INDEX THEORY AND NON-COMMUTATIVE GEOMETRY II. DIRAC OPERATORS AND INDEX BUNDLES October 17, 2007

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ABSTRACT. When the index bundle of a longitudinal Dirac type operator is transversely smooth, we define its Chern character in Haefliger cohomology and relate it to the Chern character of the $K-$theory index. This result gives a concrete connection between the topology of the foliation and the longitudinal index formula. Moreover, the usual spectral assumption on the Novikov-Shubin invariants of the operator is improved.

CONTENTS

INTRODUCTION

In this paper, we continue our systematic study of the index theorem in Haefliger cohomology of foliations. In [BH04], we defined a Chern character for leafwise elliptic pseudodifferential operators on foliations. By using Connes' extension in [Con86], we then translated the Connes-Skandalis $K-$theory index theorem [CS84] into Haefliger cohomology, thus proving scalar index theorems in the presence of holonomy invariant currents.

In order to get more insight into topological invariants of foliations, we extend here the results of [He95] and [HL99], which tie the indices of a leafwise operator on a foliation of a compact manifold to the so-called index bundle of the operator. In particular, we show that for a generalized Dirac operator D along the leaves of a Riemannian foliation, the Chern character of the analytic index of D coincides with the Chern character of the index bundle of D. In [He95] and [HL99], the groupoid $\mathcal{G}$ was assumed to be Hausdorff, but for Riemannian foliations that is automatic. As in [He95] and [HL99], we assume that the projection onto the kernel of D is transversely smooth, and that the spectral projections of D^2 for the intervals $(0, \epsilon)$ are transversely smooth, for ϵ sufficiently small. In those two papers, we assumed that the Novikov-Shubin invariants of D were greater than three times the codimension of F. Here we use the K $-$theory index and we need only assume that they are greater than half the codimension of F. More precisely, the pairings of these Chern characters with a given Haefliger $2k-$current agree whenever the Novikov-Shubin invariants of D are greater than k. We conjecture that this theorem is still true provided only that the Novikov-Shubin invariants are positive. Note that in the heat equation proof of the classical Atiyah-Singer families index theorem, [B86], it is assumed that there is a uniform gap about zero in the spectrum of the operator, which implies the conditions we assume on the spectral projections.

In [Con79, Con81], Connes extended the classical construction of Atiyah [A75] of the L^2 covering index theorem to leafwise elliptic operators on compact foliated manifolds. To do so he replaced the lifting and deck transformations used by Atiyah by a lifting to the holonomy covers of the leaves invariant under the natural action of the holonomy groupoid. Moreover, he defined an analytic index map from the K-theory of the tangent bundle of the foliation to the K-theory of the C* algebra of the foliation, which plays the role of the K-theory of the space of leaves. In [CS84], Connes and Skandalis defined a push forward map in K-theory for any K-oriented map from a manifold to the space of leaves of a foliation of a compact manifold. This allowed them to define a topological index map from the K-theory of the tangent bundle of the foliation to the K-theory of its C* algebra. Their main result is that the analytic and topological index maps are equal, an extension of the classical Atiyah-Singer families index theorem. This theorem does not lead in general to a relation between the index of the operator and its index bundle, by which we mean the (graded) projection onto the kernel of the operator, even when this latter is transversely smooth and when its Chern character is well defined. This index bundle, which lives in a von Neumann algebra of the foliation, carries important information about the foliation.

In this paper, we extend the Chern character to the index bundle of D, provided the projection onto the kernel of D is transversely smooth. Our main result is that, with the conditions given in the second paragraph, the Chern character of D equals the Chern character of the index bundle of D. Since the Chern character of the index bundle equals the superconnection index defined in [He95], we obtain as a corollary the coincidence of the superconnection index with the Chern character of the analytic and topological indices. This Chern character is readily computable and directly relates the index of D with the topology of the foliation.

Here is a brief outline of the paper. In Section 1., we fix notation and briefly review some necessary material. In Section 2., we extend our Chern character to the $K-$theory of the space of super-exponentially decaying operators on the leaves of a foliation, and recall the construction of Dirac operators and the heat index idempotent. In Section 3., we review the construction of the Chern character we use, and extend it to the index bundle of a leafwise Dirac operator. In Section 4., we prove our main theorem, Theorem 4.1. In Section 5., we show that the Chern character of the index bundle for D defined here is the same as that defined in [He95] using Bismut superconnections.

It is also worth pointing out that our results are valid if we replace the holonomy groupoid $\mathcal{G}$ by any smooth groupoid between the monodromy and holonomy groupoids, see [Ph87]. We point out the papers [GL03, GL05] where Gorokhovsky and Lott prove, by a different method, an index theorem for longitudinal Dirac operators.

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1. Notation and Review

Throughout this paper M denotes a smooth compact Riemannian manifold of dimension n, and F denotes an oriented Riemannian foliation of M of dimension p and codimension q. So $n = p+q$. We assume that the metric on M, when restricted to the normal bundle ν of F, is bundle like, so the holonomy maps of ν and its dual ν^* are isometries. The tangent bundle of F will be denoted TF. If $E \to N$ is a vector bundle over a manifold N, we denote the space of smooth sections by $C^\infty(E)$ or by $C^\infty(N;E)$ if we want to emphasize the base space of the bundle. The compactly supported sections are denoted by $C^\infty_c(E)$ or $C^\infty_c(N;E)$. The space of differential $k-$forms on N is denoted $\mathcal{A}^k(N)$, and we set $\mathcal{A}(N) = \oplus_{k\geq 0}\mathcal{A}^k(N)$. The space of compactly supported $k-$forms is denoted $\mathcal{A}^k_c(N)$, and $\mathcal{A}_c(N) = \oplus_{k\geq 0}\mathcal{A}^k_c(N)$.

The holonomy groupoid $\mathcal{G}$ of F consists of equivalence classes of paths $\gamma : [0,1] \to M$ such that the image of γ is contained in a leaf of F. Two such paths γ_1 and γ_2 are equivalent if $\gamma_1(0) = \gamma_2(0)$, $\gamma_1(1) = \gamma_2(1)$, and the holonomy germ along them is the same. Two classes may be composed if the first ends where the second begins, and the composition is just the juxtaposition of the two paths. This makes $\mathcal{G}$ a groupoid. The space $\mathcal{G}^{(0)}$ of units of $\mathcal{G}$ consists of the equivalence classes of the constant paths, and we identify $\mathcal{G}^{(0)}$ with M.

For Riemannian foliations, $\mathcal{G}$ is a Hausdorff dimension $2p+q$ manifold, in fact a fibration. The basic open sets defining its manifold structure are given as follows. Let $\mathcal{U}$ be a finite good cover of M by foliation charts as defined in [HL90]. Given U and V in this cover and a leafwise path γ starting in U and ending in V, define (U,γ,V) to be the set of equivalence classes of leafwise paths starting in U and ending in V which are homotopic to γ through a homotopy of leafwise paths whose end points remain in U and V respectively. It is easy to see, using the holonomy defined by γ from a transversal in U to a transversal in V, that if $U,V \simeq \mathbb{R}^p \times \mathbb{R}^q$, then $(U,\gamma,V) \simeq \mathbb{R}^p \times \mathbb{R}^p \times \mathbb{R}^q$.

The source and range maps of the groupoid $\mathcal{G}$ are the two natural maps $s,\, r : \mathcal{G} \to M$ given by $s\big([\gamma]\big) = \gamma(0)$, $r\big([\gamma]\big) = \gamma(1)$. $\mathcal{G}$ has two natural transverse foliations F_s and F_r whose leaves are respectively $\widetilde{L}_x = s^{-1}(x)$, $\widetilde{L}^x = r^{-1}(x)$ for $x \in M$. Note that $r : \widetilde{L}_x \to L$ is the holonomy covering of L. Recall that there is a canonical lift of the normal bundle ν of F to a bundle $\nu_{\mathcal{G}} \subset T\mathcal{G}$ so that $T\mathcal{G} = TF_s \oplus TF_r \oplus \nu_{\mathcal{G}}$ and $r_*\nu_{\mathcal{G}} = \nu$, $s_*\nu_{\mathcal{G}} = \nu$. It is given as follows. Let $[\gamma] \in \mathcal{G}$ with $s\big([\gamma]\big) = x$, $r\big([\gamma]\big) = y$. Denote by $\exp : \nu \to M$ the exponential map. Given $X \in \nu_x$ and $t \in \mathbb{R}$ sufficiently small, there is a unique leafwise curve $\gamma_t : [0,1] \to M$ so that

$$\text{i) } \gamma_t(0) = \exp tX \qquad \text{ii) } \gamma_t(s) \in \exp(\nu_{\gamma(s)}).$$

In particular $\gamma_0 = \gamma$. Thus the family $[\gamma_t]$ in $\mathcal{G}$ defines a tangent vector $\widehat{X} \in T\mathcal{G}_{[\gamma]}$. It is easy to check that $s_*(\widehat{X}) = X$ and $r_*(\widehat{X})$ is the parallel translate of X along γ to ν_y.

The metric g_0 on M induces a canonical metric g_0 on $\mathcal{G}$ as follows. $T\mathcal{G} = TF_s \oplus TF_r \oplus \nu_{\mathcal{G}}$ and these bundles are mutually orthogonal. On TF_r we define g_0 to be $s^*\big(g_0|TF\big)$ and on $TF_s \oplus \nu_{\mathcal{G}} \simeq r^*TM$ we define g_0 to be r^*g_0. So the normal bundle ν_s of TF_s is $\nu_s = TF_r \oplus \nu_{\mathcal{G}}$.

The (reduced) Haefliger cohomology of F, [H80], is given as follows. For each $U_i \in \mathcal{U}$, let $T_i \subset U_i$ be a transversal and set $T = \bigcup T_i$. We may assume that the closures of the T_i are disjoint. Let $\mathcal{H}$ be the holonomy pseudogroup induced by F on T. Give $\mathcal{A}_c^k(T)$ the usual C^∞ topology, and denote the exterior derivative by $d_T : \mathcal{A}_c^k(T) \to \mathcal{A}_c^{k+1}(T)$. The usual Haefliger cohomology is defined using the quotient of $\mathcal{A}_c^k(T)$ by the vector subspace L^k generated by elements of the form $\alpha - h^*\alpha$ where $h \in \mathcal{H}$ and $\alpha \in \mathcal{A}_c^k(T)$ has support contained in the range of h. The (reduced) Haefliger cohomology uses the quotient of $\mathcal{A}_c^k(T)$ by the closure $\overline{L^k}$ of L^k. Set $\mathcal{A}_c^k(M/F) = \mathcal{A}_c^k(T)/\overline{L^k}$. The exterior derivative d_T induces a continuous differential $d_H : \mathcal{A}_c^k(M/F) \to \mathcal{A}_c^{k+1}(M/F)$. Note that $\mathcal{A}_c^k(M/F)$ and d_H are independent of the choice of cover $\mathcal{U}$. In this paper, the complex $\{\mathcal{A}_c(M/F), d_H\}$ and its cohomology $\mathrm{H}_c^*(M/F)$ will be called, respectively, the Haefliger forms and Haefliger cohomology of F.

As the bundle TF is oriented, there is a continuous open surjective linear map, called integration over the leaves,

$$\int_F : \mathcal{A}_c^{p+k}(M) \longrightarrow \mathcal{A}_c^k(M/F)$$

which commutes with the exterior derivatives d_M and d_H. Given $\omega \in \mathcal{A}_c^{p+k}(M)$, write $\omega = \sum \omega_i$ where $\omega_i \in \mathcal{A}_c^{p+k}(U_i)$. Integrate ω_i along the fibers of the submersion $\pi_i : U_i \to T_i$ to obtain $\displaystyle\int_{U_i} \omega_i \in \mathcal{A}_c^k(T_i)$. Define $\displaystyle\int_F \omega \in \mathcal{A}_c^k(M/F)$ to be the class of $\displaystyle\sum_i \int_{U_i} \omega_i$. It is independent of the choice of the ω_i and of the cover $\mathcal{U}$. As $\displaystyle\int_F$ commutes with d_M and d_H, it induces the map $\displaystyle\int_F : \mathrm{H}^{p+k}(M;\mathbb{R}) \to \mathrm{H}_c^k(M/F)$.

2. The $K-$theory index

In this section, we recall the definition of the analytic index of a Dirac operator defined along the leaves of a foliation. We begin with some general remarks about operators along the leaves of foliations.

Let E_1 and E_1' be two complex vector bundles over M with Hermitian metrics and connections, and set $E = r^*E_1$ and $E' = r^*E_1'$ with the pulled back metrics and connections. A pseudo-differential $\mathcal{G}$-operator with uniform support acting from E to E' is a *smooth* family $(P_x)_{x\in M}$ of $\mathcal{G}$-invariant pseudo-differential operators, where for each x, P_x is an operator acting from $E\,|\,\widetilde{L}_x$ to $E'\,|\,\widetilde{L}_x$. The $\mathcal{G}$-invariance property means that for any $\gamma \in \widetilde{L}_x^y = \widetilde{L}_x \cap \widetilde{L}^y$, we have

$$(\gamma \cdot P)_y = U_\gamma \circ P_x \circ U_\gamma^{-1} = P_y,$$

where U_γ denotes the operator on sections of any bundle induced by the isomorphism $\gamma : \widetilde{L}_y \to \widetilde{L}_x$ given by composition with γ; for instance

$$U_\gamma : C_c^\infty(\widetilde{L}_x; E) \longrightarrow C_c^\infty(\widetilde{L}_y; E).$$

The smoothness assumption is rigorously defined in [NWX96]. If we denote by K_x the Schwartz kernel of P_x, then the $\mathcal{G}$-invariance assumption implies that the family $(K_x)_{x\in M}$ induces a distributional section K of $\operatorname{Hom}(E, \widehat{E'})$ over $\mathcal{G}$ which is smooth outside $\mathcal{G}^{(0)} = M$. Here $\widehat{E'} = s^*E_1'$, which is also the pullback bundle of E' under the diffeomorphism $\gamma \mapsto \gamma^{-1}$. Since M is compact, the uniform support condition becomes the assumption that the support of K is compact in $\mathcal{G}$. The space of uniformly supported pseudo-differential $\mathcal{G}-$operators from E to E' is denoted $\Psi^\infty(\mathcal{G}; E, E')$, and the space of uniformly supported regularizing $\mathcal{G}$-operators is denoted by $\Psi^{-\infty}(\mathcal{G}; E, E')$. When $E' = E$ we simply denote the corresponding spaces by $\Psi^\infty(\mathcal{G}; E)$ and $\Psi^{-\infty}(\mathcal{G}; E)$. The Schwartz Kernel Theorem identifies $\Psi^{-\infty}(\mathcal{G}; E, E')$ with $C_c^\infty(\mathcal{G}, \operatorname{Hom}(E, \widehat{E'}))$, see [Con79, NWX96].

An element of $\Psi^\infty(\mathcal{G}; E, E')$ is elliptic if it is elliptic when restricted to each leaf of F_s. The parametrix theorem can be extended to the foliated case and we have

Proposition 2.1. [Con79] *Let P be a uniformly supported elliptic pseudo-differential $\mathcal{G}-$operator acting from E to E'. Then there exists a uniformly supported pseudo-differential $\mathcal{G}-$operator Q acting from E' to E such that*

$$I_E - Q \circ P \in \Psi^{-\infty}(\mathcal{G}; E) \text{ and } I_{E'} - P \circ Q \in \Psi^{-\infty}(\mathcal{G}; E').$$

Here I_E and $I_{E'}$ denote the identity operators of E and E' respectively.

A classical $K-$theory construction assigns to any uniformly supported elliptic pseudo-differential $\mathcal{G}-$operator P from E to E', a $K-$theory class

$$\operatorname{Ind}_a(P) \in \mathrm{K}_0(\Psi^{-\infty}(\mathcal{G}; E \oplus E')) = \mathrm{K}_0(C_c^\infty(\mathcal{G}; \operatorname{Hom}(E \oplus E')))$$

called the analytic index of P, [CM91, BH04]. It will be useful to define this index class using functional calculus in a wider space of smoothing operators, so we now relax the uniform support condition and extend the above pseudodifferential calculus.

A super-exponentially decaying $\mathcal{G}-$operator from E to E' is a family $P = (P_x)_{x\in M}$ of smoothing $\mathcal{G}-$operators so that its Schwartz kernel $P_x(y,z)$ is smooth in x, y, and z, and satisfies

2.2. *Given non-negative integer multi indices α, β, and γ, there are positive constants ϵ, C_1, and C_2, such that for all $x \in M$, $y, z \in \widetilde{L}_x$,*

$$\|\frac{\partial^{|\alpha|+|\beta|+|\gamma|} P_x(y,z)}{\partial x^\alpha \partial y^\beta \partial z^\gamma}\| \leq C_1 \exp\Big[\frac{-d_x(y,z)^{1+\epsilon}}{C_2}\Big].$$

Here $\partial/\partial x$, $\partial/\partial y$, and $\partial/\partial z$ come from coordinates obtained from the finite good cover $\mathcal{U}$ of M and $d_x(\ ,\ )$ is the distance on $\widetilde{L}_x$. The space of all such operators is denoted $\Psi_{\mathfrak{S}}^{-\infty}(\mathcal{G}; E, E')$ or $C_{\mathfrak{S}}^\infty(\mathcal{G}; \operatorname{Hom}(E, \widehat{E'}))$. Again when $E' = E$ we denote the corresponding spaces by $\Psi_{\mathfrak{S}}^{-\infty}(\mathcal{G}; E)$ and $C_{\mathfrak{S}}^\infty(\mathcal{G}; \operatorname{Hom}(E))$ for simplicity. When E and E' are trivial line bundles, we omit them and denote the corresponding spaces by $\Psi_{\mathfrak{S}}^{-\infty}(\mathcal{G})$ and $C_{\mathfrak{S}}^\infty(\mathcal{G})$.

Lemma 2.3. *When $E' = E$, the space $\Psi_{\mathfrak{S}}^{-\infty}(\mathcal{G}; E)$ is an algebra.*

Proof. Let P and $Q \in \Psi^{-\infty}_{\mathfrak{S}}(\mathcal{G}; E)$, with constants ϵ_1, C_1, C_2 and ϵ_2, D_1, D_2 respectively, for the estimate given by Equation 2.2. We may replace ϵ_1 and ϵ_2 by $\epsilon = \min(\epsilon_1, \epsilon_2)$. Set $\alpha = 1 + \epsilon$, $C = C_1 D_1$ and $D = C_2 + D_2$. Then for $y, z \in \widetilde{L}_x$,

$$|P_x \circ Q_x(y,z)| = |\int_{\widetilde{L}_x} P_x(y,w) Q_x(w,z)\, dw| \leq \int_{\widetilde{L}_x} C_1 e^{-d(y,w)^\alpha/C_2} D_1 e^{-d(w,z)^\alpha/D_2}\, dw \leq$$

$$\int_{\widetilde{L}_x} C e^{-d(y,w)^\alpha/D} e^{-d(w,z)^\alpha/D}\, dw = C e^{-(d(y,z)^\alpha/2^\alpha D)} \int_{\widetilde{L}_x} e^{-(d(y,w)^\alpha + d(w,z)^\alpha - (d(y,z)/2)^\alpha)/D}\, dw \leq$$

$$C e^{-(d(y,z)^\alpha/2^\alpha D)} \Big[\int_{S_z} e^{-d(y,w)^\alpha/D}\, dw + \int_{S_y} e^{-d(w,z)^\alpha/D}\, dw \Big] \leq$$

$$C e^{-(d(y,z)^\alpha/2^\alpha D)} \Big[\int_{\widetilde{L}_x} e^{-d(y,w)^\alpha/D}\, dw + \int_{\widetilde{L}_x} e^{-d(w,z)^\alpha/D}\, dw \Big],$$

where

$$S_z = \{w \in \widetilde{L}_x \,|\, d(w,z) \geq d(y,z)/2\} \quad \text{and} \quad S_y = \{w \in \widetilde{L}_x \,|\, d(y,w) \geq d(y,z)/2\}.$$

Now each of the integrals $\displaystyle\int_{\widetilde{L}_x} e^{-d(y,w)^\alpha/D}\, dw$ and $\displaystyle\int_{\widetilde{L}_x} e^{-d(w,z)^\alpha/D}\, dw$ is bounded independently of x, y, and z. This is a standard argument for foliations of compact manifolds. Since M is compact, the leaves $\widetilde{L}_x$ have at most (uniformly bounded) exponential growth, and the integrands are super-exponentially decaying with uniform super-exponential bounds. This gives us the estimate in 2.2 for $P \circ Q$.

To get the estimate for the derivatives $\partial^{|\alpha|+|\beta|+|\gamma|}(P \circ Q)_x(y,z)/\partial x^\alpha \partial y^\beta \partial z^\gamma$ we need only note that these are finite sums of the form

$$\sum_{\alpha_1+\alpha_2=\alpha} \int_{\widetilde{L}_x} \Big(\frac{\partial^{|\alpha_1|+|\beta|} P_x(y,w)}{\partial x^{\alpha_1} \partial y^\beta} \Big) \Big(\frac{\partial^{|\alpha_2|+|\gamma|} Q_x(w,z)}{\partial x^{\alpha_2} \partial z^\gamma} \Big)\, dw.$$

We can then repeat the argument above, using the estimates for the individual integrands.

□

There is a continuous embedding of algebras

$$j_{\mathfrak{S}} : C^\infty_c(\mathcal{G}; \mathrm{Hom}(E \oplus E')) \hookrightarrow C^\infty_{\mathfrak{S}}(\mathcal{G}; \mathrm{Hom}(E \oplus E')),$$

and we define the Schwartz analytic index $\mathrm{Ind}^{\mathfrak{S}}_a$ as the composition of the analytic index Ind_a and the induced morphism $j_{\mathfrak{S}*} : \mathrm{K}_0(C^\infty_c(\mathcal{G}; \mathrm{Hom}(E \oplus E'))) \to \mathrm{K}_0(C^\infty_{\mathfrak{S}}(\mathcal{G}; \mathrm{Hom}(E \oplus E')))$. So if P is a uniformly supported elliptic pseudo-differential $\mathcal{G}-$operator,

$$\mathrm{Ind}^{\mathfrak{S}}_a(P) = j_{\mathfrak{S}*}(\mathrm{Ind}_a(P)) \quad \in \quad \mathrm{K}_0(C^\infty_{\mathfrak{S}}(\mathcal{G}; \mathrm{Hom}(E \oplus E'))).$$

By classical arguments, see for instance [MN96], it is easy to check that $\Psi^{-\infty}_{\mathfrak{S}}(\mathcal{G}; E, E')$ is a right module over the algebra $\Psi^{-\infty}(\mathcal{G})$. The extended pseudodifferential calculus is defined by:

$$\Psi^\infty_{\mathfrak{S}}(\mathcal{G}; E, E') := \Psi^{-\infty}_{\mathfrak{S}}(\mathcal{G}; E, E') \otimes_{\Psi^{-\infty}(\mathcal{G})} \Psi^\infty(\mathcal{G}; E, E').$$

It is generated by $\Psi^\infty(\mathcal{G}; E, E')$ and $\Psi^{-\infty}_{\mathfrak{S}}(\mathcal{G}; E, E')$. When $E' = E$, we obtain in this way an algebra of pseudodifferential operators. The subspace $\Psi^{-\infty}_{\mathfrak{S}}(\mathcal{G}; E)$ is then an ideal in the algebra $\Psi^\infty_{\mathfrak{S}}(\mathcal{G}; E)$. This is due to the estimate given in 2.2. In particular, we may define $\mathrm{Ind}^{\mathfrak{S}}_a(P)$ directly using a parametrix $Q \in \Psi^\infty_{\mathfrak{S}}(\mathcal{G}; E', E)$ and the classical construction, and it is obvious that the two definitions agree.

