

## PRACTICE PROBLEMS FOR THE MATH 330 FINAL

Here is a list of sample problems that you should be comfortable solving quickly and accurately. I have not included problems involving material from Chapters 18, 20, 21, 22 since we have not covered that material yet.

- (1) Find the lcm and the gcd of the integers 1240 and 820.
- (2) Find all the subgroups of the group  $\mathbb{Z}/240\mathbb{Z}$  and describe their orders.
- (3) Find the number of elements of order 20 in  $\mathbb{Z}/60\mathbb{Z} \oplus \mathbb{Z}/80\mathbb{Z} \oplus \mathbb{Z}/25\mathbb{Z}$ .
- (4) Find the number of subgroups of order 9 in  $\mathbb{Z}/27\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$ . (Caution: They need not all be cyclic.)
- (5) Let  $p$  be a prime number. Calculate the inner automorphism groups of  $S_{2p}$ ,  $A_{2p}$  and  $D_{2p}$  up to isomorphism.
- (6) Calculate the automorphism group of  $\mathbb{Z}/1800\mathbb{Z}$  up to isomorphism. Is this group cyclic? Express the group as a direct product of cyclic groups.
- (7) List the conjugacy classes of the symmetric group  $S_6$ . Determine how many elements each of the conjugacy class has. Do the same problem for the alternating group  $A_5$ .
- (8) Let  $G$  be an abelian group of order 225.
  - (a) Suppose  $G$  has 24 elements of order 45. Determine the isomorphism class of  $G$ .
  - (b) Suppose  $G$  has no elements of order 45, but has exactly 4 elements of order 5. Determine the isomorphism class of  $G$ .
  - (c) Suppose  $G$  has more than 24 elements of order 45. Determine the isomorphism class of  $G$ .
- (9) Find all the group homomorphisms  $f : \mathbb{Z}/160\mathbb{Z} \rightarrow \mathbb{Z}/100\mathbb{Z}$  and determine their kernels.
- (10) Consider the group homomorphism  $f : \mathbb{Z} \oplus \mathbb{Z} \rightarrow \mathbb{Z} \oplus \mathbb{Z}$  with the property that  $f((1, 0)) = (6, 12)$  and  $f((0, 1)) = (2, 4)$ . Determine the image of  $f$ . Determine the kernel of  $f$ . Consider the factor group  $\mathbb{Z} \oplus \mathbb{Z}/f(\mathbb{Z} \oplus \mathbb{Z})$ . