novikov conjecture. in *Novikov Conjectures, Index Theorems and Rigidity, Vol. 2 (Oberwolfach, 1993).* London Math. Soc. Lecture Note Ser., 227: 255–276, 1995.

[HK] Steven Hurder and Anatole Katok. Ergodic theory and Weil measures for foliations. *Annals of Math.*, 126:221–275, 1987.

[M] G. W. Mackey. Ergodic theory, group theory, and differential geometry. *Proc. Nat. Acad. Sci., U. S. A.*, 50:1184–1191, 1963.

[MS] C. C. Moore and C. Schochet. Analysis on Foliated Spaces. *Math. Sci. Res. Inst. Publ.*, 9, 1988.

[P] Joseph Plante. Foliations with measure preserving holonony. *Annals of Mathematics*, 102:327–361, 1975.

[R1] John Roe. Exotic cohomology and index theory. *Bulletin Amer. Math. Soc.*, 23:447–453, 1990.

[R2] John Roe. Hyperbolic metric spaces and the exotic cohomology Novikov Conjecture. *K-Theory*, 4:501–512, 1991.

[R3] John Roe. Coarse cohomology and index theory on complete Riemannian manifolds. *Memoirs of AMS*, 104 (497), 1993.

[S] Dennis Sullivan. A counter-example to the periodic orbit conjecture. *Publ. Math. Inst. Hautes Etudes Sci.*, 46:5–14, 1976.

[Vo1] Elmar Vogt. Foliations of codimension 2 with all leaves compact. *Manuscripta Math.*, 18:187–212, 1976.

[Vo2] Elmar Vogt. Foliations of codimension 2 on closed 3, 4 and 5-manifolds. *Math. Zeit.*, 157:201–223, 1977.

[Vo3] Elmar Vogt. A periodic flow with infinite Epstein hierarchy. *Manuscripta Math.*, 22:403–412, 1977.

[W] Andre Weil. Sur les théorèmes de de Rham. *Comment. Math. Helv.*, 26:119–145, 1952.

[Wi] Elmar Winkelnkemper. The graph of a foliation. *Ann. Global Ann. Geo.*, 1:53–75, 1983.