The construction of the Chern character $\mathrm{ch}_a : \mathrm{K}_0(C^\infty_c(\mathcal{G}; \mathrm{Hom}(E \oplus E'))) \to \mathrm{H}^*_c(M/F)$ in [BH04] is reviewed and extended to this case in Section 3 below. Thus we have

$$\mathrm{ch}^{\mathfrak{S}}_a : \mathrm{K}_0(C^\infty_{\mathfrak{S}}(\mathcal{G}; \mathrm{Hom}(E \oplus E'))) \longrightarrow \mathrm{H}^*_c(M/F)$$

and

$$\mathrm{ch}^{\mathfrak{S}}_a \circ j_{\mathfrak{S}*} = \mathrm{ch}_a\,.$$

Finally, the formula for ch_a in Definition 3.3 below also holds for $\mathrm{ch}^{\mathfrak{S}}_a$.

Now assume that the dimension p of F is even and denote by D a generalized Dirac operator for the foliation F. One of the most important examples of such an operator is given by the leafwise Dirac operator with coefficients in a vector bundle over M. It is defined as follows. As above, let E_1 be a complex vector bundle over M with Hermitian metric and connection, and set $E = r^*(E_1)$ with the pulled back metric and connection. Assume that the tangent bundle TF of F is spin with a fixed spin structure. Then TF_s is also spin, and we endow it with the pulled back spin structure from TF. Denote by $\mathcal{S} = \mathcal{S}^+ \oplus \mathcal{S}^-$ the bundle of spinors along the leaves of F_s. Denote by ∇^0 the connection on TF_s given by the orthogonal projection of the Levi-Civita connection for g_0 on $T\mathcal{G}$. ∇^0 in then the Levi-Civita connection on each leaf of F_s for the induced metric. For all $x \in M$, ∇^0 induces a connection ∇^0 on $\mathcal{S}|\widetilde{L}_x$ and we denote also by ∇^0 the tensor product connection on $\mathcal{S} \otimes E|\widetilde{L}_x$. These data determine a smooth family $D = \{D_x\}$ of Dirac operators, where D_x acts on sections of $\mathcal{S} \otimes E|\widetilde{L}_x$ as follows. Let $X_1, \ldots, X_p$ be a local oriented orthonormal basis of $T\widetilde{L}_x$, and set

$$D_x = \sum_{i=1}^{p} \rho(X_i)\nabla^0_{X_i}$$

where $\rho(X_i)$ is the Clifford action of X_i on the bundle $\mathcal{S} \otimes E|\widetilde{L}_x$. Then D_x does not depend on the choice of the X_i, and it is an odd operator for the $\mathbb{Z}_2$ grading of $\mathcal{S} \otimes E = (\mathcal{S}^+ \otimes E) \oplus (\mathcal{S}^- \otimes E)$. Set $D^+ = D : C_c^\infty(\mathcal{S}^+ \otimes E) \to C_c^\infty(\mathcal{S}^- \otimes E)$ and $D^- = D : C_c^\infty(\mathcal{S}^- \otimes E) \to C_c^\infty(\mathcal{S}^+ \otimes E)$. For more on generalized Dirac operators, see [LM89].

A super-exponentially decaying $\mathcal{G}-$operator on $\mathcal{S} \otimes E$ is defined to be an operator of the form

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix},$$

where each A_{ij} is a smoothing operator whose Schwartz kernel $A_{ij,x}(y,z)$ is smooth in x, y, and z, and satisfies the estimate in 2.2. A_{11} maps sections of $\mathcal{S}^+ \otimes E$ to itself, A_{12} maps sections of $\mathcal{S}^- \otimes E$ to sections of $\mathcal{S}^+ \otimes E$, etc. The set of all such operators is denoted $\Psi_{\mathfrak{S}}^{-\infty}(\mathcal{G}; \mathcal{S} \otimes E)$ or $C_{\mathfrak{S}}^\infty(\mathcal{G}; \mathrm{Hom}(\mathcal{S} \otimes E))$. If we unitalize $\Psi_{\mathfrak{S}}^{-\infty}(\mathcal{G}; \mathcal{S} \otimes E)$ by adding two copies of $\mathbb{C}$ corresponding to the projections $\pi_\pm : C_c^\infty(\mathcal{S} \otimes E) \to C_c^\infty(\mathcal{S}^\pm \otimes E)$, then we get a unital algebra that we denote by $\widetilde{\Psi}_{\mathfrak{S}}^{-\infty}(\mathcal{G}; \mathcal{S} \otimes E)$. Note that $\pi_+ = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}$ and $\pi_- = \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}$. Since the grading operator α for $\mathcal{S} = \mathcal{S}^+ \oplus \mathcal{S}^-$ satisfies $\alpha = \pi_+ - \pi_-$, α belongs to $\widetilde{\Psi}_{\mathfrak{S}}^{-\infty}(\mathcal{G}; \mathcal{S} \otimes E)$.

The odd operator D is elliptic, so its analytic index is defined using a parametrix Q for D which is also odd, i.e.

$$Q = Q^\pm : C_c^\infty(\mathcal{S}^\pm \otimes E) \longrightarrow C_c^\infty(\mathcal{S}^\mp \otimes E).$$

Set

$$S_+ = I - Q^- \circ D^+ \text{ and } S_- = I - D^+ \circ Q^-$$

so

$$S_\pm : C_c^\infty(\mathcal{S}^\pm \otimes E) \longrightarrow C_c^\infty(\mathcal{S}^\pm \otimes E).$$

Using embeddings of our bundles in trivial bundles and computing the boundary map in $K-$theory, it is easy to see that the analytic index of D is the $K-$theory class, see [CM91], in $\mathrm{K}_0(\Psi^{-\infty}(\mathcal{G}; \mathcal{S} \otimes E)) = \mathrm{K}_0(C_c^\infty(\mathcal{G}; \mathrm{Hom}(\mathcal{S} \otimes E))$,

$$\mathrm{Ind}_a(D^+) = [e] - [\pi_-],$$

where the idempotent e is given by

2.4.
$$e = \begin{pmatrix} S_+^2 & -Q^- \circ (S_- + S_-^2) \\ -S_- \circ D^+ & I - S_-^2 \end{pmatrix}.$$

The class $[e]-[\pi_-]$ lives in the K_0-group of the unital algebra $\widetilde{\Psi}^{-\infty}(\mathcal{G};\mathcal{S}\otimes E)$ but its image in the K_0-group of $\mathbb{C}\oplus\mathbb{C}$ under the map induced by

$$p:\begin{pmatrix} A_{11}+\lambda I_{\mathcal{S}^+\otimes E} & A_{12} \\ A_{21} & A_{22}+\mu I_{\mathcal{S}^+\otimes E}\end{pmatrix}\longmapsto(\lambda,\mu),$$

is trivial. Since this epimorphism admits a splitting homomorphism, it is clear that the kernel of the induced map p_* is isomorphic to the K_0-group of the non-unital algebra $\Psi^{-\infty}(\mathcal{G};\mathcal{S}\otimes E)$. Hence, the index $\mathrm{Ind}_a(D^+)=[e]-[\pi_-]$ is well defined.

Proposition 2.5. *Set*

$$P(tD)=\begin{bmatrix} e^{-tD^-D^+} & (-e^{-tD^-D^+/2})\frac{I-e^{-tD^-D^+}}{tD^-D^+}\sqrt{t}D^- \\ -e^{-tD^+D^-/2}\sqrt{t}D^+ & I-e^{-tD^+D^-}\end{bmatrix}.$$

Then, for all $t>0$, $P(tD)$ is an idempotent in $\widetilde{\Psi}^{-\infty}_{\mathfrak{S}}(\mathcal{G};\mathcal{S}\otimes E)$ and

$$[P(tD)]-[\pi_-]=\mathrm{Ind}_a^{\mathfrak{S}}(D^+)\in \mathrm{K}_0(C^\infty_{\mathfrak{S}}(\mathcal{G};\mathrm{Hom}(\mathcal{S}\otimes E))).$$

Proof. It is classical that all the operators in $P(tD)$ (with the possible exception of the term π_-) are smoothing when restricted to any $\widetilde{L}_x$, so their Schwartz kernels are smooth when restricted to any $\widetilde{L}_x$. Thus to check for smoothness, we need only check that they are smooth transversely, i.e. smooth in the variable $x\in M$. The coefficients of the $D^\pm$ are smooth, and Corollary 3.11 of [He95], says that the $e^{-tD^\pm D^\mp}$ are transversely smooth. We will show presently that $e^{-tD^-D^+/2}\dfrac{I-e^{-tD^-D^+}}{tD^-D^+}\sqrt{t}D^- = \sqrt{t}D^-e^{-tD^+D^-/2}\dfrac{I-e^{-tD^+D^-}}{tD^+D^-}$ is also transversely smooth.

By [He95], the Schwartz kernels $P^\pm_{t,x}(y,z)$ of the $e^{-tD^\pm D^\mp}$ satisfy the following estimate. Given a non-negative integer i and non-negative integer multi indices α, β, and γ, and a real number $T>0$, there is a constant $C>0$ such that for all $x\in M$, $y,z\in\widetilde{L}_x$, and $0\le t\le T$,

2.6.
$$\Big\|\frac{\partial^{i+|\alpha|+|\beta|+|\gamma|}P^\pm_{t,x}(y,z)}{\partial t^i\partial x^\alpha\partial y^\beta\partial z^\gamma}\Big\|\le Ct^{-(p/2+i+|\alpha|+|\beta|+|\gamma|)}\exp\Big[\frac{-d_x(y,z)^2}{4t}\Big].$$

It follows immediately that the $e^{-tD^\pm D^\mp}$ and the $e^{-tD^\pm D^\mp/2}$ satisfy the estimate in Equation 2.2, and so also $e^{-tD^+D^-/2}\sqrt{t}D^+=\sqrt{t}D^+e^{-tD^-D^+/2}$, since the derivatives of the coefficients of $D^\pm$ are uniformly bounded on $\mathcal{G}$.

To handle $e^{-tD^-D^+/2}\dfrac{I-e^{-tD^-D^+}}{tD^-D^+}\sqrt{t}D^-=\sqrt{t}D^-e^{-tD^+D^-/2}\dfrac{I-e^{-tD^+D^-}}{tD^+D^-}$, note that

$$\frac{d}{ds}\Big[\frac{I-e^{-sD^+D^-}}{D^+D^-}\Big]=e^{-sD^+D^-},\quad\text{so}\quad \frac{I-e^{-tD^+D^-}}{tD^+D^-}=\frac{1}{t}\int_0^t e^{-sD^+D^-}\,ds.$$

Thus

$$\sqrt{t}D^-e^{-tD^+D^-/2}\frac{I-e^{-tD^+D^-}}{tD^+D^-}=\frac{\sqrt{t}D^-}{t}\int_0^t e^{-(t/2+s)D^+D^-}\,ds=\frac{\sqrt{t}D^-}{t}\int_{t/2}^{3t/2}e^{-sD^+D^-}\,ds.$$

A simple calculation using Equation 2.6 above then shows that for fixed t, $\sqrt{t}D^-e^{-tD^+D^-/2}\dfrac{I-e^{-tD^+D^-}}{tD^+D^-}$ is transversely smooth and that it satisfies the estimate in Equation 2.2.

It is easy to check that the operator $Q(tD)=Q^\pm(tD)$ where $Q^-(tD)=\dfrac{I-e^{-tD^-D^+/2}}{tD^-D^+}\sqrt{t}D^-$ and $Q^+(tD)=\dfrac{I-e^{-tD^+D^-/2}}{tD^+D^-}\sqrt{t}D^+$, is a parametrix for $\sqrt{t}D$. The corresponding idempotent e given by Equation 2.4 is then $P(tD)$, so the Schwartz analytic index of tD is just $[P(tD)]-[\pi_-]$. For $t=1$ it is by definition $\mathrm{Ind}_a^{\mathfrak{S}}(D^+)$. Since $P(tD)$ is a smooth family of idempotents, it follows from results of [BH04] that the K-theory class $[P(tD)]-[\pi_-]$ is independent of t, and so is $\mathrm{Ind}_a^{\mathfrak{S}}(D^+)$ for all $t>0$. $\square$

Remark 2.7. *The above representation of the analytic $K-$theory index uses the isomorphism between the $K-$group of the algebra $C^\infty_{\mathfrak{S}}(\mathcal{G};\mathrm{Hom}(\mathcal{S}\otimes E))$ and the kernel of the homomorphism induced by the surjection*

$$p : \widetilde{\Psi}^{-\infty}_{\mathfrak{S}}(\mathcal{G};\mathcal{S}\otimes E) \longrightarrow \mathbb{C}\oplus\mathbb{C},$$

as described in Section 3.

3. The Chern character in Haefliger cohomology

In this section we review and extend the construction of the Chern-Connes character in Haefliger cohomology given in [BH04]. In view of our definition of the analytic index through the $K-$group of the unitalization $\widetilde{\Psi}^{-\infty}_{\mathfrak{S}}(\mathcal{G};\mathcal{S}\otimes E)$, the Chern character is easy to express in terms of heat kernels.

Denote the connection on the Hermitian bundle E_1 over M by ∇^{E_1}. Extend the leafwise Levi-Civita connection on the bundle of spinors along the leaves of F to a connection ∇^F on this bundle as a bundle over M. Then $\nabla = r^*(\nabla^{F\otimes E_1})$ is an extension of the leafwise connection ∇^0 on $\mathcal{S}\otimes E$ to a connection on $\mathcal{S}\otimes E$ as a bundle over $\mathcal{G}$. We may regard ∇ as an operator of degree one on $C^\infty(\mathcal{S}\otimes E\otimes\wedge T^*\mathcal{G})$ where on decomposable sections $\phi\otimes\omega$, $\nabla(\phi\otimes\omega) = (\nabla\phi)\wedge\omega + \phi\otimes d\omega$. The foliation F_s has normal bundle $\nu_s = TF_r\oplus\nu_{\mathcal{G}}$ and dual normal bundle $\nu_s^* = s^*(T^*M)$, and ∇ defines a *quasi-connection* ∇^ν acting on $C^\infty(\mathcal{S}\otimes E\otimes\wedge\nu_s^*)$ by the composition

$$C^\infty(\mathcal{S}\otimes E\otimes\wedge\nu_s^*) \xrightarrow{i} C^\infty(\mathcal{S}\otimes E\otimes\wedge T^*\mathcal{G}) \xrightarrow{\nabla} C^\infty(\mathcal{S}\otimes E\otimes\wedge T^*\mathcal{G}) \xrightarrow{p_\nu} C^\infty(\mathcal{S}\otimes E\otimes\wedge\nu_s^*),$$

where i is the inclusion and p_ν is induced by the projection $p_\nu : T^*\mathcal{G}\to\nu_s^*$ determined by the decomposition $T\mathcal{G} = TF_s\oplus TF_r\oplus\nu_{\mathcal{G}}$.

Note that $C^\infty(\mathcal{S}\otimes E\otimes\wedge\nu_s^*)$ is an $\mathcal{A}(M)$-module where for $\phi\in C^\infty(\mathcal{S}\otimes E\otimes\wedge\nu_s^*)$, and $\omega\in\mathcal{A}(M)$, we set

$$\omega\cdot\phi = s^*(\omega)\phi.$$

Recall $\Psi^{-\infty}(\mathcal{G};\mathcal{S}\otimes E)\simeq C^\infty_c(\mathcal{G};\mathrm{Hom}(\mathcal{S}\otimes E))$ the space of uniformly supported regularizing $\mathcal{G}$-operators. We may consider the algebra

$$\mathcal{A}_c(\mathcal{G},\mathcal{S}\otimes E) := \Psi^{-\infty}(\mathcal{G};\mathcal{S}\otimes E)\widehat{\otimes}_{C^\infty(M)}\mathcal{A}(M)$$

as a subspace of the space of $\mathcal{A}(M)$-equivariant endomorphisms of $C^\infty(\mathcal{S}\otimes E\otimes\wedge\nu_s^*)$ by using the $\mathcal{A}(M)$ module structure of $C^\infty(\mathcal{S}\otimes E\otimes\wedge\nu_s^*)$. More specifically, given $\phi\in C^\infty(\mathcal{S}\otimes E\otimes\wedge\nu_s^*)$, write it as

$$\phi = \sum_j \phi_j\otimes s^*(\omega_j),$$

where the $\phi_j\in C^\infty(\mathcal{S}\otimes E)$ and the $\omega_j\in\mathcal{A}(M)$. Then for $A\otimes\omega\in\Psi^{-\infty}(\mathcal{G};\mathcal{S}\otimes E)\widehat{\otimes}_{C^\infty(M)}\mathcal{A}(M)$,

$$(A\otimes\omega)(\phi) := \sum_j A(\phi)\otimes s^*(\omega\wedge\omega_j).$$

It is easy to check that this is well defined.

Denote by $\partial_\nu : \mathrm{End}(C^\infty(\mathcal{S}\otimes E\otimes\wedge\nu_s^*))\to\mathrm{End}(C^\infty(\mathcal{S}\otimes E\otimes\wedge\nu_s^*))$ the linear operator given by the graded commutator

$$\partial_\nu(T) = [\nabla^\nu, T].$$

The operator ∂_ν maps the space $\mathcal{A}_c(\mathcal{G},\mathcal{S}\otimes E)$ to itself, and $(\partial_\nu)^2$ is given by the commutator with the curvature $\theta = (\nabla^\nu)^2$ of ∇^ν. The operator θ is a leafwise differential operator. To see this, let (U,γ,V) be a basic open set for $\mathcal{G}$ where $U,V\in\mathcal{U}$ have coordinates $x_1,...,x_p,w_1,...,w_q$ and $y_1,...,y_p,z_1,...,z_q$. The x_i and y_i are the leaf coordinates for F, and the w_i and z_i are the normal coordinates. Then $x_1,...,x_p,y_1,...,y_p,z_1,...,z_q$ are coordinates for (U,γ,V), and TF_s is spanned by the $\partial/\partial y_j$, ν_s^* is spanned by the dx_i and the dz_i, and ν_s is spanned by the $\partial/\partial x_i$ (which span TF_r) and vector fields of the form $\partial/\partial z_i + \sum_{j=1}^p a_{ij}\partial/\partial y_j$ (since vector fields of the form $\partial/\partial z_i + \sum_{j=1}^p a_{ij}\partial/\partial y_j + \sum_{j=1}^p b_{ij}\partial/\partial x_j$ span $\nu_{\mathcal{G}}$). The a_{ij} are locally defined functions on (U,γ,V) which only depend on $y_1,...,y_p,z_1,...,z_q$, i.e. they are pull backs of functions on V. In particular, on V, the vector fields $\partial/\partial z_i + \sum_{j=1}^p a_{ij}\partial/\partial y_j$ span ν. A simple computation then shows that

$$p_\nu(dx_i) = dx_i, \quad p_\nu(dz_i) = dz_i, \quad \text{and} \quad p_\nu(dy_j) = a_{ij}dz_i,$$

where we use the Einstein convention of summing over repeated indices. Suppose that ϕ is a local section of $C^\infty(\mathcal{S}\otimes E\otimes\wedge\nu_s^*)$ and f is a smooth function defined on (U,γ,V). Then another simple computation shows that

$$\theta(f\phi) = p_\nu d(p_\nu df)\phi + f\theta(\phi).$$

Thus we need to show that the operator $(p_\nu d)^2$ is a leafwise differential operator. Now

$$p_\nu df = \frac{\partial f}{\partial x_i}dx_i + \frac{\partial f}{\partial z_i}dz_i + \frac{\partial f}{\partial y_j}a_{ij}dz_i,$$

so

$$p_\nu d(p_\nu df) = \frac{\partial^2 f}{\partial x_k\partial x_i}dx_k\wedge dx_i + \frac{\partial^2 f}{\partial x_k\partial z_i}dx_k\wedge dz_i + \frac{\partial^2 f}{\partial x_k\partial y_j}a_{ij}dx_k\wedge dz_i + \frac{\partial f}{\partial y_j}\frac{\partial a_{ij}}{\partial x_k}dx_k\wedge dz_i +$$
$$\frac{\partial^2 f}{\partial z_k\partial x_i}dz_k\wedge dx_i + \frac{\partial^2 f}{\partial z_k\partial z_i}dz_k\wedge dz_i + \frac{\partial^2 f}{\partial z_k\partial y_j}a_{ij}dz_k\wedge dz_i + \frac{\partial f}{\partial y_j}\frac{\partial a_{ij}}{\partial z_k}dz_k\wedge dz_i +$$
$$\frac{\partial^2 f}{\partial y_k\partial x_i}a_{\ell k}dz_\ell\wedge dx_i + \frac{\partial^2 f}{\partial y_k\partial z_i}a_{\ell k}dz_\ell\wedge dz_i + \frac{\partial^2 f}{\partial y_k\partial y_j}a_{\ell k}a_{ij}dz_\ell\wedge dz_i + \frac{\partial f}{\partial y_j}\frac{\partial a_{ij}}{\partial y_k}a_{\ell k}dz_\ell\wedge dz_i.$$

The first, sixth, and eleventh terms are zero as usual, the fourth term is zero as $\partial a_{ij}/\partial x_k = 0$, the second and fifth terms cancel, the third and ninth terms cancel, and the seventh and tenth terms cancel. Thus

$$(p_\nu d)^2 = \left[\frac{\partial a_{ij}}{\partial z_k} + \frac{\partial a_{ij}}{\partial y_\ell}a_{k\ell}\right]dz_k\wedge dz_i\otimes\frac{\partial}{\partial y_j},$$

which is a first order leafwise differential operator, with coefficients in $\wedge^*\nu_s^*$. In fact θ has coefficients in $r^*(\wedge^*\nu^*)$, as it is the pull back under r of an analogous operator on M. In particular, we may form the operator θ^F on M with respect to the foliation F and the normal bundle ν in complete analogy with the operator θ. Using the coordinates above, it is clear that $\theta\,|\,(U,\gamma,V) = r^*\theta^F\,|\,V$. This fact will allow us to handle θ in the estimates used in the proof of the main theorem.

In the same way as above, we consider the algebra

$$\mathcal{A}_{\mathfrak{S}}(\mathcal{G},\mathcal{S}\otimes E) := \Psi_{\mathfrak{S}}^{-\infty}(\mathcal{G};\mathcal{S}\otimes E)\widehat{\otimes}_{C^\infty(M)}\mathcal{A}(M)$$

as a subspace of the space of $\mathcal{A}(M)$-equivariant endomorphisms of $C^\infty(\mathcal{S}\otimes E\otimes\wedge\nu_s^*)$, where $\Psi_{\mathfrak{S}}^{-\infty}(\mathcal{G};\mathcal{S}\otimes E)$ is the algebra of superexponentially decaying $\mathcal{G}-$operators defined in the previous section. To extend our Chern-Connes character to $\mathrm{K}_0(\Psi_{\mathfrak{S}}^{-\infty}(\mathcal{G};\mathcal{S}\otimes E))$, we need the following.

Lemma 3.1. *∂_ν preserves $\mathcal{A}_{\mathfrak{S}}(\mathcal{G},\mathcal{S}\otimes E)$, and $(\partial_\nu)^2$ is given by the commutator with θ.*

Proof. This is a local (on M) question, so we may restrict attention to $U\in\mathcal{U}$. If $H\in\Psi_{\mathfrak{S}}^{-\infty}(\mathcal{G};\mathcal{S}\otimes E)\,|_U$ and ω is a differential form on U, then $\partial_\nu(H\otimes s^*(\omega)) = \partial_\nu(H)\otimes s^*(\omega)\pm H\otimes p_\nu ds^*(\omega)$. As any element of $\mathcal{A}_{\mathfrak{S}}(\mathcal{G},\mathcal{S}\otimes E)\,|_U$ may be written as a sum of elements of the form $H\otimes s^*(\omega)$, to show that ∂_ν preserves $\mathcal{A}_{\mathfrak{S}}(\mathcal{G},\mathcal{S}\otimes E)$, we need only show that $i_X\partial_\nu H\in\Psi_{\mathfrak{S}}^{-\infty}(\mathcal{G};\mathcal{S}\otimes E)\,|_U$, for any bounded vector field X on U. It then follows immediately from the proof of Lemma 3.1 in [BH04] that $(\partial_\nu)^2$ is given by the commutator with θ.

The Schwartz kernel of H (also denoted H) is a section of a bundle over the double graph $\mathcal{G}_{[2]} = \bigcup_{x\in M}\widetilde{L}_x\times\widetilde{L}_x$. Given basic open sets (U,γ_1,V_1) and (U,γ_2,V_2) of $\mathcal{G}$, the basic open set $(U,\gamma_1,V_1,\gamma_2,V_2)$ on $\mathcal{G}_{[2]}$ consists of ordered pairs $([\alpha_1],[\alpha_2])$, where $[\alpha_j]\in(U,\gamma_j,V_j)$, and $s([\alpha_1]) = s([\alpha_2])$. Let $x_1,...,x_p,w_1,...,w_q$ be coordinates on U, and $y_1^j,...,y_p^j,z_1^j,...,z_q^j$ coordinates on V_j, where the x_i and y_i^j are the leaf coordinates, and the w_i and z_i^j are the normal coordinates. Note carefully that we now use $x_1,...,x_p,w_1,...,w_q,y_1^j,...,y_p^j$ as coordinates for (U,γ_j,V_j), and $x_1,...,x_p,w_1,...,w_q,y_1^1,...,y_p^1,y_1^2,...,y_p^2$ as coordinates for $(U,\gamma_1,V_1,\gamma_2,V_2)$. On V_j, ν is spanned by the vector fields $\partial/\partial z_i^j + \sum_{k=1}^p a_{ik}^j\partial/\partial y_k^j$, where the a_{ik}^j are locally defined functions, each of whose derivatives is uniformly bounded over all V_j in the good cover $\mathcal{U}$. On (U,γ_j,V_j), we still have the one forms dz_i^j, and as above

$$p_\nu(dx_i)=dx_i \quad p_\nu(dw_i)=dw_i, \quad p_\nu(dz_i^j)=dz_i^j, \quad \text{and} \quad p_\nu(dy_i^j)=a_{ki}^j dz_k^j.$$

Denote by $\mathcal{S}_0$ the bundle of spinors along TF, and suppose that on V_j, $\nabla^{F\otimes E_1}$ is given by

$$\nabla^{F\otimes E_1}\,|_{V_j} = d_M + A_i^j dy_i^j + B_k^j dz_k^j,$$

where $A_i^j, B_k^j \in C^\infty(V_j;\mathrm{Hom}(\mathcal{S}_0\otimes E_1))$. The coefficients of the A_i^j and B_k^j (with respect to orthonormal bases of $\mathcal{S}_0\otimes E_1$) have each of their derivatives uniformly bounded over all V_j in the good cover $\mathcal{U}$. Then on (U,γ_j,V_j), $\nabla = d_{\mathcal{G}} + A_i^j dy_i^j + B_k^j dz_k^j$ and

$$\nabla^\nu = \partial/\partial x_i \otimes p_\nu dx_i + \partial/\partial w_i\otimes p_\nu dw_i + \partial/\partial y_i^j \otimes p_\nu dy_i^j + A_i^j p_\nu dy_i^j + B_k^j p_\nu dz_k^j,$$

where A_i^j, B_k^j are now in $C^\infty((U,\gamma_j,V_j);\mathrm{Hom}(\mathcal{S}\otimes E))$, and are really $A_i^j\circ r$ and $B_k^j\circ r$. Denote by $\pi_j: (U,\gamma_1,V_1,\gamma_2,V_2)\to(U,\gamma_j,V_j)$ the obvious projections. Then a straight forward computation shows that the Schwartz kernel $\partial_\nu H$ on $(U,\gamma_1,V_1,\gamma_2,V_2)$ with respect to the corresponding local bases, is given by

$$\begin{aligned}\partial_\nu H = \pi_1^* p_\nu(\nabla\,|_{(U,\gamma_1,V_1)})\circ H - H\circ \pi_2^* p_\nu(\nabla\,|_{(U,\gamma_2,V_2)}) = \\ \frac{\partial H}{\partial x_i}\pi_1^* p_\nu dx_i + \frac{\partial H}{\partial w_i}\pi_1^* p_\nu dw_i + \frac{\partial H}{\partial y_i^1}\pi_1^* p_\nu dy_i^1 + \frac{\partial H}{\partial y_i^2}\pi_2^* p_\nu dy_i^2 + \\ \pi_1^*\Big(A_i^1 p_\nu dy_i^1 + B_k^1 p_\nu dz_k^1\Big)\circ H - H\circ\pi_2^*\Big(A_i^2 p_\nu dy_i^2 + B_k^2 p_\nu dz_k^2\Big) = \\ \frac{\partial H}{\partial w_i}dw_i + \frac{\partial H}{\partial y_i^1}\pi_1^*(a_{ki}^1 dz_k^1) + \frac{\partial H}{\partial y_i^2}\pi_2^*(a_{ki}^2 dz_k^2) + \pi_1^*\Big((A_i^1 a_{ki}^1 + B_k^1)dz_k^1\Big)\circ H - H\circ\pi_2^*\Big((A_i^2 a_{ki}^2 + B_k^2)dz_k^2\Big),\end{aligned}$$

as $\partial H/\partial x_i = 0$ since H is $\mathcal{G}-$invariant. Since the w_i are functions of the z_k^1, $\partial_\nu H$ depends only on the y_k^j and z_k^j, so it is also $\mathcal{G}$ invariant. In addition,

$$\pi_j^* dz_k^j = h_{\gamma_j}^* dz_k^j = c_{km}^j dw_m,$$

where $h_{\gamma_j}^*$ is the holonomy map defined by γ_j from $\nu^*\,|_{V_j}$ to $\nu^*\,|_U$. As $h_{\gamma_j}^*$ is an isometry, the c_{km}^j have each of their derivatives uniformly bounded over all $(U,\gamma_1,V_1,\gamma_2,V_2)$. Since each of the derivatives of the A_i^j, B_i^j and a_{ki}^j are also uniformly bounded, and H is super-exponentially decaying, it follows easily that for any bounded vector field X on U, $i_X\partial_\nu H$ is also super-exponentially decaying. □

By the Schwartz kernel theorem, $\mathcal{A}_{\mathfrak{S}}(\mathcal{G},\mathcal{S}\otimes E)$ is isomorphic to the algebra

$$C_{\mathfrak{S}}^\infty(\mathcal{G};\mathrm{Hom}(\mathcal{S}\otimes E))\widehat{\otimes}_{C^\infty(M)}\mathcal{A}(M).$$

For any $T\in\mathcal{A}_{\mathfrak{S}}(\mathcal{G},\mathcal{S}\otimes E)$, define the trace of T to be the (compactly supported) Haefliger k-form $\mathrm{Tr}(T)$ given by

$$\mathrm{Tr}(T) = \int_F \mathrm{tr}(T_x(\overline{x},\overline{x}))dx,$$

where $T_x(\overline{x},\overline{x})$ is the smooth Schwartz kernel of T, $\overline{x}$ is the class of the constant path at x, $\mathrm{tr}(T_x(\overline{x},\overline{x}))$ is the usual trace of $T_x(\overline{x},\overline{x})\in\mathrm{End}((\mathcal{S}\otimes E)_{\overline{x}})\otimes\wedge TM_x^*$ and so belongs to $\wedge TM_x^*$, and dx is the leafwise volume form associated with the fixed orientation of the foliation F. The map

$$\mathrm{Tr}:\mathcal{A}_{\mathfrak{S}}(\mathcal{G},\mathcal{S}\otimes E)\longrightarrow\mathcal{A}_c(M/F)$$

is then a graded trace which satisfies $\mathrm{Tr}\circ\partial_\nu = d_H\circ\mathrm{Tr}$. See [HL02], Lemma 2.5, and [BH04], Lemma 3.2. Note that Lemma 2.5 of [HL02] requires one of the elements to be uniformly exponentially decaying while the other must have uniformly bounded coefficients. But if an operator is uniformly exponentially decaying it does have uniformly bounded coefficients.

Since ∂_ν^2 is not necessarily zero, we used Connes' $X-$trick to construct a new graded differential algebra $(\widetilde{\mathcal{A}}_{\mathfrak{S}},\delta)$ out of the graded quasi-differential algebra $(A_{\mathfrak{S}}(\mathcal{G},\mathcal{S}\otimes E),\partial_\nu)$, see [Con94], p. 229. By Lemma 3.1, the curvature operator θ preserves $\mathcal{A}_{\mathfrak{G}}(\mathcal{G}, \mathcal{S}\otimes E)$. As a vector space $\widetilde{\mathcal{A}}_{\mathfrak{G}} = M_2(A_{\mathfrak{G}}(\mathcal{G}, \mathcal{S}\otimes E))$. An element $\widetilde{T} = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} \in \widetilde{\mathcal{A}}_{\mathfrak{G}}$ is homogeneous of degree $\partial\widetilde{T} = k$ if

$$k = \partial T_{11} = \partial T_{12} + 1 = \partial T_{21} + 1 = \partial T_{22} + 2.$$

On homogeneous elements of $\widetilde{\mathcal{A}}_{\mathfrak{G}}$, δ is given by

$$\delta\widetilde{T} = \begin{pmatrix} \partial_\nu T_{11} & \partial_\nu T_{12} \\ -\partial_\nu T_{21} & -\partial_\nu T_{22} \end{pmatrix} + \begin{pmatrix} 0 & -\theta \\ 1 & 0 \end{pmatrix}\widetilde{T} + (-1)^{\partial\widetilde{T}}\widetilde{T}\begin{pmatrix} 0 & 1 \\ -\theta & 0 \end{pmatrix},$$

and is extended to non-homogenous elements by linearity. A straightforward computation gives $\delta^2 = 0$. For homogeneous $T \in A_{\mathfrak{G}}(\mathcal{G}, \mathcal{S}\otimes E)$, the differential δ on $\begin{pmatrix} T & 0 \\ 0 & 0 \end{pmatrix} \in \widetilde{\mathcal{A}}_{\mathfrak{G}}$ is given by

$$\delta\begin{pmatrix} T & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \partial_\nu T & (-1)^{\partial T}T \\ T & 0 \end{pmatrix}.$$

Set

$$\Theta = \begin{pmatrix} 1 & 0 \\ 0 & \theta \end{pmatrix}$$

and define a new product on $\widetilde{\mathcal{A}}_{\mathfrak{G}}$ by

$$\widetilde{T} * \widetilde{T}' = \widetilde{T}\Theta\widetilde{T}'.$$

This makes $(\widetilde{\mathcal{A}}_{\mathfrak{G}}, \delta)$ a graded differential algebra.

The graded algebra $A_{\mathfrak{G}}(\mathcal{G}, \mathcal{S}\otimes E)$ embeds as a subalgebra of $\widetilde{\mathcal{A}}_{\mathfrak{G}}$ by using the map

$$T \hookrightarrow \begin{pmatrix} T & 0 \\ 0 & 0 \end{pmatrix}.$$

We shall therefore also denote by T the image in $\widetilde{\mathcal{A}}_{\mathfrak{G}}$ of any $T \in A_{\mathfrak{G}}(\mathcal{G}, \mathcal{S}\otimes E)$.

For homogeneous $\widetilde{T} \in \widetilde{\mathcal{A}}_{\mathfrak{G}}$ define

$$\Phi(\widetilde{T}) = \operatorname{Tr}(T_{11}) - (-1)^{\partial\widetilde{T}}\operatorname{Tr}(T_{22}\theta),$$

and extend to arbitrary elements by linearity. The results of [BH04] extend easily to show that the map $\Phi : \widetilde{\mathcal{A}}_{\mathfrak{G}} \to \mathcal{A}^*_c(M/F)$ is a graded trace, and that $\Phi\circ\delta = d_H\circ\Phi$.

The (algebraic) Chern-Connes character in the even case is the morphism

$$\mathrm{ch}_a : \mathrm{K}_0(C^\infty_{\mathfrak{G}}(\mathcal{G}, \mathcal{S}\otimes E)) = \mathrm{K}_0(\Psi^{-\infty}_{\mathfrak{G}}(\mathcal{G}; \mathcal{S}\otimes E)) \longrightarrow \mathrm{H}^*_c(M/F)$$

defined as follows. Denote by $\widehat{\Psi}^{-\infty}_{\mathfrak{G}}(\mathcal{G}; \mathcal{S}\otimes E)$ the minimal unitalization of $\Psi^{-\infty}_{\mathfrak{G}}(\mathcal{G}; \mathcal{S}\otimes E)$. This amounts to adding a copy of the complex numbers $\mathbb{C}$, so

$$\widehat{\Psi}^{-\infty}_{\mathfrak{G}}(\mathcal{G}; \mathcal{S}\otimes E) = \Psi^{-\infty}_{\mathfrak{G}}(\mathcal{G}; \mathcal{S}\otimes E) \oplus \mathbb{C}.$$

Let $M_N(\widehat{\Psi}^{-\infty}_{\mathfrak{G}}(\mathcal{G}; \mathcal{S}\otimes E))$ be the space of $N\times N$ matrices with coefficients in $\widehat{\Psi}^{-\infty}_{\mathfrak{G}}(\mathcal{G}; \mathcal{S}\otimes E)$. Denote by $\mathrm{tr} : M_N(\Psi^{-\infty}_{\mathfrak{G}}(\mathcal{G}; \mathcal{S}\otimes E)) \to \Psi^{-\infty}_{\mathfrak{G}}(\mathcal{G}; \mathcal{S}\otimes E)$ the usual trace.

The results in [BH04] again extend easily to give the following.

Theorem 3.2. *Let $B = [\tilde{e}_1] - [\tilde{e}_2]$ be an element of $\mathrm{K}_0(\Psi^{-\infty}_{\mathfrak{G}}(\mathcal{G}; \mathcal{S}\otimes E))$, where $\tilde{e}_1 = (e_1, \lambda_1)$ and $\tilde{e}_2 = (e_2, \lambda_2)$ are idempotents in $M_N(\widehat{\Psi}^{-\infty}_{\mathfrak{G}}(\mathcal{G}; \mathcal{S}\otimes E))$. Then the Haefliger forms*

$$(\Phi\circ\mathrm{tr})\Big(e_1\exp\Big(\frac{-(\delta e_1)^2}{2i\pi}\Big)\Big) \text{ and } (\Phi\circ\mathrm{tr})\Big(e_2\exp\Big(\frac{-(\delta e_2)^2}{2i\pi}\Big)\Big)$$

are closed and the Haefliger cohomology class of their difference depends only on B.

Definition 3.3. *The algebraic Chern character $\mathrm{ch}_a(B)$ of B is the Haefliger cohomology class*

3.4.
$$\mathrm{ch}_a(B) = \Big[(\Phi\circ\mathrm{tr})\Big(e_1\exp\Big(\frac{-(\delta e_1)^2}{2i\pi}\Big)\Big)\Big] - \Big[(\Phi\circ\mathrm{tr})\Big(e_2\exp\Big(\frac{-(\delta e_2)^2}{2i\pi}\Big)\Big)\Big].$$

In order to effectively compute the Chern character of the index of a generalized Dirac operator for F, we need some further results. The exact sequence of algebras

$$0 \to \Psi^{-\infty}_{\mathfrak{G}}(\mathcal{G}; \mathcal{S} \otimes E) \xhookrightarrow{i} \widetilde{\Psi}^{-\infty}_{\mathfrak{G}}(\mathcal{G}; \mathcal{S} \otimes E) \xrightarrow{p} \mathbb{C}^2 \to 0$$

has a splitting homomorphism $\varrho : \mathbb{C}^2 \to \widetilde{\Psi}^{-\infty}_{\mathfrak{G}}(\mathcal{G}; \mathcal{S} \otimes E)$ given by $\varrho(\lambda, \mu) = \lambda\pi_+ + \mu\pi_-$. Therefore the kernel of the induced map

$$p_* : K_0(\widetilde{\Psi}^{-\infty}_{\mathfrak{G}}(\mathcal{G}; \mathcal{S} \otimes E)) \longrightarrow K_0(\mathbb{C}^2) \simeq \mathbb{Z}^2,$$

is isomorphic to the group $K_0(\Psi^{-\infty}_{\mathfrak{G}}(\mathcal{G}; \mathcal{S} \otimes E))$. Denote by p_0 the obvious projection of $\widehat{\Psi}^{-\infty}_{\mathfrak{G}}(\mathcal{G}; \mathcal{S} \otimes E)$ onto $\mathbb{C}$. Then the inclusion map

$$\beta : \widehat{\Psi}^{-\infty}_{\mathfrak{G}}(\mathcal{G}; \mathcal{S} \otimes E) \longrightarrow \widetilde{\Psi}^{-\infty}_{\mathfrak{G}}(\mathcal{G}; \mathcal{S} \otimes E),$$

given by $\beta(T, \lambda) = T + \lambda\pi_+ + \lambda\pi_-$ induces the isomorphism

$$\beta_* : K_0(\Psi^{-\infty}_{\mathfrak{G}}(\mathcal{G}; \mathcal{S} \otimes E)) = \operatorname{Ker}(p_{0,*}) \longrightarrow \operatorname{Ker}(p_*) \subset K_0(\widetilde{\Psi}^{-\infty}_{\mathfrak{G}}(\mathcal{G}; \mathcal{S} \otimes E)).$$

We shall use the universal graded algebra in the proof of Proposition 3.5 below, so we recall its definition. To any algebra $\mathcal{C}$, there corresponds a (universal) differential graded algebra $\Omega(\mathcal{C}) = \oplus_{n\geq 0}\Omega^n(\mathcal{C})$ which is defined by

$$\Omega^0(\mathcal{C}) := \mathcal{C} \oplus \mathbb{C}, \text{ and for } n \geq 1, \Omega^n(\mathcal{C}) := (\mathcal{C} \oplus \mathbb{C}) \otimes \mathcal{C}^{\otimes n}.$$

The differential $d : \Omega^n(\mathcal{C}) \to \Omega^{n+1}(\mathcal{C})$ is defined for $a^j \in \mathcal{C}$ and $c \in \mathbb{C}$ by

$$d\left[(a^0 + c) \otimes a^1 \otimes \cdots \otimes a^n)\right] := 1 \otimes a^0 \otimes a^1 \otimes \cdots \otimes a^n.$$

It is clear that by definition $d^2 = 0$. The space $\Omega^n(\mathcal{C})$ is endowed with a natural right $\mathcal{C}$-module structure (and hence right $\mathcal{C} \oplus \mathbb{C}$-module structure) defined by

$$((a^0 + c) \otimes a^1 \otimes \cdots \otimes a^n)a^{n+1} := (-1)^n \sum_{j=0}^{n} (-1)^j (a^0 + c) \otimes \cdots \otimes a^j a^{j+1} \otimes \cdots \otimes a^{n+1}.$$

The algebra structure of $\Omega(\mathcal{C})$ is defined by setting

$$((a^0 + c) \otimes a^1 \otimes \cdots \otimes a^n)(b^0 \otimes b^1 \otimes \cdots \otimes b^k) := ((a^0 + c) \otimes a^1 \otimes \cdots \otimes a^n)b^0 \otimes b^1 \otimes \cdots \otimes b^k$$

and

$$((a^0 + c) \otimes a^1 \otimes \cdots \otimes a^n)c' \otimes b^1 \otimes \cdots \otimes b^k) := c'[(a^0 + c) \otimes a^1 \otimes \cdots \otimes a^n \otimes b^1 \otimes \cdots \otimes b^k].$$

A straightforward verification shows that $(\Omega(\mathcal{C}), d)$ is a differential graded algebra, see [Con85]. We point out that by definition

$$(a^0 + c)da^1 \cdots da^n = (a^0 + c) \otimes a^1 \otimes \cdots \otimes a^n.$$

The following is known to experts. We give the proof for completeness, since it will be used in the sequel.

Proposition 3.5. *Let $\tilde{e}$ and $\tilde{e}'$ be two idempotents in $M_N(\widetilde{\Psi}^{-\infty}_{\mathfrak{G}}(\mathcal{G}; \mathcal{S} \otimes E))$ such that $[\tilde{e}] - [\tilde{e}']$ belongs to the kernel of p_*. Then the Haefliger forms*

$$(\Phi \circ \operatorname{tr})\Big((\tilde{e} - (\varrho \circ p)(\tilde{e})) \exp\left(\frac{-(\delta(\tilde{e} - (\varrho \circ p)(\tilde{e})))^2}{2i\pi}\right)\Big) \text{ and }$$

$$(\Phi \circ \operatorname{tr})\Big((\tilde{e}' - (\varrho \circ p)(\tilde{e}')) \exp\left(\frac{-(\delta(\tilde{e}' - (\varrho \circ p)(\tilde{e}')))^2}{2i\pi}\right)\Big)$$

are closed and we have the following equality in Haefliger cohomology:

$$(\operatorname{ch}_a \circ \beta_*^{-1})([\tilde{e}] - [\tilde{e}']) = \left[(\Phi \circ \operatorname{tr})\Big((\tilde{e} - (\varrho \circ p)(\tilde{e})) \exp\left(\frac{-(\delta(\tilde{e} - (\varrho \circ p)(\tilde{e})))^2}{2i\pi}\right)\Big)\right]$$
$$- \left[(\Phi \circ \operatorname{tr})\Big((\tilde{e}' - (\varrho \circ p)(\tilde{e}')) \exp\left(\frac{-(\delta(\tilde{e}' - (\varrho \circ p)(\tilde{e}')))^2}{2i\pi}\right)\Big)\right].$$

Proof. We define for every $k \geq 0$ a multilinear functional $\widetilde{\Phi}$ on the unital algebra $\widetilde{\Psi}_{\mathfrak{S}}^{-\infty}(\mathcal{G};\mathcal{S}\otimes E)$ by the equality

$$\widetilde{\Phi}(\widetilde{T}^0,\cdots,\widetilde{T}^k) := \Phi(T^0\delta T^1\cdots\delta T^k) + \Phi(\delta(\Lambda^0 T^1)\delta T^2\cdots\delta T^k),$$

where $\widetilde{T}^j = T^j + \Lambda^j \in \widetilde{\Psi}_{\mathfrak{S}}^{-\infty}(\mathcal{G};\mathcal{S}\otimes E)$ with

$$T^j = \widetilde{T}^j - (\varrho\circ p)(\widetilde{T}^j) \in \Psi_{\mathfrak{S}}^{-\infty}(\mathcal{G};\mathcal{S}\otimes E) \text{ and } \Lambda^j = \varrho\circ p(\widetilde{T}^j) = \begin{pmatrix} \lambda^j & 0 \\ 0 & \mu^j \end{pmatrix} = \lambda^j\pi_+ + \mu^j\pi_-.$$

Then $\widetilde{\Phi}$ is a functional on the universal differential graded algebra associated with $\widetilde{\Psi}_{\mathfrak{S}}^{-\infty}(\mathcal{G};\mathcal{S}\otimes E)$, see [Con85] and also the bivariant constructions in [CQ97, Nis93]. More precisely, we set:

$$\widetilde{\Phi}((\widetilde{T}^0 + c)d\widetilde{T}^1\cdots d\widetilde{T}^k) := \widetilde{\Phi}(\widetilde{T}^0,\cdots,\widetilde{T}^k).$$

We then have by definition

$$(\widetilde{\Phi}\circ d) = 0$$

on the universal differential graded algebra associated with $\widetilde{\Psi}_{\mathfrak{S}}^{-\infty}(\mathcal{G};\mathcal{S}\otimes E)$.

For $\widetilde{T}^j = T^j + \Lambda^j \in \widetilde{\Psi}_{\mathfrak{S}}^{-\infty}(\mathcal{G};\mathcal{S}\otimes E)$, we have

$$\begin{aligned}
(-1)^k\widetilde{\Phi}([\widetilde{T}^0 d\widetilde{T}^1\cdots d\widetilde{T}^k,\widetilde{T}^{k+1}]) &= (-1)^k\widetilde{\Phi}(\widetilde{T}^0 d\widetilde{T}^1\cdots d\widetilde{T}^k\widetilde{T}^{k+1}) - (-1)^k\widetilde{\Phi}(\widetilde{T}^{k+1}\widetilde{T}^0 d\widetilde{T}^1\cdots d\widetilde{T}^k)\\
&= \widetilde{\Phi}(\widetilde{T}^0\widetilde{T}^1 d\widetilde{T}^2\cdots d\widetilde{T}^{k+1}) + \sum_{j=1}^k(-1)^j\widetilde{\Phi}(\widetilde{T}^0 d\widetilde{T}^1\cdots d\widetilde{T}^{j-1}d(\widetilde{T}^j\widetilde{T}^{j+1})d\widetilde{T}^{j+2}\cdots d\widetilde{T}^{k+1})\\
&- (-1)^k\widetilde{\Phi}(\widetilde{T}^{k+1}\widetilde{T}^0 d\widetilde{T}^1\cdots d\widetilde{T}^k)\\
&= \Phi((T^0T^1 + \Lambda^0T^1 + T^0\Lambda^1)\delta T^2\cdots\delta T^{k+1}) + \Phi(\delta(\Lambda^0\Lambda^1T^2)\delta T^3\cdots\delta T^{k+1})\\
&+ \sum_{j=1}^k(-1)^j\Phi(T^0\delta T^1\cdots\delta T^{j-1}\delta(T^jT^{j+1} + \Lambda^jT^{j+1} + T^j\Lambda^{j+1})\delta T^{j+2}\cdots\delta T^{k+1})\\
&- \Phi(\delta(\Lambda^0(T^1T^2 + \Lambda^1T^2 + T^1\Lambda^2))\delta T^3\cdots\delta T^{k+1})\\
&+ \sum_{j=2}^k(-1)^j\Phi(\delta(\Lambda^0T^1)\delta T^2\cdots\delta T^{j-1}\delta(T^jT^{j+1} + \Lambda^jT^{j+1} + T^j\Lambda^{j+1})\delta T^{j+2}\cdots\delta T^{k+1})\\
&- (-1)^k\Phi((T^{k+1}T^0 + T^{k+1}\Lambda^0 + \Lambda^{k+1}T^0)\delta T^1\cdots\delta T^k)\\
&- (-1)^k\Phi(\delta(\Lambda^{k+1}\Lambda^0T^1)\delta T^2\cdots\delta T^k).
\end{aligned}$$

By using a connection which commutes with the grading we insure that $\partial^\nu(\Lambda) = 0$ for any $\Lambda \in \mathbb{C}\pi_+\oplus\mathbb{C}\pi_-$. Thus, using the definitions of the product and the differential δ, we can easily deduce the following relations for all Λ, T, Λ', and T':

3.6. $\partial^\nu(\Lambda T) = \Lambda(\partial^\nu T)$, $\partial^\nu(T\Lambda) = (\partial^\nu T)\Lambda$, $\theta\Lambda T = \Lambda\theta T$, $T\Lambda\delta(T') = T\delta(\Lambda T')$, $\delta(T\Lambda)T' = (\delta T)(\Lambda T')$,

$$\delta(T\Lambda)\delta(T') = \delta(T)\delta(\Lambda T'),\quad T\Lambda\delta(\Lambda'T') = T\delta(\Lambda\Lambda'T') \quad \textit{and} \quad \delta(TT') = \delta TT' + T\delta T'.$$

It is then a straightforward calculation that

$$\Phi((T^0T^1 + \Lambda^0T^1 + T^0\Lambda^1)\delta T^2\cdots\delta T^{k+1}) +$$

$$\sum_{j=1}^k(-1)^j\Phi(T^0\delta T^1\cdots\delta T^{j-1}\delta(T^jT^{j+1} + \Lambda^jT^{j+1} + T^j\Lambda^{j+1})\delta T^{j+2}\cdots\delta T^{k+1})$$

collapses to

$$\Phi(\Lambda^0T^1\delta T^2\cdots\delta T^{k+1}) + (-1)^k\Phi(T^0\delta T^1\cdots\delta T^{k-1}(\delta T^kT^{k+1} + \delta(T^k\Lambda^{k+1}))),$$

and

$$\Phi(\delta(\Lambda^0\Lambda^1T^2)\delta T^3\cdots\delta T^{k+1}) - \Phi(\delta(\Lambda^0(T^1T^2 + \Lambda^1T^2 + T^1\Lambda^2))\delta T^3\cdots\delta T^{k+1}) +$$

$$\sum_{j=2}^{k}(-1)^j\Phi(\delta(\Lambda^0T^1)\delta T^2\cdots\delta T^{j-1}\delta(T^jT^{j+1}+\Lambda^jT^{j+1}+T^j\Lambda^{j+1})\delta T^{j+2}\cdots\delta T^{k+1})$$

collapses to

$$-\Phi(\Lambda^0T^1\delta T^2\cdots\delta T^{k+1})+(-1)^k\Phi(\delta(\Lambda^0T^1)\delta T^2\cdots\delta T^{k-1}(\delta T^kT^{k+1}+\delta(T^k\Lambda^{k+1}))).$$

Substituting and multiplying by $(-1)^k$, we get

$$\begin{aligned}
\widetilde{\Phi}([\widetilde{T}^0d\widetilde{T}^1\cdots d\widetilde{T}^k,\widetilde{T}^{k+1}]) &= (-1)^k\Phi(\Lambda^0T^1\delta T^2\cdots\delta T^{k+1})\\
&+ \Phi(T^0\delta T^1\cdots\delta T^kT^{k+1}))\\
&+ \Phi(T^0\delta T^1\cdots\delta T^{k-1}\delta(T^k\Lambda^{k+1}))\\
&- (-1)^k\Phi(\Lambda^0T^1\delta T^2\cdots\delta T^{k+1})\\
&+ \Phi(\delta(\Lambda^0T^1)\delta T^2\cdots\delta T^kT^{k+1})\\
&+ \Phi(\delta(\Lambda^0T^1)\delta T^2\cdots\delta T^{k-1}\delta(T^k\Lambda^{k+1}))\\
&- \Phi(T^{k+1}T^0\delta T^1\cdots\delta T^k)\\
&- \Phi(T^{k+1}\Lambda^0\delta T^1\cdots\delta T^k)\\
&- \Phi(\Lambda^{k+1}T^0\delta T^1\cdots\delta T^k)\\
&- \Phi(\delta(\Lambda^{k+1}\Lambda^0T^1)\delta T^2\cdots\delta T^k).
\end{aligned}$$

The first and the fourth terms on the right cancel. Using 3.6 and the trace property of Φ we have the following equations:

$$\begin{aligned}
0 &= \Phi(T^0\delta T^1\cdots\delta T^kT^{k+1})-\Phi(T^{k+1}T^0\delta T^1\cdots\delta T^k).\\
0 &= \Phi(T^0\delta T^1\cdots\delta T^{k-1}\delta(T^k\Lambda^{k+1}))-\Phi(\Lambda^{k+1}T^0\delta T^1\cdots\delta T^k).\\
0 &= \Phi(\delta(\Lambda^0T^1)\delta T^2\cdots\delta T^kT^{k+1})-\Phi(T^{k+1}\Lambda^0\delta T^1\cdots\delta T^k).\\
0 &= \Phi(\delta(\Lambda^0T^1)\delta T^2\cdots\delta T^{k-1}\delta(T^k\Lambda^{k+1}))-\Phi(\delta(\Lambda^{k+1}\Lambda^0T^1)\delta T^2\cdots\delta T^k).
\end{aligned}$$

Thus

$$\widetilde{\Phi}([\widetilde{T}^0d\widetilde{T}^1\cdots d\widetilde{T}^k,\widetilde{T}^{k+1}])=0.$$

Hence $\widetilde{\Phi}$ is a closed graded trace on the whole universal algebra associated with $\widetilde{\Psi}^{-\infty}_{\mathfrak{S}}(\mathcal{G};\mathcal{S}\otimes E)$ which commutes with the differentials.

Given the above, we know that for any idempotent $\tilde{e}$ in the matrix algebra $M_N(\widetilde{\Psi}^{-\infty}_{\mathfrak{S}}(\mathcal{G};\mathcal{S}\otimes E))$, the expression

$$(\widetilde{\Phi}\circ\mathrm{tr})\Big(\tilde{e}\exp\left(\frac{-\delta(\tilde{e})^2}{2i\pi}\right)\Big)$$

is a closed Haefliger form and that its cohomology class only depends on the $K-$theory class $[\tilde{e}]$ of the idempotent $\tilde{e}$, see for instance [BH04]. But note that this Haefliger differential form coincides with the differential form

$$(\Phi\circ\mathrm{tr})\Big((\tilde{e}-(\varrho\circ p)(\tilde{e}))\exp\left(\frac{-(\delta(\tilde{e}-(\varrho\circ p)(\tilde{e})))^2}{2i\pi}\right)\Big)$$

which is then also closed and represents the same Haefliger cohomology class. Thus we deduce that the Haefliger class

$$\Big[(\Phi\circ\mathrm{tr})\Big((\tilde{e}-(\varrho\circ p)(\tilde{e}))\exp\left(\frac{-(\delta(\tilde{e}-(\varrho\circ p)(\tilde{e})))^2}{2i\pi}\right)\Big)\Big]-$$
$$\Big[(\Phi\circ\mathrm{tr})\Big((\tilde{e}'-(\varrho\circ p)(\tilde{e}'))\exp\left(\frac{-(\delta(\tilde{e}'-(\varrho\circ p)(\tilde{e}')))^2}{2i\pi}\right)\Big)\Big],$$

is well defined and only depends on the $K-$theory class $[\tilde{e}]-[\tilde{e}']$. We denote it by $\widetilde{\mathrm{ch}}_a([\tilde{e}]-[\tilde{e}'])$. So we have the following morphism

$$\widetilde{\mathrm{ch}}_a : K_0(\widetilde{\Psi}_{\mathfrak{S}}^{-\infty}(\mathcal{G};\mathcal{S}\otimes E)) \longrightarrow \mathrm{H}_c^*(M/F).$$

The above construction applies also to the minimal unitalization $\widehat{\Psi}_{\mathfrak{S}}^{-\infty}(\mathcal{G};\mathcal{S}\otimes E)$ of the algebra $\Psi_{\mathfrak{S}}^{-\infty}(\mathcal{G};\mathcal{S}\otimes E)$ and yields a morphism

$$\widehat{\mathrm{ch}}_a : K_0(\widehat{\Psi}_{\mathfrak{S}}^{-\infty}(\mathcal{G};\mathcal{S}\otimes E)) \longrightarrow \mathrm{H}_c^*(M/F),$$

whose restriction to $K_0(\Psi_{\mathfrak{S}}^{-\infty}(\mathcal{G};\mathcal{S}\otimes E))$ is by definition the Chern character ch_a. Note that $\widehat{\mathrm{ch}}_a$ is given by the same formula (3.4), except that the $K-$theory element is no longer supposed to live in the kernel of

$$p_{0,*} : K_0(\widehat{\Psi}_{\mathfrak{S}}^{-\infty}(\mathcal{G};\mathcal{S}\otimes E)) \longrightarrow K_0(\Psi_{\mathfrak{S}}^{-\infty}(\mathcal{G};\mathcal{S}\otimes E)).$$

Now the map $\beta : \widehat{\Psi}_{\mathfrak{S}}^{-\infty}(\mathcal{G};\mathcal{S}\otimes E) \to \widetilde{\Psi}_{\mathfrak{S}}^{-\infty}(\mathcal{G};\mathcal{S}\otimes E)$ induces a well defined morphism of short exact sequences

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K_0(\Psi_{\mathfrak{S}}^{-\infty}(\mathcal{G};\mathcal{S}\otimes E)) & \xrightarrow{i_{0,*}} & K_0(\widehat{\Psi}_{\mathfrak{S}}^{-\infty}(\mathcal{G};\mathcal{S}\otimes E)) & \xrightarrow{p_{0,*}} & K_0(\mathbb{C})\simeq\mathbb{Z} & \longrightarrow & 0 \\
 & & \downarrow id & & \downarrow \beta_* & & \downarrow [\beta]_* & & \\
0 & \longrightarrow & K_0(\Psi_{\mathfrak{S}}^{-\infty}(\mathcal{G};\mathcal{S}\otimes E)) & \xrightarrow{i_*} & K_0(\widetilde{\Psi}_{\mathfrak{S}}^{-\infty}(\mathcal{G};\mathcal{S}\otimes E)) & \xrightarrow{p_*} & K_0(\mathbb{C}^2)\simeq\mathbb{Z}^2 & \longrightarrow & 0.
\end{array}$$

Hence composing with $\widetilde{\mathrm{ch}}_a$ gives the following diagram which is commutative by the very definition of the maps:

$$\begin{array}{ccccccc}
 & & & & K_0(\widehat{\Psi}_{\mathfrak{S}}^{-\infty}(\mathcal{G};\mathcal{S}\otimes E)) & & \\
 & & & \nearrow^{i_{0,*}} & & \searrow^{\widehat{\mathrm{ch}}_a} & \\
0 & \longrightarrow & K_0(\Psi_{\mathfrak{S}}^{-\infty}(\mathcal{G};\mathcal{S}\otimes E)) & & \downarrow \beta_* & & \mathrm{H}_c(M/F). \\
 & & & \searrow_{i_*} & & \nearrow_{\widetilde{\mathrm{ch}}_a} & \\
 & & & & K_0(\widetilde{\Psi}_{\mathfrak{S}}^{-\infty}(\mathcal{G};\mathcal{S}\otimes E)) & &
\end{array}$$

In particular, $\widetilde{\mathrm{ch}}_a\circ\beta_* = \widehat{\mathrm{ch}}_a$, so

$$\widetilde{\mathrm{ch}}_a\circ\beta_*\circ i_{0,*} = \widehat{\mathrm{ch}}_a\circ i_{0,*} = \mathrm{ch}_a\,.$$

But,

$$\beta_*\circ i_{0,*} : K_0(\Psi_{\mathfrak{S}}^{-\infty}(\mathcal{G};\mathcal{S}\otimes E)) \longrightarrow \mathrm{Ker}\, p_*,$$

is an isomorphism, so we may define the Chern character directly on the group $K_0(\Psi_{\mathfrak{S}}^{-\infty}(\mathcal{G};\mathcal{S}\otimes E)) = \mathrm{Ker}\, p_*$. The proof is thus complete. □

Corollary 3.7. *Let D be a generalized Dirac operator for the foliation F acting on the sections of the $\mathbb{Z}_2-$graded bundle $\mathcal{S}\otimes E$. Let $P(tD)$ be the associated idempotent in the algebra $\widetilde{\Psi}_{\mathfrak{S}}^{-\infty}(\mathcal{G};\mathcal{S}\otimes E)$, as in Proposition 2.5. Set $P_t = P(tD)-\pi_-$. Then for all $t>0$, the Haefliger form*

$$(\Phi\circ\mathrm{tr})\Big(P_t\exp\left[\frac{-(\delta P_t)^2}{2i\pi}\right]\Big),$$

is closed and as Haefliger classes, we have the equality

$$\mathrm{ch}_a(\mathrm{Ind}_a(D^+)) = \left[(\Phi\circ\mathrm{tr})\Big(P_t\exp\left[\frac{-(\delta P_t)^2}{2i\pi}\right]\Big)\right].$$

Proof. The analytic $K-$theory index of D in the $K-$theory group $K_0(\Psi_{\mathfrak{S}}^{-\infty}(\mathcal{G};\mathcal{S}\otimes E))$ of superexeponentially decaying operators is given by

$$\mathrm{Ind}_a(D^+) = [P(tD)]-[\pi_-] \quad \in \quad \mathrm{Ker}\Big(K_0(\widetilde{\Psi}_{\mathfrak{S}}^{-\infty}(\mathcal{G};\mathcal{S}\otimes E))\to\mathbb{Z}^2\Big).$$

Since the splitting map $\varrho : \mathbb{C}^2 \to \widetilde{\Psi}_{\mathfrak{G}}^{-\infty}(\mathcal{G}; \mathcal{S} \otimes E)$ is $\varrho(\lambda, \mu) = \lambda\pi_+ + \mu\pi_-$, we have that

$$P(tD) - (\varrho \circ p)(P(tD)) = P_t \text{ and } \pi_- - (\varrho \circ p)(\pi_-) = 0.$$

Now apply Proposition 3.5. □

In [BH04] we proved that the Chern character ch_a composed with the topological and analytic index maps of Connes-Skandalis [CS84] yield the same map. As a particular case, for any generalized Dirac operator D with coefficients in a Hermitian bundle E_1 over M, the Chern character of the topological index of D, denoted $\mathrm{ch}_a(\mathrm{Ind}_t(D^+))$, coincides with the Chern character of the analytic index of D, i.e.

$$\mathrm{ch}_a(\mathrm{Ind}_t(D^+)) = \mathrm{ch}_a(\mathrm{Ind}_a(D^+)),$$

and the common value of this Haefliger cohomology class is

$$\mathrm{ch}_a(\mathrm{Ind}_t(D^+)) = \mathrm{ch}_a(\mathrm{Ind}_a(D^+)) = \int_F \widehat{A}(TF)\,\mathrm{ch}(E_1).$$

Here $\widehat{A}(TF)$ is the usual $\widehat{A}$ genus of the tangent bundle of F, and ch is the usual Chern character of E_1.

In order to define the Chern character of the index bundle of D, we need the concept of "transverse smoothness" for $\mathcal{A}(M)$ equivariant bounded leafwise smoothing operators on $\mathcal{S} \otimes E \otimes \wedge\nu_s^*$.

The spaces we consider carry a natural Sobolev structure due to the compactness of the ambient manifold M. For each leaf $\widetilde{L}_x$ of the foliation F_s of $\mathcal{G}$, and $k \in \mathbb{R}$, denote by $\mathcal{H}_k(\mathcal{S} \otimes E \,|\, \widetilde{L}_x)$ the Sobolev space which is the completion of $C_c^\infty(\mathcal{S} \otimes E \,|\, \widetilde{L}_x)$ with respect to the norm $\|\sigma\|_k = \|(1 + D^2)^{k/2}\sigma\|$, where $\| \ \|$ is the L^2 norm on $C_c^\infty(\mathcal{S} \otimes E \,|\, \widetilde{L}_x)$. Because all of the objects we use are the pull-backs of objects on the compact manifold M which are smooth as objects on M, the Sobolev spaces $\mathcal{H}_k(\mathcal{S} \otimes E \,|\, \widetilde{L}_x)$ do not depend on the choices made. An operator

$$A : C^\infty(\mathcal{S} \otimes E) \to C^\infty(\mathcal{S} \otimes E)$$

is a bounded leafwise smoothing operator provided that for all k and ℓ, and all $x \in M$, A defines a bounded operator

$$A : \mathcal{H}_k(\mathcal{S} \otimes E \,|\, \widetilde{L}_x) \to \mathcal{H}_\ell(\mathcal{S} \otimes E \,|\, \widetilde{L}_x),$$

with bound independent of x, but perhaps depending on k and ℓ. A prime example of such an operator is $g(D^2)$, where g is a Borel function on $[0, \infty)$ so that for all k, $(1 + x)^{k/2}g(x)$ is bounded on $[0, \infty)$.

An $\mathcal{A}(M)$ equivariant bounded leafwise smoothing operator H on $C^\infty(\mathcal{S}\otimes E\otimes\wedge\nu_s^*)$ is first of all a leafwise operator

$$H : C^\infty(\mathcal{S} \otimes E \otimes \wedge\nu_s^*) \to C^\infty(\mathcal{S} \otimes E \otimes \wedge\nu_s^*)$$

which is equivariant with respect to the $\mathcal{A}(M)$ module structure of $C^\infty(\mathcal{S} \otimes E \otimes \wedge\nu_s^*)$. As such it can be written as

$$H = H_{[0]} + H_{[1]} + \cdots + H_{[n]},$$

where $H_{[d]}$ is homogeneous of degree d, that is, for all t,

$$H_{[d]} : C^\infty(\mathcal{S} \otimes E \otimes \wedge^t\nu_s^*) \to C^\infty(\mathcal{S} \otimes E \otimes \wedge^{t+d}\nu_s^*).$$

Then $H_{[d]}$ may be written as

$$H_{[d]} = \sum_j H_{[d],j} \otimes s^*(\omega_j),$$

where the $\omega_j \in C^\infty(\wedge^d TM^*)$ and $H_{[d],j}$ is a leafwise operator on $\mathcal{S} \otimes E$. We further require that for any $X \in C^\infty(\wedge^d TM)$, $i_X H_{[d]}$ is a bounded leafwise smoothing operator on $\mathcal{S}\otimes E$. The k, ℓ norm $||H||_{k,\ell}$ of such an operator is given by

$$||H||_{k,\ell} = \sup_{x,d,X} ||i_X H^x_{[d]}||_{k,\ell},$$

where $X \in \wedge^d TM_x$ has norm 1, and $||i_X H^x_{[d]}||_{k,\ell}$ is the norm of the operator

$$i_X H_{[d]} : \mathcal{H}_k(\mathcal{S} \otimes E \,|\, \widetilde{L}_x) \to \mathcal{H}_\ell(\mathcal{S} \otimes E \,|\, \widetilde{L}_x).$$

The norm $||H||_{0,0}$ will also be denoted $||H||$. Since M is compact, it is easy to prove that $||H||_{k,\ell} < \infty$ for all k, ℓ.

For $X \in C^\infty(\wedge^d TM)$, and $Y \in C^\infty(TM)$, set

$$\partial_\nu^Y(i_X H_{[d]}) = i_Y(\partial_\nu(i_X H_{[d]})),$$

which (if it exists) is an operator on $\mathcal{S} \otimes E$.

Definition 3.8. *Suppose H is a $\mathcal{A}(M)$ equivariant bounded leafwise smoothing operator on $\mathcal{S} \otimes E \otimes \wedge\nu_s^*$. We say H is transversely smooth provided that for any $X \in C^\infty(\wedge^d TM)$, and any $Y_1, ..., Y_\ell \in C^\infty(TM)$, the operator*

$$\partial_\nu^{Y_1} ... \partial_\nu^{Y_\ell}(i_X H_{[d]})$$

is a bounded leafwise smoothing operator on $\mathcal{S} \otimes E$.

If H is transversely smooth, so is $\partial_\nu H$. Note that as in the proof of Lemma 3.1, this is a local question on M. On $U \in \mathcal{U}$, H is a sum of operators of the form $i_X H_{[d]} \otimes s^*(\omega_X)$ where $\omega_X \in C^\infty(\wedge^d T^*U)$ is a closed form and $X \in C^\infty(\wedge^d TU)$, both of which extend to M. Then on U, $\partial_\nu H$ is a sum of operators of the form $\partial_\nu(i_X H_{[d]}) \otimes s^*(\omega_X)$, and the result is immediate.

Suppose that ϕ and $\psi \in C_c^\infty(\mathcal{G})$, with associated multiplication operators, M_ϕ and M_ψ. If H is transversely smooth, then $M_\phi \circ H \circ M_\psi$ is also transversely smooth. Standard techniques using the local expression for ∂_ν^Y then show that the Schwartz kernel $H_x(y,z)$ is smooth in all its variables, where $x \in M$, and $y, z \in \widetilde{L}_x$.

It follows from Lemma 3.1 that any element $A \in \mathcal{A}_{\mathfrak{S}}(\mathcal{G}, \mathcal{S} \otimes E)$ is transversely smooth.

We shall assume that P_0, the projection onto the kernel of D, is transversely smooth. Note that classical results imply that P_0 is a smoothing operator when restricted to any leaf $\widetilde{L}_x$.

Recall that $\alpha = \pi_+ - \pi_-$ is the grading involution for $\mathcal{S} \otimes E = (\mathcal{S}^+ \otimes E) \oplus (\mathcal{S}^- \otimes E)$. Then

$$P_0 = \begin{bmatrix} P_0^+ & 0 \\ 0 & P_0^- \end{bmatrix}, \quad \text{so} \quad \alpha P_0 = \begin{bmatrix} P_0^+ & 0 \\ 0 & -P_0^- \end{bmatrix}$$

is the super-projection onto the leafwise kernel of D, where $P_0^\pm$ is projection onto the kernel of $D^\pm$. Note that $\partial_\nu \pi_\pm = 0$, provided we use a connection which preserves the splitting $\mathcal{S} = \mathcal{S}^+ \oplus \mathcal{S}^-$, which we assume that we do, so $\partial_\nu \alpha = 0$, and $\alpha\theta = \theta\alpha$. Note also that $\alpha P_0 = P_0 \alpha$, so

$$(\partial_\nu(\alpha P_0))^2 = \alpha^2(\partial_\nu P_0)^2 = (\partial_\nu P_0)^2 \text{ and } \alpha P_0 \theta \alpha P_0 = \alpha^2 P_0 \theta P_0 = P_0 \theta P_0, \text{ which implies } (\delta(\alpha P_0))^2 = (\delta P_0)^2.$$

Proposition 3.9. *The Haefliger form* $(\Phi \circ \mathrm{tr})\Big(\alpha P_0 \exp(\frac{-(\delta(\alpha P_0))^2}{2i\pi})\Big) = (\Phi \circ \mathrm{tr})\Big(\alpha P_0 \exp(\frac{-(\delta P_0)^2}{2i\pi})\Big)$ *is closed, and the Haefliger class it defines depends only on P_0.*

Proof. Set $U = 2P_0 - 1$ then

$$\alpha U = U\alpha, U^2 = I, UP_0 = P_0 = P_0 U \text{ and } U(\delta P_0) = \frac{1}{2}U(\delta U) = -\frac{1}{2}(\delta U)U = -(\delta P_0)U.$$

Thus, for any $k \geq 0$,

$$(d_H \circ \Phi \circ \mathrm{tr})\Big(\alpha P_0 (\delta P_0)^{2k}\Big) = (\Phi \circ \mathrm{tr})\Big(\alpha (\delta P_0)^{2k+1}\Big) =$$

$$(\Phi \circ \mathrm{tr})\Big(U^2 \alpha (\delta P_0)^{2k+1}\Big) = (-1)^{2k+1} (\Phi \circ \mathrm{tr})\Big(U \alpha (\delta P_0)^{2k+1} U\Big) = -(\Phi \circ \mathrm{tr})\Big(U \alpha (\delta P_0)^{2k+1} U\Big).$$

Lemma 3.10. $(\Phi \circ \mathrm{tr})\Big(U\alpha(\delta P_0)^{2k+1}U\Big) = (\Phi \circ \mathrm{tr})\Big(\alpha(\delta P_0)^{2k+1}\Big)$.

This immediately implies that

$$(d_H \circ \Phi \circ \mathrm{tr})\Big(\alpha P_0 (\delta P_0)^{2k}\Big) = 0.$$

Proof. Using $U = 2P_0 - 1$, we get by multiplying out

$$(\Phi\circ\mathrm{tr})\Big(U\alpha(\delta P_0)^{2k+1}U\Big) = 4(\Phi\circ\mathrm{tr})\Big(P_0\alpha(\delta P_0)^{2k+1}P_0\Big) - 2(\Phi\circ\mathrm{tr})\Big(\alpha(\delta P_0)^{2k+1}P_0\Big) -$$
$$2(\Phi\circ\mathrm{tr})\Big(P_0\alpha(\delta P_0)^{2k+1}\Big) + (\Phi\circ\mathrm{tr})\Big(\alpha(\delta P_0)^{2k+1}\Big).$$

Thus we need to show that

$$(\Phi\circ\mathrm{tr})\Big(\alpha(\delta P_0)^{2k+1}P_0\Big) = (\Phi\circ\mathrm{tr})\Big(P_0\alpha(\delta P_0)^{2k+1}P_0\Big) = (\Phi\circ\mathrm{tr})\Big(P_0\alpha(\delta P_0)^{2k+1}\Big).$$

As $P_0 = P_0 e^{-tD^2}$, we have

$$(\Phi\circ\mathrm{tr})\Big(\alpha(\delta P_0)^{2k+1}P_0\Big) = (\Phi\circ\mathrm{tr})\Big(\alpha(\delta P_0)^{2k+1}P_0e^{-tD^2}\Big).$$

The operator $\alpha(\delta P_0)^{2k+1}P_0$ is bounded, and e^{-tD^2} is super exponentially decaying, so we may apply Lemma 2.5 of [HL02] to get that this last equals

$$(\Phi\circ\mathrm{tr})\Big(e^{-tD^2}\alpha(\delta P_0)^{2k+1}P_0\Big),$$

which is true for all $t > 0$. Now the proof of Theorem 2.3.17 [HL90] allows us to conclude that

$$\lim_{t\to\infty}(\Phi\circ\mathrm{tr})\Big(e^{-tD^2}\alpha(\delta P_0)^{2k+1}P_0\Big) = (\Phi\circ\mathrm{tr})\Big(\lim_{t\to\infty}e^{-tD^2}\alpha(\delta P_0)^{2k+1}P_0\Big) = (\Phi\circ\mathrm{tr})\Big(P_0\alpha(\delta P_0)^{2k+1}P_0\Big).$$

Similarly for the second equality. □

In order to show the independence of the choice of connection, we use the relevant parts of the proof of Theorem 4.1 of [BH04]. Indeed, it is obvious that the Poincaré argument developed there still applies to the regularizing operator P_0 even though it may be non-compactly supported. □

Definition 3.11. *The analytic Chern character* $\mathrm{ch}_a([P_0])$ *of the index bundle of* D *is the class of the Haefliger form* $(\Phi\circ\mathrm{tr})\Big(\alpha P_0\exp(\dfrac{-(\delta(\alpha P_0))^2}{2i\pi})\Big) = (\Phi\circ\mathrm{tr})\Big(\alpha P_0\exp(\dfrac{-(\delta P_0)^2}{2i\pi})\Big)$.

Finally, an easy induction argument using the fact that for any idempotent e, $e(\partial_\nu e)^{2\ell-1}e = 0$ for all $\ell > 0$, shows that

$$e(\delta e)^{2j} = \begin{pmatrix} e\left((\partial_\nu e)^2 + e\theta e\right)^j & 0 \\ 0 & 0\end{pmatrix}.$$

Thus

3.12. $$\mathrm{ch}_a([P_0]) =)\Big[(\mathrm{Tr}\circ\mathrm{tr})\Big(\alpha P_0\exp(\frac{-((\partial_\nu P_0)^2 + P_0\theta P_0)}{2i\pi})\Big)\Big].$$

4. Proof of Main Theorem

Denote by P_ϵ the spectral projection for D^2 for the interval $(0,\epsilon)$. Recall that the Novikov-Shubin invariants of D are greater than $k \geq 0$ provided that there is $\beta > k$ so that

$$(\mathrm{Tr}\circ\mathrm{tr})(P_\epsilon) = (\Phi\circ\mathrm{tr})(P_\epsilon) \text{ is } \mathcal{O}(\epsilon^\beta) \text{ as } \epsilon\to 0.$$

When we say a Haefliger form Ψ depending on ϵ is $\mathcal{O}(\epsilon^\beta)$ as $\epsilon\to 0$ we mean that there is a representative $\psi\in\Psi$ defined on a transversal T, and a constant $C > 0$, so that the function on T, $\|\psi\|_T \leq C\epsilon^\beta$ as $\epsilon\to 0$. Here $\|\ \ \|_T$ is the pointwise norm on forms on the transversal T induced from the metric on M.

We now prove our main theorem.

Theorem 4.1. *Assume that* F *is Riemannian, and that the Novikov-Shubin invariants of* D *are greater than* $q/2$. *Assume further that the leafwise operators* P_0, *and (for* ϵ *sufficiently small)* P_ϵ *are transversely smooth. Then the analytic Chern character of the K-theory index of* D *equals the analytic Chern character of the index bundle of* D, *that is*

$$\mathrm{ch}_a(\mathrm{Ind}_a(D^+)) = \mathrm{ch}_a([P_0]).$$

Theorem 4.1 uses estimates on Novikov-Shubin invariants of D to deduce the equality of the whole Chern character of the index bundle with that of the analytic index. We will actually prove the following stronger theorem.

Theorem 4.2. *Assume that F is Riemannian, and that the leafwise operators P_0, and (for ϵ sufficiently small) P_ϵ are transversely smooth. For a fixed integer k with $0 \leq k \leq q/2$, assume that the Novikov-Shubin invariants of D are greater than k. Then the k^{th} component of the Chern character of the K-theory index of D equals the k^{th} component of the Chern character of the index bundle of D, that is*

$$\mathrm{ch}_a^k(\mathrm{Ind}_a(D^+)) = \mathrm{ch}_a^k([P_0]) \quad \in \quad \mathrm{H}_c^{2k}(M/F).$$

The proof of this theorem is rather long and involves a number of complicated estimates. For easier reading, we will split it into a series of propositions and lemmas. Note that Theorem 4.2 implies Theorem 4.1.

For the rest of this section, let k be a fixed integer in the interval $[0, q/2]$. By Corollary 3.7, we need only show that,

$$\lim_{t\to\infty} (\Phi \circ \mathrm{tr})\Big(P_t(\delta P_t)^{2k}\Big) = (\Phi \circ \mathrm{tr})\Big(\alpha P_0(\delta(\alpha P_0))^{2k}\Big).$$

If we ignore the minus signs in P_t, we see that the diagonal terms give e^{-tD^2}, and the off diagonal terms are given by $(P_t)_{21} = (e^{-tD^2/2}\sqrt{t}D)_{21}$ and $(P_t)_{12} = (e^{-tD^2/2}\dfrac{I - e^{-tD^2}}{tD^2}\sqrt{t}D)_{12}$. Thus

$$P_t = \pi_+ e^{-tD^2}\pi_+ - \pi_- e^{-tD^2}\pi_- - \pi_- e^{-tD^2/2}\sqrt{t}D\pi_+ - \pi_+ e^{-tD^2/2}\frac{I - e^{-tD^2}}{tD^2}\sqrt{t}D\pi_-.$$

As the connection ∇ used in the definition of ∂_ν preserves the splitting $\mathcal{S} \otimes E = (\mathcal{S}^+ \otimes E) \oplus (\mathcal{S}^- \otimes E)$, $\partial_\nu \pi_\pm = 0$, and we may work with the operators e^{-tD^2}, $e^{-tD^2/2}\sqrt{t}D$, and $e^{-tD^2/2}\dfrac{I - e^{-tD^2}}{tD^2}\sqrt{t}D$ in what follows instead of the (more notationally complicated) entries of P_t.

We will assume that the reader is familiar with the Spectral Mapping Theorem, see for instance [RS80], and how to use it to compute bounds on norms, strong convergence, etc. This theorem gives that for $\ell \geq 0$, the norms of the operators $D^\ell e^{-tD^2}$, $D^\ell e^{-tD^2/2}\sqrt{t}D$ and $D^\ell e^{-tD^2/2}\dfrac{I - e^{-tD^2}}{tD^2}\sqrt{t}D$ are uniformly bounded as $t \to \infty$. In addition, as $t \to \infty$, $D^\ell e^{-tD^2/2}\sqrt{t}D$ and $D^\ell e^{-tD^2/2}\dfrac{I - e^{-tD^2}}{tD^2}\sqrt{t}D$ converge in norm to zero for $\ell \geq 0$, and for $\ell > 0$, $D^\ell e^{-tD^2}$ also converges in norm to zero.

Choose δ so that

$$-1 < \delta < \frac{-k}{\beta} < 0$$

and couple ϵ to t by setting

$$\epsilon = t^\delta.$$

Because of the uniformly bounded geometry of the leaves of F_s, which follows from the fact that all the structures we use on $\mathcal{G}$ are pulled back from the compact manifold M, the leafwise estimates we give below are uniform over all leaves of F_s.

Denote by Q_ϵ the spectral projection for D^2 for the interval $[\epsilon, \infty)$. Since $I = P_0 + P_\epsilon + Q_\epsilon$, the operator $\partial_\nu Q_\epsilon$ is bounded. Now consider

$$P_t = P_0 P_t P_0 + P_\epsilon P_t P_\epsilon + Q_\epsilon P_t Q_\epsilon = \alpha P_0 + P_\epsilon P_t P_\epsilon + Q_\epsilon P_t Q_\epsilon.$$

Proposition 4.3. *As $t \to \infty$,*

(i) $||Q_\epsilon P_t Q_\epsilon||$ is bounded by a multiple of $e^{-(t^{(1+\delta)}/16)}$,

(ii) $||\partial_\nu(Q_\epsilon P_t Q_\epsilon)||$ is bounded by a multiple of $e^{-(t^{(1+\delta)}/16)}$,

(iii) $||P_\epsilon P_t P_\epsilon||$ is bounded,

(iv) $||\partial_\nu(P_\epsilon P_t P_\epsilon)||$ is bounded by a multiple of $t^{(\frac{1}{2}+a)}$, for any $a > 0$.

Remark 4.4. *The coefficient* $\frac{1}{16}$ *in (i) and (ii) can be improved very easily but this does not allow us to improve the assumption on the Novikov-Shubin invariants.*

Proof. Note that the element

$$\partial_\nu(Q_\epsilon P_t Q_\epsilon) = \partial_\nu(Q_\epsilon) P_t Q_\epsilon + Q_\epsilon P_t \partial_\nu(Q_\epsilon) + Q_\epsilon \partial_\nu(P_t) Q_\epsilon$$

and $||\partial_\nu(Q_\epsilon)||$ is bounded. We may write $P_t = e^{-tD^2/4}\widehat{P}_t = \widehat{P}_t e^{-tD^2/4}$ where

$$\widehat{P}_t = \begin{bmatrix} e^{-3tD^-D^+/4} & (-e^{-tD^-D^+/4})\dfrac{I - e^{-tD^-D^+}}{tD^-D^+}\sqrt{t}D^- \\ -e^{-tD^+D^-/4}\sqrt{t}D^+ & -e^{-3tD^+D^-/4} \end{bmatrix}.$$

$\widehat{P}_t$ has essentially the same properties as P_t, in particular its norm is bounded independently of t. Since $||e^{-tD^2/4}Q_\epsilon|| = ||Q_\epsilon e^{-tD^2/4}|| \leq e^{-t\epsilon/4} = e^{(-t^{(1+\delta)}/4)}$, we have that $||P_t Q_\epsilon||$ and $||Q_\epsilon P_t||$ (so also $||Q_\epsilon P_t Q_\epsilon||$, $||\partial_\nu(Q_\epsilon) P_t Q_\epsilon||$ and $||Q_\epsilon P_t \partial_\nu(Q_\epsilon)||$) are bounded by a multiple of $e^{(-t^{(1+\delta)}/4)}$. Thus we have (i) of the Proposition, and to establish (ii) we need only consider the term $Q_\epsilon \partial_\nu(P_t) Q_\epsilon$. First however, we need the following result on the operators $\partial_\nu(D)$ and $\partial_\nu(D^2)$.

Lemma 4.5. *Suppose that* H *is a bounded leafwise smoothing operator on* $\mathcal{S}\otimes E$*, and extend it to an* $\mathcal{A}(M)$ *equivariant bounded leafwise smoothing operator on* $\mathcal{S}\otimes E \otimes \wedge\nu_s^*$*. Then* $H\partial_\nu(D)$*,* $\partial_\nu(D)H$*,* $H\partial_\nu(D^2)$ *and* $\partial_\nu(D^2)H$ *are* $\mathcal{A}(M)$ *equivariant bounded leafwise smoothing operators on* $\mathcal{S}\otimes E\otimes\wedge\nu_s^*$*.*

Proof. We may construct the partial derivative ∂_ν^F for the foliation F and its normal bundle ν^* in complete analogy with the operator ∂_ν, and we have the leafwise Dirac operator D_F for the foliation F and the bundle E_1. Then, $\partial_\nu(D) = r^*\partial_\nu^F(D_F)$. In particular, let (U,γ,V) be a basic open set for $\mathcal{G}$ where $U, V \in \mathcal{U}$ have coordinates $x_1, ..., x_p, w_1, ..., w_q$ and $y_1, ..., y_p, z_1, ..., z_q$. The x_i and y_i are the leaf coordinates for F, and the w_i and z_i are the normal coordinates. We use $x_1, ..., x_p, y_1, ..., y_p, z_1, ..., z_q$ as coordinates for (U,γ,V), and ν_s^* is spanned by the dx_i and the dz_i. On V, ν^* is spanned by the dz_i, and $\partial_\nu^F(D_F)$ is given by an expression of the form

$$\partial_\nu^F(D_F) = A_{ij}(y,z)dz_j \otimes \partial/\partial y_i + B_j(y,z)dz_j,$$

where $A_{ij}, B_j \in C^\infty(V; \mathrm{Hom}(\mathcal{S}_0\otimes E_1))$. The coefficients of the A_{ij} and B_j (with respect to an orthonormal basis of $\mathcal{S}_0\otimes E_1\,|\,V$) have each of their derivatives uniformly bounded over all V in the good cover $\mathcal{U}$. Because D is independent of $x_1, ..., x_p$, and $\nabla\,|_{(U,\gamma,V)}$ has the form given in the proof of Lemma 3.1, $\partial_\nu(D)$ has exactly the same form on (U,γ,V), that is

$$\partial_\nu(D) = A_{ij}(y,z)dz_j \otimes \partial/\partial y_i + B_j(y,z)dz_j,$$

where A_{ij}, B_j are now in $C^\infty((U,\gamma,V); \mathrm{Hom}(\mathcal{S}\otimes E))$, and are really $A_{ij}\circ r$ and $B_j \circ r$.

Now suppose that $x \in U$ and $X \in TM_x$ with $||X|| = 1$, and consider the leafwise operator $i_X\partial_\nu(D)$ on $\widetilde{L}_x$. Write $X = X^F + X^\nu$ where $X^F \in TF_x$ and $X^\nu \in \nu_x$. Then the local expression for $i_X\partial_\nu(D)$ in (U,γ,V) is

$$i_X\partial_\nu(D) = A_{ij}(y,z)dz_j(h_\gamma(X^\nu))\partial/\partial y_i + B_j(y,z)dz_j(h_\gamma(X^\nu)),$$

where $h_\gamma : \nu_x \to \nu_{x'}$ is the holonomy map induced by γ and $x' = (y,z) \in V$. Since h_γ is an isometry, $||h_\gamma(X^\nu)|| \leq 1$. Thus, $i_X\partial_\nu(D)$ is a smooth first order leafwise differential operator on $\mathcal{S}\otimes E$ with uniformly bounded coefficients. It follows that for any $k \in \mathbb{R}$, the operator $i_X\partial_\nu(D)$ maps $\mathcal{H}_k(\mathcal{S}\otimes E\,|\,\widetilde{L}_x)$ to $\mathcal{H}_{k-1}(\mathcal{S}\otimes E\,|\,\widetilde{L}_x)$, and it is a bounded operator with norm independent of x. See [S92], [K91], and [K95].

If H is a bounded leafwise smoothing operator on $\mathcal{S}\otimes E$, then for all $k,\ell \in \mathbb{R}$ and all $x \in M$, $H : \mathcal{H}_{k-1}(\mathcal{S}\otimes E\,|\,\widetilde{L}_x) \to \mathcal{H}_{k+\ell}(\mathcal{S}\otimes E\,|\,\widetilde{L}_x)$ and it is a bounded operator with norm independent of x. Thus, for all $k,\ell \in \mathbb{R}$ and all $x \in M$, the composition

$$i_X H\partial_\nu(D) : \mathcal{H}_k(\mathcal{S}\otimes E\,|\,\widetilde{L}_x) \overset{i_X\partial_\nu(D)}{\longrightarrow} \mathcal{H}_{k-1}(\mathcal{S}\otimes E\,|\,\widetilde{L}_x) \overset{H}{\longrightarrow} \mathcal{H}_{k+\ell}(\mathcal{S}\otimes E\,|\,\widetilde{L}_x),$$

is a bounded operator, and its norm $\|i_X H\partial_\nu(D)\|_{k,k+\ell} \leq \|H\|_{k-1,k+\ell}\|i_X\partial_\nu(D)\|_{k,k-1}$, which is independent of x. Thus $H\partial_\nu(D)$ is an $\mathcal{A}(M)$ equivariant bounded leafwise smoothing operator on $\mathcal{S}\otimes E\otimes\wedge\nu_s^*$, in particular $\|H\partial_\nu(D)\| < \infty$.

The same argument using the composition

$$i_X\partial_\nu(D)H : \mathcal{H}_k(\mathcal{S}\otimes E\,|\,\widetilde{L}_x) \xrightarrow{H} \mathcal{H}_{k+\ell+1}(\mathcal{S}\otimes E\,|\,\widetilde{L}_x) \overset{i_X\partial_\nu(D)}{\longrightarrow} \mathcal{H}_{k+\ell}(\mathcal{S}\otimes E\,|\,\widetilde{L}_x),$$

shows that $\|\partial_\nu(D)H\| < \infty$.

The same proof gives the result for $\partial_\nu(D^2)$ if we replace $\partial_\nu(D)$ by $\partial_\nu(D^2)$, and $\partial_\nu^F(D_F)$ by $\partial_\nu^F(D_F^2)$. The fact that $i_X\partial_\nu(D^2)$ is a smooth second order leafwise differential operator on $\mathcal{S}\otimes E$ with uniformly bounded coefficients gives that for any $k\in\mathbb{R}$, the operator $i_X\partial_\nu(D^2)$ maps $\mathcal{H}_k(\mathcal{S}\otimes E\,|\,\widetilde{L}_x)$ to $\mathcal{H}_{k-2}(\mathcal{S}\otimes E\,|\,\widetilde{L}_x)$, and it is a bounded operator with norm independent of x. □

Now we establish (ii) by considering the individual elements making up the term $Q_\epsilon\partial_\nu(P_t)Q_\epsilon$.

Lemma 4.6. *$||Q_\epsilon\partial_\nu(e^{-tD^2/k})Q_\epsilon||$ is bounded by a multiple of $e^{-(t^{1+\delta}/8k)}$.*

Proof. Recall the foliation Duhamel formula of [He95] (which requires that $\mathcal{G}$ be Hausdorff) which states that

$$\partial_\nu(e^{-tD^2}) = -\int_0^t e^{-sD^2}\partial_\nu(D^2)e^{(s-t)D^2}ds.$$

Thus

$$Q_\epsilon\partial_\nu(e^{-tD^2})Q_\epsilon = -\int_0^t Q_\epsilon e^{-sD^2}\partial_\nu(D^2)e^{(s-t)D^2}Q_\epsilon ds =$$

$$-\int_{t/2}^t Q_\epsilon e^{-sD^2}\partial_\nu(D^2)e^{(s-t)D^2}Q_\epsilon ds - \int_0^{t/2} Q_\epsilon e^{-sD^2}\partial_\nu(D^2)e^{(s-t)D^2}Q_\epsilon ds.$$

The norm of the first integral satisfies

$$\begin{aligned} ||\int_{t/2}^t Q_\epsilon e^{-sD^2}\partial_\nu(D^2)e^{(s-t)D^2}Q_\epsilon ds|| &\leq \int_{t/2}^t ||Q_\epsilon e^{-sD^2}\partial_\nu(D^2)e^{(s-t)D^2}Q_\epsilon||ds \\ &\leq \int_{t/2}^t ||Q_\epsilon e^{-\frac{s}{2}D^2}||\,||e^{-\frac{s}{2}D^2}\partial_\nu(D^2)||\,||e^{(s-t)D^2}Q_\epsilon||ds. \end{aligned}$$

Now $||e^{(s-t)D^2}Q_\epsilon|| \leq 1$, and the operator $e^{-\frac{1}{2}D^2}$ is a bounded leafwise smoothing operator, so by Lemma 4.5, $\|e^{-\frac{1}{2}D^2}\partial_\nu(D^2)\|$ is bounded. Thus

$$||e^{-\frac{s}{2}D^2}\partial_\nu(D^2)|| \leq ||e^{-\frac{s-1}{2}D^2}||\,||e^{-\frac{1}{2}D^2}\partial_\nu(D^2)|| \leq ||e^{-\frac{1}{2}D^2}\partial_\nu(D^2)||$$

for $t > 2$, as then $||e^{-\frac{s-1}{2}D^2}|| \leq 1$ for all $s \geq t/2$. Finally, $||Q_\epsilon e^{-\frac{s}{2}D^2}|| \leq e^{-s\epsilon/2}$, so the last integral is bounded by a multiple of

$$\epsilon^{-1}(e^{-(t\epsilon/4)} - e^{-(t\epsilon/2)}) = t^{-\delta}(e^{-(t^{1+\delta}/4)} - e^{-(t^{1+\delta}/2)}) < t^{-\delta}e^{-(t^{1+\delta}/4)}.$$

This in turn is bounded by a multiple of $e^{-(t^{1+\delta}/8)}$, for t sufficiently large.

The change of variables $s\to t-s$ transforms the integral $\int_0^{t/2} Q_\epsilon e^{-sD^2}\partial_\nu(D^2)e^{(s-t)D^2}Q_\epsilon ds$ to the integral $\int_{t/2}^t Q_\epsilon e^{(s-t)D^2}\partial_\nu(D^2)e^{-sD^2}Q_\epsilon ds$, so this satisfies the same estimate. Replacing D^2 by D^2/k then gives the estimate of the lemma. □

Lemma 4.7. *As $t\to\infty$, $||Q_\epsilon\partial_\nu(e^{-tD^2}\sqrt{t}D)Q_\epsilon||$ is bounded by a multiple of $e^{-(t^{1+\delta}/16)}$.*

Proof. Observe that

$$Q_\epsilon\partial_\nu(e^{-tD^2}\sqrt{t}D)Q_\epsilon = Q_\epsilon\partial_\nu(e^{-tD^2/2}\sqrt{t}De^{-tD^2/2})Q_\epsilon =$$

$$Q_\epsilon\partial_\nu(e^{-tD^2/2})\sqrt{t}De^{-tD^2/2}Q_\epsilon + Q_\epsilon e^{-tD^2/2}\partial_\nu(\sqrt{t}D)e^{-tD^2/2}Q_\epsilon + Q_\epsilon e^{-tD^2/2}\sqrt{t}D\partial_\nu(e^{-tD^2/2})Q_\epsilon =$$

$$Q_\epsilon \partial_\nu(e^{-tD^2/2})Q_\epsilon\sqrt{t}De^{-tD^2/2}Q_\epsilon + Q_\epsilon e^{-tD^2/2}\sqrt{t}\partial_\nu(D)e^{-tD^2/2}Q_\epsilon + Q_\epsilon e^{-tD^2/2}\sqrt{t}DQ_\epsilon\partial_\nu(e^{-tD^2/2})Q_\epsilon.$$

By the Spectral Mapping Theorem, $\sqrt{t}De^{-tD^2/2} = e^{-tD^2/2}\sqrt{t}D$ has norm bounded by $1/\sqrt{e}$. Using Lemma 4.6 and the fact that $||Q_\epsilon|| \leq 1$, the first and third terms satisfy the estimate.

The operator $e^{-tD^2/2}$ is a bounded leafwise smoothing operator, so by Lemma 4.5, $\partial_\nu(D)e^{-tD^2/2}$ is an $\mathcal{A}(M)$ equivariant bounded leafwise smoothing operator. As $e^{-(s+t)D^2} = e^{-sD^2}e^{-tD^2}$ and $||e^{-tD^2}|| \leq 1$, it follows easily that its norm is bounded independently of t, for t large. The fact that $||Q_\epsilon e^{-tD^2/2}\sqrt{t}|| \leq \sqrt{t}e^{-t\epsilon/2} = \sqrt{t}e^{-(t^{1+\delta}/2)} \leq e^{-(t^{1+\delta}/4)}$, for t large, gives the estimate for the middle term. $\square$

Lemma 4.8. *As $t \to \infty$, $||Q_\epsilon\partial_\nu(e^{-tD^2}\frac{I-e^{-tD^2}}{tD^2}\sqrt{t}D)Q_\epsilon||$ is bounded by a multiple of $e^{-(t^{1+\delta}/16)}$.*

Proof.

$$||Q_\epsilon\partial_\nu(e^{-tD^2}\frac{I-e^{-tD^2}}{tD^2}\sqrt{t}D)Q_\epsilon|| = ||Q_\epsilon\partial_\nu(e^{-tD^2}\sqrt{t}D\frac{I-e^{-tD^2}}{tD^2})Q_\epsilon|| \leq$$
$$||Q_\epsilon\partial_\nu(e^{-tD^2}\sqrt{t}D)Q_\epsilon\frac{I-e^{-tD^2}}{tD^2}Q_\epsilon|| + ||Q_\epsilon e^{-tD^2}\sqrt{t}D\partial_\nu(\frac{I-e^{-tD^2}}{tD^2})Q_\epsilon||$$

and $||\frac{I-e^{-tD^2}}{tD^2}|| \leq 1$, so by Lemma 4.7 the first term immediately above satisfies the lemma. If G is the Green's operator for D, the second term may be written as $Q_\epsilon(G/\sqrt{t})Q_\epsilon e^{-tD^2}tD^2\partial_\nu(\frac{I-e^{-tD^2}}{tD^2})Q_\epsilon$, and $||Q_\epsilon G/\sqrt{t}|| \leq (t\epsilon)^{-1/2} = t^{-(1+\delta)/2}$, which is bounded for t large since $1+\delta > 0$. The operator $tD^2\frac{I-e^{-tD^2}}{tD^2} = I - e^{-tD^2}$, so

$$tD^2\partial_\nu(\frac{I-e^{-tD^2}}{tD^2}) = -\partial_\nu(tD^2)\frac{I-e^{-tD^2}}{tD^2} - \partial_\nu(e^{-tD^2}),$$

and

$$Q_\epsilon e^{-tD^2}tD^2\partial_\nu(\frac{I-e^{-tD^2}}{tD^2})Q_\epsilon = -Q_\epsilon e^{-tD^2}\partial_\nu(tD^2)\frac{I-e^{-tD^2}}{tD^2}Q_\epsilon - Q_\epsilon e^{-tD^2}\partial_\nu(e^{-tD^2})Q_\epsilon.$$

Now,

$$Q_\epsilon e^{-tD^2}\partial_\nu(tD^2) = Q_\epsilon te^{-tD^2/2}e^{-tD^2/2}\partial_\nu(D^2),$$

and, as in the proof of Lemma 4.7, $e^{-tD^2/2}\partial_\nu(D^2)$ has norm bounded independently of t, for t large. As

$$||Q_\epsilon te^{-tD^2/2}|| \leq te^{-t\epsilon/2} = te^{-(t^{1+\delta}/2)} < e^{-(t^{1+\delta}/4)}$$

for t large, the term $Q_\epsilon e^{-tD^2}\partial_\nu(tD^2)\frac{I-e^{-tD^2}}{tD^2}Q_\epsilon$ has norm bounded by a multiple of $e^{-(t^{1+\delta}/4)}$. By Lemma 4.6, the term $Q_\epsilon e^{-tD^2}\partial_\nu(e^{-tD^2})Q_\epsilon = Q_\epsilon e^{-tD^2}Q_\epsilon\partial_\nu(e^{-tD^2})Q_\epsilon$ is bounded by a multiple of $e^{-(t^{1+\delta}/8)}$ (actually $e^{-t^{1+\delta}}$ if we use the estimate $||Q_\epsilon e^{-tD^2}|| \leq e^{-t\epsilon} = e^{-t^{1+\delta}}$). $\square$

Thus we have the second inequality of Proposition 4.3. The third estimate follows immediately from the fact that both P_t and P_ϵ are bounded.

Lemma 4.9. *$||P_\epsilon\partial_\nu(e^{-tD^2})P_\epsilon||$ is bounded by a multiple of $t^{1+(\delta/2)}$.*

Note that $1+(\delta/2) > 1/2$, but by choosing δ close to -1, we can make $1+(\delta/2)$ as close to $1/2$ as we please.

Proof.

$$P_\epsilon\partial_\nu(e^{-tD^2})P_\epsilon = -\int_0^t P_\epsilon e^{-sD^2}\partial_\nu(D^2)e^{(s-t)D^2}P_\epsilon ds = -\int_0^t P_\epsilon e^{-sD^2}P_\epsilon[\partial_\nu(D)D + D\partial_\nu(D)]P_\epsilon e^{(s-t)D^2}P_\epsilon ds.$$

As P_ϵ is a bounded leafwise smoothing operator, $||P_\epsilon\partial_\nu(D)||$ and $||\partial_\nu(D)P_\epsilon||$ are bounded (Lemma 4.5 again). Since $\epsilon \to 0$ as $t \to \infty$, their norms are bounded independently of t for t large. This follows since for $\epsilon_1 \leq \epsilon$, $P_{\epsilon_1} = P_{\epsilon_1}P_\epsilon = P_\epsilon P_{\epsilon_1}$ and $||P_{\epsilon_1}|| \leq 1$. Both $||P_\epsilon e^{-sD^2}||$ and $||e^{(s-t)D^2}P_\epsilon||$ are bounded by 1, and both $||P_\epsilon D||$ and $||DP_\epsilon||$ are bounded by $\sqrt{\epsilon}$. Thus $||P_\epsilon \partial_\nu(e^{-tD^2})P_\epsilon||$ is bounded by a multiple of $\int_0^t \sqrt{\epsilon} ds = \sqrt{\epsilon} t = t^{1+(\delta/2)}$. □

Lemma 4.10. *$||P_\epsilon \partial_\nu(e^{-tD^2}\sqrt{t}D)P_\epsilon||$ is bounded by a multiple of $t^{(3/2)+\delta}$.*

Again note that we can make $3/2+\delta$ as close to $1/2$ as we please.

Proof.

$$\begin{aligned} ||P_\epsilon \partial_\nu(e^{-tD^2}\sqrt{t}D)P_\epsilon|| &\leq ||P_\epsilon \partial_\nu(e^{-tD^2})P_\epsilon \sqrt{t}DP_\epsilon|| + ||P_\epsilon e^{-tD^2}P_\epsilon \partial_\nu(\sqrt{t}D)P_\epsilon|| \\ &\leq ||P_\epsilon \partial_\nu(e^{-tD^2})P_\epsilon||\,\sqrt{t}\,||DP_\epsilon|| + \sqrt{t}\,||P_\epsilon e^{-tD^2}P_\epsilon||\,||\partial_\nu(D)P_\epsilon|| \\ &\leq C_1(t^{1+(\delta/2)})\sqrt{t\epsilon} + C_2 t^{1/2} = C_1 t^{(3/2)+\delta} + C_2 t^{1/2} \leq Ct^{(3/2)+\delta}. \end{aligned}$$

□

Lemma 4.11. *$||P_\epsilon \partial_\nu(e^{-tD^2}\frac{I-e^{-tD^2}}{tD^2}\sqrt{t}D)P_\epsilon||$ is bounded by a multiple of $t^{(3/2)+\delta}$.*

Proof.

$$P_\epsilon \partial_\nu(e^{-tD^2}\frac{I-e^{-tD^2}}{tD^2}\sqrt{t}D)P_\epsilon = P_\epsilon \partial_\nu(e^{-tD^2}\sqrt{t}D)P_\epsilon \frac{I-e^{-tD^2}}{tD^2}P_\epsilon + P_\epsilon e^{-tD^2}\sqrt{t}DP_\epsilon \partial_\nu(\frac{I-e^{-tD^2}}{tD^2})P_\epsilon,$$

so by the Lemma 4.10 and the fact that $||\frac{I-e^{-tD^2}}{tD^2}|| \leq 1$, we need only consider the term

$$||P_\epsilon e^{-tD^2}\sqrt{t}DP_\epsilon \partial_\nu(\frac{I-e^{-tD^2}}{tD^2})P_\epsilon|| \leq ||e^{-tD^2}\sqrt{t}DP_\epsilon||\,||P_\epsilon \partial_\nu(\frac{I-e^{-tD^2}}{tD^2})P_\epsilon|| \leq$$

$$C\sqrt{t\epsilon}\,||P_\epsilon \partial_\nu(\frac{I-e^{-tD^2}}{tD^2})P_\epsilon|| = Ct^{(1+\delta)/2}||P_\epsilon \partial_\nu(\frac{I-e^{-tD^2}}{tD^2})P_\epsilon||.$$

Thus we need only show that $||P_\epsilon \partial_\nu(\frac{I-e^{-tD^2}}{tD^2})P_\epsilon||$ is bounded by a multiple of $t^{1+(\delta/2)}$. Note that

$$\frac{d}{dr}(\frac{I-e^{-rD^2}}{D^2}) = e^{-rD^2}$$

so

$$\frac{d}{dr}(\partial_\nu(\frac{I-e^{-rD^2}}{D^2})) = \partial_\nu(\frac{d}{dr}(\frac{I-e^{-rD^2}}{D^2})) = \partial_\nu(e^{-rD^2}) = -\int_0^r e^{-sD^2}\partial_\nu(D^2)e^{(s-r)D^2}ds.$$

Thus

$$\partial_\nu(\frac{I-e^{-tD^2}}{D^2}) = \int_0^t \frac{d}{dr}(\partial_\nu(\frac{I-e^{-rD^2}}{D^2}))dr = -\int_0^t\int_0^r e^{-sD^2}\partial_\nu(D^2)e^{(s-r)D^2}ds\,dr,$$

and

$$||P_\epsilon \partial_\nu(\frac{I-e^{-tD^2}}{tD^2})P_\epsilon|| = ||\frac{1}{t}P_\epsilon \partial_\nu(\frac{I-e^{-tD^2}}{D^2})P_\epsilon|| = ||\frac{1}{t}\int_0^t\int_0^r e^{-sD^2}P_\epsilon \partial_\nu(D^2)P_\epsilon e^{(s-r)D^2}ds\,dr|| \leq$$

$$\frac{1}{t}\int_0^t\int_0^r ||e^{-sD^2}||\,||P_\epsilon[\partial_\nu(D)D + D\partial_\nu(D)]P_\epsilon||\,||e^{(s-r)D^2}||ds\,dr \leq \frac{1}{t}\int_0^t\int_0^r C\sqrt{\epsilon}\,ds\,dr = Ct^{1+(\delta/2)}.$$

□

This finishes the proof of Proposition 4.3 □

To finish the proof of Theorem 4.1, first note that the estimates of Proposition 4.3 remain true with ∂_ν replaced by δ. This follows from the fact that for $T \in A_{\mathfrak{G}}(\mathcal{G}, \mathcal{S} \otimes E) \subset \widetilde{\mathcal{A}}_{\mathfrak{G}}$, δT involves only T and $\partial_\nu T$. Similarly, δP_0, $\delta(\alpha P_0)$, δP_ϵ, and δQ_ϵ are bounded operators.

Next, recall that for $\widetilde{T}_1, \widetilde{T}_2 \in \widetilde{\mathcal{A}}_{\mathfrak{G}}$, $\widetilde{T}_1 * \widetilde{T}_2 = \widetilde{T}_1 \Theta \widetilde{T}_2$, so we must take the operator $\Theta = \begin{pmatrix} 1 & 0 \\ 0 & \theta \end{pmatrix}$ (which is in general an unbounded operator) into account. The calculation made at the beginning of Section 3 shows that the operator $\theta = r^*(\theta^F)$, where θ^F is (at worst) a first order leafwise differential operator which is globally smooth on M. Thus it will behave in our estimates just like the operator $\partial_\nu(D)$. If H is a bounded leafwise smoothing operator on $\mathcal{S} \otimes E$, e.g. $H = P_0, P_\epsilon$, or $\widehat{P}_t$, it follows, just as in the proof of Lemma 4.5, that the composition θH is an $\mathcal{A}(M)$ equivariant bounded leafwise smoothing operator on $\mathcal{S} \otimes E \otimes \wedge \nu_s^*$. In particular, θP_0 and θP_ϵ are bounded operators, as is $\theta \alpha P_0 = (\theta P_0)\alpha$. As $\|\theta P_\epsilon P_t P_\epsilon\| = \|\theta P_\epsilon P_\epsilon P_t P_\epsilon\| \leq \|\theta P_\epsilon\| \, \|P_\epsilon P_t P_\epsilon\|$ and $\|P_\epsilon P_t P_\epsilon\|$is bounded by (iii) of Proposition 4.3, $\|\theta P_\epsilon P_t P_\epsilon\|$ is bounded, that is, it also satisfies estimate (iii) of Proposition 4.3. Finally, $\|\theta Q_\epsilon P_t Q_\epsilon\| = \|\theta \widehat{P}_t Q_\epsilon e^{-tD^2/4} Q_\epsilon\| \leq \|\theta \widehat{P}_t\| \, \|Q_\epsilon e^{-tD^2/4} Q_\epsilon\|$, and $\|\theta \widehat{P}_t\|$ is bounded, so $\|\theta Q_\epsilon P_t Q_\epsilon\|$ satisfies estimate (i) of Proposition 4.3, since $\|Q_\epsilon e^{-tD^2/4} Q_\epsilon\|$ satisfies that estimate. In the argument below, wherever Θ occurs non-trivially in an estimate (e.g. $\Theta P_\epsilon = \begin{pmatrix} 1 & 0 \\ 0 & \theta \end{pmatrix} \begin{pmatrix} P_\epsilon & 0 \\ 0 & 0 \end{pmatrix} = P_\epsilon$ is a trivial occurrance), it occurs in the form $\Theta\delta(A)$ where A is one of αP_0, P_ϵ, $P_\epsilon P_t P_\epsilon$, or $Q_\epsilon P_t Q_\epsilon$. But,

$$\Theta\delta(A) = \begin{pmatrix} 1 & 0 \\ 0 & \theta \end{pmatrix} \begin{pmatrix} \partial_\nu(A) & \pm A \\ A & 0 \end{pmatrix} = \begin{pmatrix} \partial_\nu(A) & \pm A \\ \theta A & 0 \end{pmatrix},$$

so any norm estimate satisfied by $\partial_\nu(A)$, where $A = \alpha P_0$, P_ϵ, $P_\epsilon P_t P_\epsilon$, or $Q_\epsilon P_t Q_\epsilon$, is also satisfied by $\Theta\delta(A)$, with possibly a different constant. In particular, $\Theta\delta(Q_\epsilon P_t Q_\epsilon)$ and $\Theta\delta(P_\epsilon P_t P_\epsilon)$ satisfy estimates (ii) and (iv) respectively of Proposition 4.3.

Previously, we suppressed the occurrence of Θ in the products we considered. For the sake of clarity, we will no longer do this, except for some trivial occurrences.

Since $P_t = \alpha P_0 + P_\epsilon P_t P_\epsilon + Q_\epsilon P_t Q_\epsilon$,

$$\Phi \circ \mathrm{tr}(P_t(\Theta\delta P_t)^{2k}) = \Phi \circ \mathrm{tr}(\alpha P_0(\Theta\delta(\alpha P_0))^{2k}) + \Phi \circ \mathrm{tr}(\alpha P_0(\Theta\delta P_t)^{2k} - \alpha P_0(\Theta\delta(\alpha P_0))^{2k}) +$$
$$\Phi \circ \mathrm{tr}(P_\epsilon P_t P_\epsilon(\Theta\delta P_t)^{2k}) + \Phi \circ \mathrm{tr}(Q_\epsilon P_t Q_\epsilon(\Theta\delta P_t)^{2k}),$$

and we need to show that the limits as t goes to infinity of the last three terms on the right side are zero. For any integer $\ell \geq 0$,

$$||D^{2\ell} Q_\epsilon P_t Q_\epsilon(\Theta\delta P_t)^{2k}|| = ||D^{2\ell} Q_\epsilon e^{-tD^2/4} Q_\epsilon \widehat{P}_t(\Theta\delta P_t)^{2k}|| \leq ||D^{2\ell} Q_\epsilon e^{-tD^2/4} Q_\epsilon|| \, ||\widehat{P}_t(\Theta\delta P_t)^{2k}||.$$

Now

$$\Theta\delta(P_t) = \Theta\delta(\alpha P_0) + \Theta\delta(P_\epsilon P_t P_\epsilon) + \Theta\delta(Q_\epsilon P_t Q_\epsilon),$$

and $||\widehat{P}_t||$ is bounded independently of t. So $||\widehat{P}_t(\Theta\delta P_t)^{2k}||$ is bounded by a multiple of

$$||(\Theta\delta P_t)^{2k}|| = ||\Theta\delta(\alpha P_0) + \Theta\delta(P_\epsilon P_t P_\epsilon) + \Theta\delta(Q_\epsilon P_t Q_\epsilon)||^{2k} \leq C t^{2k(\frac{1}{2}+a)}$$

where $a > 0$ is a number to be chosen later (as close to zero as we please). On the other hand, for t sufficiently large (so that $t^{1+\delta} > 4\ell$), the maximum of $z^\ell e^{-tz/4}$ on the interval $[\epsilon, \infty)$ occurs at ϵ, so

$$||D^{2\ell} Q_\epsilon e^{-tD^2/4} Q_\epsilon|| \leq \epsilon^\ell e^{-t\epsilon/4} = t^{\delta\ell} e^{-(t^{(1+\delta)}/4)}$$

so

$$||D^{2\ell} Q_\epsilon P_t Q_\epsilon(\Theta\delta P_t)^{2k}|| \leq C t^{2k(\frac{1}{2}+a)} t^{\delta\ell} e^{-(t^{(1+\delta)}/4)}$$

which goes to zero as $t \to \infty$. The proof of Theorem 2.3.13 of [HL90] shows that this implies that $\mathrm{tr}(Q_\epsilon P_t Q_\epsilon(\Theta\delta P_t)^{2k})$ is pointwise bounded on M and converges pointwise to zero as $t \to \infty$. As Φ is integration over a compact set, the bounded convergence theorem gives

4.12. $$\lim_{t\to\infty} \Phi \circ \mathrm{tr}(Q_\epsilon P_t Q_\epsilon(\Theta\delta P_t)^{2k}) = 0.$$

To finish the proof we need the following lemma whose proof is given in the Appendix.

Lemma 4.13. *Suppose that H and K are $\mathcal{G}$ invariant $\mathcal{A}(M)$ equivariant bounded leafwise smoothing operators on $\mathcal{S}\otimes E\otimes \wedge\nu_s^*$, which are transversely smooth, then* $\mathrm{Tr}([H,K])=0$.

By assumption, P_0, P_ϵ and Q_ϵ satisfy the hypotheses of this Lemma. If H satisfies the hypotheses, so does $\partial_\nu H$. See the proof of Lemma 3.1. Since ∂_ν is a derivation, any product of operators which satisfy these hypotheses, also satisfies them. As noted above, any super-exponentially decaying operator, e.g. P_t, satisfies these hypotheses. Since the coefficients of θ and all their derivatives are uniformly bounded, the product of θ with any operator satisfying these hypotheses, also satisfies them. Finally, note that the trace property in the conclusion of the Lemma extends in the obvious way to $\Phi\circ\mathrm{tr}$ applied to terms which have elements which satisfy the hypotheses of the Lemma.

Now consider $\Phi\circ\mathrm{tr}(P_\epsilon P_t P_\epsilon(\Theta\delta P_t)^{2k}) = \Phi\circ\mathrm{tr}(P_\epsilon P_t P_\epsilon^2(\Theta\delta P_t)^{2k}) = \Phi\circ\mathrm{tr}(P_\epsilon(\Theta\delta P_t)^{2k}P_\epsilon P_t P_\epsilon)$. The proof of Proposition 12 of [HL99] shows that

$$||\Phi\circ\mathrm{tr}(P_\epsilon(\Theta\delta P_t)^{2k}P_\epsilon P_t P_\epsilon)||_T \;\leq\; C||P_\epsilon(\Theta\delta P_t)^{2k}P_\epsilon P_t||\,||\Phi\circ\mathrm{tr}(P_\epsilon)||_T.$$

Now P_ϵ and $P_\epsilon P_t$ are bounded and $||(\Theta\delta P_t)^{2k}||\leq Ct^{2k(\frac{1}{2}+\alpha)}$, so $||P_\epsilon(\Theta\delta P_t)^{2k}P_\epsilon P_t||\leq Ct^{2k(\frac{1}{2}+\alpha)}$ also. As $||\Phi\circ\mathrm{tr}(P_\epsilon)||_T$ is $\mathcal{O}(\epsilon^\beta)$, $||\Phi\circ\mathrm{tr}(P_\epsilon(\Theta\delta P_t)^{2k}P_\epsilon P_t P_\epsilon)||_T$ is bounded by a multiple of

$$t^{2k(\frac{1}{2}+a)}\epsilon^\beta \;=\; t^{2k(\frac{1}{2}+a)}t^{\delta\beta} \;=\; t^{2k(\frac{1}{2}+a)+\delta\beta}.$$

Recall that $-1<\delta<-k/\beta$ and therefore we can choose $a>0$ so small that

$$2k(\frac{1}{2}+a)+\delta\beta<0.$$

Then

4.14.
$$\lim_{t\to\infty}\Phi\circ\mathrm{tr}(P_\epsilon P_t P_\epsilon(\Theta\delta P_t)^{2k})=0.$$

Finally, consider the individual terms of

$$\Phi\circ\mathrm{tr}(\alpha P_0(\Theta\delta P_t)^{2k}-\alpha P_0(\Theta\delta(\alpha P_0))^{2k}) = \Phi\circ\mathrm{tr}(P_0[\alpha(\Theta\delta P_t)^{2k}-\alpha(\Theta\delta(\alpha P_0))^{2k}]).$$

Suppose the term P_0A contains a $\Theta\delta(Q_\epsilon P_t Q_\epsilon)$. Then

$$||P_0A||\leq ||P_0||\,||\alpha||\,||\Theta\delta(Q_\epsilon P_t Q_\epsilon)||^\mu||\Theta\delta(\alpha P_0)||^\beta||\Theta\delta(P_\epsilon P_t P_\epsilon)||^\gamma$$

where $\mu+\beta+\gamma=2k$ and $\mu>0$. Since $||\alpha||$ is bounded, Proposition 4.3, gives that as $t\to\infty$,

$$||P_0A||\leq Ce^{-\mu(t^{1+\delta}/16)}t^{\gamma(\frac{1}{2}+a)}.$$

For every positive integer ℓ, $D^{2\ell}P_0=0$, so for every integer $\ell\geq 0$, $||D^{2\ell}P_0A||\to 0$ as $t\to\infty$. Proceeding as in the proof of Equation 4.12, we have

$$\lim_{t\to\infty}\Phi\circ\mathrm{tr}(P_0A)=0.$$

Now suppose that we have one of the remaining terms. It must contain a term of the form $\Theta\delta(P_\epsilon P_t P_\epsilon)$. As $P_\epsilon^2=P_\epsilon$ and δ is a derivation, we may replace $\Theta\delta(P_\epsilon P_t P_\epsilon)$ by

$$\Theta\delta(P_\epsilon^2P_tP_\epsilon)=\Theta\delta(P_\epsilon)\Theta(P_\epsilon P_t P_\epsilon)+\Theta P_\epsilon\Theta\delta(P_\epsilon P_t P_\epsilon)=\Theta\delta(P_\epsilon)(P_\epsilon P_t P_\epsilon)P_\epsilon+P_\epsilon\delta(P_\epsilon P_t P_\epsilon).$$

Using the trace property of $\Phi\circ\mathrm{tr}$, (any terms we need to interchange have elements which satisfy the hypotheses of Lemma 4.13) we get two terms of the form $\Phi\circ\mathrm{tr}(AP_\epsilon)$. As above, the proof of Proposition 12 of [HL99] shows that

$$||\Phi\circ\mathrm{tr}(AP_\epsilon)||_T \;\leq\; C||A||\,||\Phi\circ\mathrm{tr}(P_\epsilon)||_T.$$

Now A is a product of terms of the form $\alpha, P_0, \Theta\delta(\alpha P_0), P_\epsilon, \Theta\delta(P_\epsilon), P_\epsilon P_t P_\epsilon$, and $\Theta\delta(P_\epsilon P_t P_\epsilon)$. Each of these is bounded in norm, except the last which has norm bounded by a multiple of $t^{(\frac{1}{2}+a)}$. As A can contain no more that $2k$ terms of the form $\Theta\delta(P_\epsilon P_t P_\epsilon)$, and $\Phi\circ\mathrm{tr}(P_\epsilon)$ is $\mathcal{O}(\epsilon^\beta)$, we have that $||\Phi\circ\mathrm{tr}(AP_\epsilon)||_T$ is bounded by a multiple of

$$t^{2k(\frac{1}{2}+a)}\epsilon^\beta \;=\; t^{2k(\frac{1}{2}+a)}t^{\delta\beta} \;=\; t^{2k(\frac{1}{2}+a)+\delta\beta}.$$

By our choice of a, we have that the limit as $t\to\infty$ of these terms is zero, just as in the proof of Equation 4.14.

This completes the proof of Theorem 4.2.

5. Bismut superconnections

As noted above, in [BH04] we proved that the Chern character ch_a composed with the topological and analytic index maps of Connes-Skandalis [CS84] yield the same map. In particular, for any Dirac operator D, the Chern character of the topological index of D, coincides with the Chern character of the analytic index of D, i.e.

$$\mathrm{ch}_a(\mathrm{Ind}_t(D^+)) = \mathrm{ch}_a(\mathrm{Ind}_a(D^+)).$$

In [HL99], it is proved that $\mathrm{ch}_a(\mathrm{Ind}_t(D^+))$ is equal to the Chern character of the index bundle of D in another sense. We defined a "connection" ∇ on the index bundle $[P_0]$ of D, and defined the Chern character of $[P_0]$ to be the Haefliger class of $\mathrm{Tr}(\alpha e^{-(\nabla^2/2i\pi)})$. We then used a Bismut superconnection for foliations, [He95], to show that $\mathrm{ch}_a(\mathrm{Ind}_t(D))$ contains the Haefliger form $\mathrm{Tr}(\alpha e^{-(\nabla^2/2i\pi)})$, provided that the assumptions of Theorem 4.1 are satisfied, but with the stronger assumption that the Novikov-Shubin invariants of D are greater than **three** times the codimension of F. We will now show that whenever P_0 is smooth, $\mathrm{ch}_a([P_0])$ contains the Haefliger form $\mathrm{Tr}(\alpha e^{-(\nabla^2/2i\pi)})$, so the two definitions of the Chern character of $[P_0]$ agree.

We first recall the construction of Bismut superconnections for D. See [B86], [BV87], and also [He95]. Let ∇^B be a Bott connection on ν_s^*. If $\omega_1, \ldots, \omega_n$ is a local framing for ν_s^*, then $\nabla^B \omega_i = \sum_{j=1}^n \omega_j \otimes \theta_j^i$ where θ_j^i are local one forms on $\mathcal{G}$ and the θ_j^i satisfy $d\omega_i = \displaystyle\sum_{i=1}^n \omega_j \wedge \theta_j^i$. That is, the composition

$$C^\infty(\nu_s^*) \xrightarrow{\nabla^B} C^\infty(\nu_s^* \otimes T^*\mathcal{G}) \xrightarrow{\wedge} C^\infty(\nu_s^* \wedge T^*\mathcal{G})$$

is just $\omega \to d\omega$. ∇^B induces a connection on $\wedge \nu_s^*$ also denoted ∇^B so that

$$C^\infty(\wedge\nu_s^*) \xrightarrow{\nabla^B} C^\infty(\wedge\nu_s^* \otimes T^*\mathcal{G}) \xrightarrow{\wedge} C^\infty(\wedge\nu_s^* \wedge T^*\mathcal{G})$$

is also just $\omega \to d\omega$.

Set $\mathcal{V} = TF_s \oplus \nu_s \oplus \nu_s^* = T\mathcal{G} \oplus \nu_s^*$ over $\mathcal{G}$, and define a symmetric bilinear form g on $\mathcal{V}$ as follows. TF_s and $\nu_s \oplus \nu_s^*$ are orthogonal and $g|TF_s$ is $g_0|TF_s$. The form $g|\nu_s \oplus \nu_s^*$ is given by the canonical duality, i.e. ν_s and ν_s^* are totally isotropic and $g(X, \omega) = \omega(X)$ for $X \in \nu_s$, $\omega \in \nu_s^*$. In [BV87], p. 455. it is shown that there is a unique connection ∇, the Bismut connection, on $\mathcal{V}$ so that ∇ preserves ν_s^* and g, $\nabla|\nu_s^* = \nabla^B$ and for all X, $Y \in C^\infty(T\mathcal{G})$, $\nabla_X Y - \nabla_Y X = [X, Y]$. Note that in general ∇ does not preserve $T\mathcal{G}$ but that for $X, Y \in C^\infty(T\mathcal{G})$, $\nabla_X Y - \nabla_Y X \in C^\infty(T\mathcal{G})$.

Consider the vector space $V = \mathbb{R}^p \oplus \mathbb{R}^n \oplus \mathbb{R}^{n*}$. Define a bilinear form Q on V as g was on $\mathcal{V}$, i.e. $\mathbb{R}^p$ is orthogonal to $\mathbb{R}^n \oplus \mathbb{R}^{n*}$, $Q|\mathbb{R}^p$ is the usual inner product, and $Q|\mathbb{R}^n \oplus \mathbb{R}^{n*}$ is given by the canonical duality. Let $C(V, Q)$ be the associated Clifford algebra and set $S_0 = \wedge\mathbb{R}^{n*} \otimes S$ where S is the spinor space for $\mathbb{R}^p$ with the usual inner product. Let ρ be the representation of the Clifford algebra of $\mathbb{R}^p$ in S. Then S_0 is the spinor space for $C(V, Q)$ with the Clifford multiplication being defined by

$$\begin{aligned} \rho_0(X)(\omega \otimes s) &= (-1)^{\deg \omega} \omega \otimes \rho(X)s \\ \rho_0(Y)(\omega \otimes s) &= -2i(Y)\omega \otimes s \\ \rho_0(\phi)(\omega \otimes s) &= \phi \wedge \omega \otimes s \end{aligned}$$

for $X \in \mathbb{R}^p$, $Y \in \mathbb{R}^n$, $\phi \in \mathbb{R}^{n*}$, $\omega \in \wedge\mathbb{R}^{n*}$, $s \in S$. See [BV87], p. 456 and for general facts about spinors and Clifford algebras, [LM89].

The above fact allows Berline and Vergne to give a beautiful and concise definition of Bismut superconnections for fiber bundles which was extended to foliations in [He95]. Recall that $\mathcal{S}$ is the spinor bundle along the leaves of F_s, and consider the vector bundle $\mathcal{S}_0 = \wedge\nu_s^* \otimes \mathcal{S}$ over $\mathcal{G}$ and the bundle of Clifford algebras $C(\mathcal{V})$ over $\mathcal{G}$ associated to $\mathcal{V}, g$. Then $\mathcal{S}_{0,y}$, the fiber over $y \in \mathcal{G}$ of $\mathcal{S}_0$, is a module for the algebra $C(\mathcal{V})_y$ and we denote the module action also by ρ_0. The connection ∇ on $\mathcal{V}$ induces a connection ∇ on $\mathcal{S}_0$ ([BV87], p. 456; or more generally [LM89], Ch. 4). Let E be a vector bundle with connection over $\mathcal{G}$ as in Section 2. We shall also denote by ∇ the tensor product connection on $\mathcal{S}_0 \otimes E$.

A Bismut superconnection $\mathbb{B}$ for F_s and E is the Dirac type operator on $C_c^\infty(\mathcal{S}_0 \otimes E)$ defined as follows. Let $X_1, \ldots, X_p$ be a local oriented orthonormal basis of TF_s, and $X_{p+1}, \ldots, X_{p+n}$ a local basis of ν_s. Let $X_1^*, \ldots, X_{p+n}^*$ be the dual basis in $TF_s \oplus \nu_s^*$, i.e. $X_i^* = X_i$ for $1 \leq i \leq p$, $X_i^* = \omega_i$, for $p+1 \leq i \leq p+n$ where $\omega_i \in \nu_s^*$ and $\omega_i(X_j) = \delta_{ij}$. Set

$$\mathbb{B} = \sum_{i=1}^{p+n} \big(\rho_0(X_i^*) \otimes 1\big) \nabla_{X_i} = \sum_{i=1}^{p} \rho(X_i)\nabla_{X_i} + \sum_{i=p+1}^{p+n} \omega_i \nabla_{X_i}.$$

$\mathbb{B}$ does not depend on the choice of $X_1, \ldots, X_{p+n}$.

Since $\mathcal{S} = \mathcal{S}^+ \oplus \mathcal{S}^-$ is $\mathbb{Z}_2$ graded and $\wedge \nu_s^*$ is $\mathbb{Z}$ graded, $\mathcal{S}_0 = \wedge \nu_s^* \otimes \mathcal{S}$ has a total $\mathbb{Z}_2$ grading and we write $\mathcal{S}_0 = \mathcal{S}_0^+ \oplus \mathcal{S}_0^-$. We then have an associated $\mathbb{Z}_2$ grading $\mathcal{S}_0 \otimes E = (\mathcal{S}_0^+ \otimes E) \oplus (\mathcal{S}_0^- \otimes E)$. It is immediate from the fact that ∇ preserves the grading that $\mathbb{B}$ is an odd operator, i.e. $\mathbb{B}$ maps $C^\infty(\mathcal{S}_0^+ \otimes E)$ to $C^\infty(\mathcal{S}_0^- \otimes E)$ and vice-versa.

Finally, we may use the $\mathbb{Z}$ grading on $\wedge \nu_s^*$ to grade the operator $\mathbb{B}$, i.e. $\mathbb{B} = \mathbb{B}^{[0]} + \mathbb{B}^{[1]} + \cdots$, where $\mathbb{B}^{[i]} : C^\infty(\wedge^k \nu_s^* \otimes \mathcal{S} \otimes E) \to C^\infty(\wedge^{k+i} \nu_s^* \otimes \mathcal{S} \otimes E)$.

It is straightforward to check that

Proposition 5.1. *The term $\mathbb{B}^{[1]}$ is a quasi-connection ∇^ν for $E \otimes \wedge \nu_s^*$ as defined in Section 3.*

Recall, [HL99], that a *connection* on the *index bundle* of D is defined by

$$\nabla = P_0 \mathbb{B}^{[1]} P_0.$$

For this to be well defined, we must require that P_0 is transversely smooth.

Theorem 5.2. *Suppose that P_0 is transversely smooth. Then $\mathrm{ch}_a([P_0])$ contains the Haefliger form $\mathrm{Tr}(\alpha e^{-(\nabla^2/2i\pi)})$*

Proof. First we calculate ∇^2.

$$\begin{aligned}
\nabla^2 &= P_0 \mathbb{B}^{[1]} P_0 \mathbb{B}^{[1]} P_0 \\
&= P_0 [\mathbb{B}^{[1]}, P_0] \mathbb{B}^{[1]} P_0 + P_0 (\mathbb{B}^{[1]})^2 P_0 \\
&= P_0 [\mathbb{B}^{[1]}, P_0][\mathbb{B}^{[1]}, P_0] + P_0 [\mathbb{B}^{[1]}, P_0] P_0 \mathbb{B}^{[1]} + P_0 (\mathbb{B}^{[1]})^2 P_0 \\
&= P_0 [\mathbb{B}^{[1]}, P_0][\mathbb{B}^{[1]}, P_0] + P_0 (\mathbb{B}^{[1]})^2 P_0.
\end{aligned}$$

The last equality is a consequence of the relation $P_0[\mathbb{B}^{[1]}, P_0]P_0 = 0$ which is true since $P_0^2 = P_0$ and since $[\mathbb{B}^{[1]}, \cdot]$ is a derivation. This derivation is precisely ∂_ν, so $(\mathbb{B}^{[1]})^2 = \theta$ as in Section 3. Thus

$$\nabla^2 = P_0 (\partial_\nu P_0)^2 + P_0 \theta P_0,$$

and

$$\nabla^{2k} = (P_0 (\partial_\nu P_0)^2 + P_0 \theta P_0)^k.$$

Note that

$$\partial_\nu(P_0) = \partial_\nu(P_0 P_0) = \partial_\nu(P_0) P_0 + P_0 \partial_\nu(P_0),$$

so

$$\partial_\nu(P_0) P_0 = \partial_\nu(P_0) - P_0 \partial_\nu(P_0).$$

Using this twice, one can easily show that

$$P_0 \partial_\nu(P_0) \partial_\nu(P_0) = P_0 \partial_\nu(P_0) \partial_\nu(P_0) P_0.$$

Then a simple induction argument shows that

$$(P_0 (\partial_\nu P_0)^2 + P_0 \theta P_0)^k = P_0((\partial_\nu P_0)^2 + P_0 \theta P_0)^k$$

Thus,

$$\mathrm{Tr}(\alpha \nabla^{2k}) = \mathrm{Tr}(\alpha P_0((\partial_\nu P_0)^2 + P_0 \theta P_0)^k),$$

and comparing with Equation 3.12, we see that $\mathrm{ch}_a([P_0])$ contains the Haefliger form $\mathrm{Tr}(\alpha e^{-(\nabla^2/2i\pi)})$. ☐

6. Appendix

We now prove Lemma 4.13, namely the fact that if H and K are $\mathcal{G}$ invariant $\mathcal{A}(M)$ equivariant bounded leafwise smoothing operators on $\mathcal{S}\otimes E\otimes \wedge\nu_s^*$, which are transversely smooth, then $\mathrm{Tr}([H,K])=0$. To accomplish this it suffices to construct forms in $\mathrm{Tr}([H,K])$ which are arbitrarily C^ℓ close to 0, for any ℓ. To do so, we adapt the argument in the proof of Lemma 2.6 of [HL02].

Denote by $H_x(y,z)$ and $K_x(y,z)$ the Schwartz kernels of H and K, where $x\in M$ and $y,z\in \widetilde{L}_x$. Denote by $k(x,y,z)$, the pointwise trace $k(x,y,z)=\mathrm{tr}(H_x(y,z)K_x(z,y)\pm K_x(y,z)H_x(z,y))$, where the ambiguity of signs occurs because we are using graded comutators. Then,

$$\mathrm{Tr}([H,K])=\int_F\int_{\widetilde{L}_x} k(x,\overline{x},z)dz\,dx,$$

where $\overline{x}\in\widetilde{L}_x$ is the class of the constant path at x.

Let $\mathcal{U}_\mathcal{G}=\big\{(U_i,\gamma_{ijk},U_j)\big\}$ be the cover of $\mathcal{G}$ corresponding to the good cover $\mathcal{U}$ of M. Here, U_i, $U_j\in\mathcal{U}$ and γ_{ijk} is a leafwise path from U_i to U_j. Let $\{\phi_i\}$ be a partition of unity subordinate to the cover $\mathcal{U}$. For each (U_i,γ_{ijk},U_j), define $\phi_{ijk}:\mathcal{G}\to\mathbb{R}$ by $\phi_{ijk}(z)=\phi_j(r(z))$ for $z\in(U_i,\gamma_{ijk},U_j)$, and $\phi_{ijk}(z)=0$ otherwise. Then for each $x\in M$, $\{\phi_{ijk}\}$ restricted to $\widetilde{L}_x$ is a partition of unity subordinate to the cover of $\widetilde{L}_x$ by its placques in the various (U_i,γ_{ijk},U_j).

On the transversal $T_i\subset U_i$, the class $\displaystyle\int_F\int_{\widetilde{L}_x}k(x,\overline{x},z)dz\,dx$ is represented by $\displaystyle\int_{P_x}\phi_i(x)\int_{\widetilde{L}_x}k(x,\overline{x},z)dz\,dx$, where P_x is the placque of $x\in U_i$. Because of the $\mathcal{G}$ invariance of k, this is equal to $\displaystyle\int_{P_x}\phi_i(y)\int_{\widetilde{L}_x}k(x,y,z)dz\,dy$, where now $x=P_x\cap T_i$ and $y\in P_x$ is thought of as the class of the path in P_x from x to y. Now we have

$$\int_{P_x}\int_{\widetilde{L}_x}\phi_i(y)k(x,y,z)dz\,dy=\sum_{j,k}\int_{P_x}\int_{\widetilde{L}_x}\phi_i(y)\phi_{i,j,k}(z)k(x,y,z)dz\,dy=$$

$$\sum_{j,k}\int_{P_x}\int_{P_z}\phi_i(y)\phi_j(z)k(x,y,z)dz\,dy,$$

where $z\in P_z\subset U_j$ is thought of as the element of (U_i,γ_{ijk},U_j) starting at x and ending at z. Note that with this interpretation, $\phi_{i,j,k}(z)$ becomes $\phi_j(z)$. For any chart of the form $(U_i,\overline{x},U_i)$, where $x\in U_i$,

$$\int_{P_x}\int_{P_x}\phi_i(y)\phi_i(z)k(x,y,z)dz\,dy=0$$

by symmetry. For the chart (U_i,γ_{ijk},U_j), consider the form $I_{ijk}=\displaystyle\int_{P_x}\int_{P_z}\phi_i(y)\phi_j(z)k(x,y,z)dz\,dy$ on T_i. On $T_j\subset U_j$ there is the corresponding form $I_{jik}=\displaystyle\int_{P_z}\int_{P_x}\phi_j(z)\phi_i(y)k(x,z,y)dy\,dz$ for the chart $(U_j,\gamma_{ijk}^{-1},U_i)$. The crucial point is that because we are using graded commutators, $I_{ijk}=h^*_{\gamma_{ijk}}(-I_{jik})$. Thus if we move the term I_{jik} to T_i using the holonomy map $h_{\gamma_{ijk}}$ associated to γ_{ijk}, it will cancel I_{ijk} and we obtain a new form in $\displaystyle\int_F\int_{\widetilde{L}}k$, which at $x\in T_i$ is given by

$$\int_{P_x}\int_{\widetilde{L}_x}\phi_i(y)(1-\phi_{ijk}(z))k(x,y,z)dz\,dy,$$

and for $x\in T_j$ is given by

$$\int_{P_z}\int_{\widetilde{L}_x}\phi_j(z)(1-\phi_{jik}(y))k(x,z,y)dy\,dz.$$

Doing this procedure over all γ_{ijk} of length less than or equal to R (of which there are only a finite number), we obtain the form in $\displaystyle\int_F\int_{\widetilde{L}}k$ which on T_i it is given by

6.1. $$\int_{P_x} \phi_i(y) \int_{\widetilde{L}_x} (1-\phi_R(z)) k(x,y,z) dz\, dy,$$

where $\phi_R(z) = \sum_{j,k} \phi_{ijk}(z)$, and the sum is over all j and k such that the length of $\gamma_{ijk} \leq R$.

We claim that this form and all its derivatives converge to zero as $R \to \infty$.

Lemma 6.2. $\int_{\widetilde{L}_x} |k(x,y,z)|\, dz$ *is bounded independently of* x *and* y.

Proof. We will use the notation of the proof of Theorem 2.3.9 of [HL90], and will use subscripts to distinguish the different inner products we use (e.g. $<\,,\,>_z$ on $(\mathcal{S}\otimes E)_z$, $<\,,\,>_x$ on sections of $\mathcal{S}\otimes E \,|\, \widetilde{L}_x$).

Let $X \in \wedge^d TM$ with $||X|| = 1$ and recall that $i_X H_{[d],x}(y,z) \in \mathrm{Hom}((\mathcal{S}\otimes E)_z, (\mathcal{S}\otimes E)_y)$. Let $v_1, \ldots, v_\ell$ and $w_1, \ldots, w_\ell$ be orthonormal bases of $(\mathcal{S}\otimes E)_y$ and $(\mathcal{S}\otimes E)_z$. Recall the Dirac delta sections $\delta_y^{v_j}$, and $\delta_z^{w_i}$ of $\mathcal{S}\otimes E \,|\, \widetilde{L}_x$, and set

$$\psi^{X,v_j}_{[d],y}(z) = \sum_i < i_X H_{[d],x} \delta_z^{w_i}, \delta_y^{v_j} >_x \, w_i.$$

Then $\psi^{X,v_j}_{[d],y}(z)$ is independent of the choice of basis w_i and so defines a section $\psi^{X,v_j}_{[d],y}$ of $\mathcal{S}\otimes E \,|\, \widetilde{L}_x$. Standard techniques show that $\psi^{X,v_j}_{[d],y} \in C^\infty(\widetilde{L}_x; \mathcal{S}\otimes E)$. For $\xi \in L^2(\widetilde{L}_x; \mathcal{S}\otimes E)$, it is easy to see that

$$i_X H_{[d],x} \xi(y) = \sum_j \Big(\int_{\widetilde{L}_x} < \psi^{X,v_j}_{[d],y}(z), \xi(z) >_z \, dz \Big) v_j,$$

that is

$$i_X H_{[d],x}(y,z) = \sum_j \psi^{X,v_j}_{[d],y}(z) \otimes v_j.$$

We claim that $\psi^{X,v_j}_{[d],y}(z) \in L^2(\widetilde{L}_x; \mathcal{S}\otimes E)$, and that its L^2 norm is uniformly bounded over all x, y, v_j, and X. This is true provided there is a constant C, independent of x, y, v_j, and X, so that for all sections ξ of $L^2(\widetilde{L}_x; \mathcal{S}\otimes E)$,

$$| < \psi^{X,v_j}_{[d],y}, \xi >_x | \leq C\, ||\xi||_0.$$

Since the $\widetilde{L}_x$ have uniformly bounded geometry, there is a $k > 0$ so that the Sobolev norms $||\delta_y^{v_j}||_{-k}$ are bounded independently of x, y and v_j. Finally, since H is transversely smooth and $||X|| = 1$, the Sobolev norm $||i_X H_{[d],x}||_{0,k}$ is bounded, independently of x and X (by $||H||_{0,k}$). Now for any $\xi \in L^2(\widetilde{L}_x; \mathcal{S}\otimes E)$, we have

$$| < \psi^{X,v_j}_{[d],y}, \xi >_x | \;=\; \Big| \int_{\widetilde{L}_x} < \psi^{X,v_j}_{[d],y}(z), \xi(z) >_z \, dz \Big| \;=\; \Big| < \sum_i \Big(\int_{\widetilde{L}_x} < \psi^{X,v_i}_{[d],y}(z), \xi(z) >_z \, dz \Big) v_i, v_j >_y \Big| \;=$$
$$| < i_X H_{[d],x}\xi(y), v_j >_y | \;=\; | < i_X H_{[d],x}\xi, \delta_y^{v_j} >_x | \;\leq\; ||i_X H_{[d],x}\xi||_k \, ||\delta_y^{v_j}||_{-k} \;\leq$$
$$||i_X H_{[d],x}||_{0,k} ||\xi||_0 \, ||\delta_y^{v_j}||_{-k} \;\leq\; C\, ||\xi||_0,$$

and the constant C is independent of x, y, v_j, and X.

Similarly, if $Y \in \wedge^e TM$ with $||Y|| = 1$, we have

$$(i_Y K_{[e],x})^*(y,z) = \sum_j \phi^{Y,v_j}_{[e],y}(z) \otimes w_j,$$

where $(i_Y K_{[e],x})^*$ is the adjoint of $i_Y K_{[e],x}$, and so is a bounded smoothing operator with bound independent of x and Y. As above, the $\phi^{Y,v_j}_{[e],y}(z) \in L^2(\widetilde{L}_x; \mathcal{S}\otimes E)$, and have L^2 norms uniformly bounded over all x, y, v_j, and Y. A standard argument gives that

$$\mathrm{tr}(i_X H_{[d],x}(y,z) i_Y K_{[e],x}(z,y)) = \sum_j < \psi^{X,v_j}_{[d],y}(z), \phi^{Y,v_j}_{[e],y}(z) >_z,$$

so

$$\int_{\widetilde{L}_x} \mathrm{tr}(i_X H_{[d],x}(y,z) i_Y K_{[e],x}(z,y))\, dz = \sum_j < \psi^{X,v_j}_{[d],y}, \phi^{Y,v_j}_{[e],y} >_x .$$

As $\psi^{X,v_j}_{[d],y}$ and $\phi^{Y,v_j}_{[e],y} \in L^2(\widetilde{L}_x;\mathcal{S}\otimes E)$, with norm bounds independent of x, y, v_j, X, and Y, the function $z \to \operatorname{tr}(i_X H_{[d],x}(y,z) i_Y K_{[e],x}(z,y)) \in L^1(\widetilde{L}_x)$, and $\displaystyle\int_{\widetilde{L}_x} |\operatorname{tr}(i_X H_{[d],x}(y,z) i_Y K_{[e],x}(z,y))|\, dz$ has bound independent of x, y, X and Y. The same holds for $\displaystyle\int_{\widetilde{L}_x} |\operatorname{tr}(i_Y K_{[e],x}(y,z) i_X H_{[d],x}(z,y))|\, dz$, so $\displaystyle\int_{\widetilde{L}_x} |k(x,y,z)|\, dz$, which is bounded by a finite sum of terms with these forms, is bounded independently of x and y. □

As

$$\int_{\widetilde{L}_x} (1-\phi_R(z))|k(x,y,z)|dz \;\leq\; \int_{\widetilde{L}_x} |k(x,y,z)|\, dz,$$

$\displaystyle\int_{\widetilde{L}_x} (1-\phi_R(z))|k(x,y,z)|dz$ is also uniformly bounded. In addition, it is monotonically decreasing as a function of R. An application of the Dominated Convergence Theorem gives

$$\lim_{R\to\infty} \int_{P_x} \phi_i(y) \int_{\widetilde{L}_x} (1-\phi_R(z))|k(x,y,z)|dz\, dy = \int_{P_x} \phi_i(y) \Big[\lim_{R\to\infty} \int_{\widetilde{L}_x} (1-\phi_R(z))|k(x,y,z)|dz\Big]\, dy.$$

Now, $0 \leq 1-\phi_R(z) \leq 1$, and $\lim\limits_{R\to\infty} 1-\phi_R(z) = 0$. It follows immediately that

$$\lim_{R\to\infty} \int_{\widetilde{L}_x} (1-\phi_R(z))|k(x,y,z)|dz = 0,$$

and the convergence in monotonic in R. So

$$\lim_{R\to\infty} \int_{P_x} \phi_i(y) \int_{\widetilde{L}_x} (1-\phi_R(z))|k(x,y,z)|dz\, dy = 0,$$

and the convergence in monotonic in R. Thus given any $\epsilon > 0$, and $x \in T$, there is an $R(x)$ so that if $R > R(x)$

$$\int_{P_x} \phi_i(y) \int_{\widetilde{L}_x} (1-\phi_R(z))|k(x,y,z)|dz\, dy < \epsilon/2.$$

For each R, the function $\displaystyle\int_{P_x} \phi_i(y) \int_{\widetilde{L}_x} (1-\phi_R(z))k(x,y,z)dz\, dy$ is in $\mathcal{A}^*_c(T)$, since $\displaystyle\int_{P_x} \phi_i(y) \int_{\widetilde{L}_x} k(x,y,z)dz\, dy$ and all the terms I_{ijk} are in $\mathcal{A}^*_c(T)$. Thus $\displaystyle\int_{P_x} \phi_i(y) \int_{\widetilde{L}_x} (1-\phi_R(z))|k(x,y,z)|dz\, dy$ is at least continuous, so given $\epsilon > 0$, and $x \in T$, there is $\delta(x)$ so that if $w \in T$ and $|x-w| < \delta(x)$, then for all $R > R(x)$ (due to monotonicity),

$$\int_{P_w} \phi_i(y) \int_{\widetilde{L}_w} (1-\phi_R(z))|k(w,y,z)|dz\, dy < \epsilon.$$

We may assume that the closure $\overline{T}$ of T in M is an embedded compact transverse submanifold, that is a smooth compact submanifold with boundary. The open sets $U(x) = \{w \,|\, |x-w| < \delta(x)\}$ form a cover of $\overline{T}$. Let $U(x_1), ..., U(x_k)$ be a finite subcover, and set $R = \max(R(x_1), ..., R(x_k))$. Then for all $x \in T$,

$$|\int_{P_x} \phi_i(y) \int_{\widetilde{L}_x} (1-\phi_R(z))k(x,y,z)dz\, dy| \leq \int_{P_x} \phi_i(y) \int_{\widetilde{L}_x} (1-\phi_R(z))|k(x,y,z)|dz\, dy < \epsilon.$$

Thus $\displaystyle\int_{P_x} \phi_i(y) \int_{\widetilde{L}_x} (1-\phi_R(z))k(x,y,z)dz\, dy$, which is in $\operatorname{Tr}([H,K])$, is arbitrarily C^0 close to 0.

To treat the derivatives, we need the following lemma.

Lemma 6.3. *Suppose that H is an $\mathcal{A}(M)$ equivariant bounded leafwise smoothing operator on $\mathcal{S}\otimes E\otimes\wedge\nu^*_s$, which is transversely smooth, and that its Schwartz kernel is zero on $\widetilde{L}_x \times \widetilde{L}_x$, if $\phi_i(x) = 0$. Then on T_i*

$$\int_{P_x} \operatorname{tr}((\partial_\nu H)_x(\overline{y},\overline{y}))\, dy = d_{T_i} \int_{P_x} \operatorname{tr}(H_x(\overline{y},\overline{y}))\, dy.$$

Proof. The Schwartz kernel of H, also denoted H, is a section of a bundle over the double graph $\mathcal{G}_{[2]}$. Denote by $\widetilde{U_i} = \{(\alpha,\beta) \in \mathcal{G}_{[2]} \,|\, \alpha,\beta \in (U_i,\overline{x},U_i)\}$, and set $W = \{(\overline{a},\overline{a}) \,|\, a \in \operatorname{supp}\phi_i\}$. Then W is a compact subset of the open set $\widetilde{U_i}$. As ∂_ν is a local operator and we are integrating over W, we may assume that $\operatorname{supp}(H)$, the support of the Schwartz kernel of H, is a subset of $\widetilde{U_i}$.

Note that the lemma is a coordinate free statement, so we are free to choose coordinates as we please, and we need only confirm the statement at a single point $x \in T_i$. We use the exponential map $\exp : \nu \to M$ restricted to P_x, to construct coordinates on a neighborhood of P_x in U_i, which we may assume is all of U_i. Then $U_i \simeq \mathbb{R}^p \times \mathbb{R}^q$ has coordinates (y,z), with $P_x = \mathbb{R}^p \times \{0\}$. We may identify T_i with $(0,0)\times\mathbb{R}^q$. That is, we use the coordinates induced on T_i, thought of simply as a manifold in its own right, by the natural diffeomorphism $T_i \simeq (0,0)\times\mathbb{R}^q$. Thus we have $U_i \simeq \mathbb{R}^p \times T_i$, and the fact that we have used the exponential map to define the coordinates means that $\nu_{(y,0)} = T(T_i)_{(y,0)}$, i.e. at $(y,0)$, ν is spanned by $\partial/\partial z_1,\ldots,\partial/\partial z_q$.

The set $(U_i,\overline{x},U_i) \simeq \mathbb{R}^p \times \mathbb{R}^q \times \mathbb{R}^p$ has coordinates (y,z,w). On U_i, with the coordinates (w,z), where $w = y$, ν is spanned by vector fields of the form $\partial/\partial z_i + \sum_j a_{ij}\partial/\partial w_j$, and $a_{ij} \,|\, P_x \equiv 0$. On $(U_i,\overline{x},U_i)$, $\nabla^\nu = d_y + d_z + \sum_{ij} a_{ij} dz_i \otimes \partial/\partial w_j + A$ where $A \in C^\infty((U_i,\overline{x},U_i);\operatorname{Hom}(\mathcal{S}\otimes E))\widehat{\otimes}_{C^\infty(U_i)}\mathcal{A}^1(U_i)$. See the proof of Lemma 3.1. Thus

$$\int_{P_x} \operatorname{tr}(\partial_\nu H)\,dy = \int_{P_x} \operatorname{tr}([d_y + d_z + \sum_{ij} a_{ij} dz_i \otimes \partial/\partial w_j + A, H]\,dy =$$

$$\int_{P_x} \operatorname{tr}(d_y H)\,dy + \int_{P_x} tr(d_z H)\,dy + \int_{P_x} tr([A,H])\,dy,$$

since $a_{ij} \,|\, P_x \equiv 0$. The term $\int_{P_x} \operatorname{tr}(d_y H)\,dy$ is obviously zero. A direct computation, using the fact that $\operatorname{supp}(H) \subset \widetilde{U_i}$, shows that $\int_{P_x} \operatorname{tr}([A,H])\,dy = 0$. Finally, note that in these coordinates, d_{T_i} is d_z. Thus

$$\int_{P_x} \operatorname{tr}(d_z H)\,dy = \int_{P_x} d_z \operatorname{tr}(H)\,dy = d_{T_i}\int_{P_x} \operatorname{tr}(H)\,dy.$$

□

The form in 6.1 may be written as

$$\int_{P_x}\Big(\int_{\widetilde{L}_x} \operatorname{tr}(M_{\phi_i}\circ H \circ M_{1-\phi_R}\circ K \pm M_{\phi_i}\circ K\circ M_{1-\phi_R}\circ H)\Big)dy$$

where the integrand satisfies the hypothesis of Lemma 6.3. To simplify the notation, we may assume that both H and K are degree zero $\mathcal{A}(M)$ equivariant operators. We want to show that the derivatives of the coefficients of this differential form on T_i go to zero as $R \to \infty$. It is sufficient to estimate the norm of

$$i_{\partial/\partial y_{j_\ell}} d_{T_i} \cdots i_{\partial/\partial y_{j_1}} d_{T_i} \int_{P_x}\Big(\int_{\widetilde{L}_x} \operatorname{tr}(M_{\phi_i}\circ H \circ M_{1-\phi_R}\circ K \pm M_{\phi_i}\circ K\circ M_{1-\phi_R}\circ H)\Big)dy,$$

where $1 \le j_1 \le \cdots \le j_\ell \le q$. Let Y be the vector field $\partial/\partial y_r$ on T_i, and denote also by Y its extension to U_i. Then

$$i_Y d_{T_i}\int_{P_x}\Big(\int_{\widetilde{L}_x} \operatorname{tr}(M_{\phi_i}\circ H \circ M_{1-\phi_R}\circ K \pm M_{\phi_i}\circ K\circ M_{1-\phi_R}\circ H)\Big)dy \;=$$
$$\int_{P_x}\Big(\int_{\widetilde{L}_x} \operatorname{tr}(\partial_\nu^Y(M_{\phi_i}\circ H \circ M_{1-\phi_R}\circ K \pm M_{\phi_i}\circ K\circ M_{1-\phi_R}\circ H))\Big)dy \;=$$
$$\int_{P_x}\Big(\int_{\widetilde{L}_x} \operatorname{tr}(M_{(Y\phi_i)}\circ H \circ M_{1-\phi_R}\circ K \pm M_{(Y\phi_i)}\circ K\circ M_{1-\phi_R}\circ H)\Big)dy \;+$$
$$\int_{P_x}\Big(\int_{\widetilde{L}_x} \operatorname{tr}(M_{\phi_i}\circ \partial_\nu^Y H \circ M_{1-\phi_R}\circ K \pm M_{\phi_i}\circ K\circ M_{1-\phi_R}\circ \partial_\nu^Y H)\Big)dy \;-$$
$$\int_{P_x}\Big(\int_{\widetilde{L}_x} \operatorname{tr}(M_{\phi_i}\circ H \circ M_{(i_Y p_\nu d\phi_R)}\circ K \pm M_{\phi_i}\circ K\circ M_{(i_Y p_\nu d\phi_R)}\circ H)\Big)dy \;+$$

$$\int_{P_x}\Big(\int_{\widetilde{L}_x}\operatorname{tr}(M_{\phi_i}\circ H\circ M_{1-\phi_R}\circ\partial_\nu^Y K\pm M_{\phi_i}\circ\partial_\nu^Y K\circ M_{1-\phi_R}\circ H)\Big)dy.$$

The first, second and fourth integrands above all have the same properties as the integrand in 6.1. So the same proof shows that as $R\to\infty$, these terms converge uniformly on T_i to zero. To obtain monotone estimates for the third integrand, first note that since ϕ_R is constructed out of the partition of unity on the compact manifold M, there is a constant $C_1>0$ so that the derivatives of ϕ_R have norm bounded by C_1. Because M is compact, there is a constant C_2 so that $\operatorname{supp}(i_Y d_\nu\phi_R)\cap\operatorname{supp}(\phi_{R-C_2})=\emptyset$. Then the third integral is majorized by the integral $\displaystyle\int_{P_x}\phi_i(y)\int_{\widetilde{L}_x}C_1(1-\phi_{R-C_2}(z))|k(x,y,z)|\,dz\,dy$. This integrand also has the same properties as the integrand in 6.1. An obvious induction argument finishes the proof.

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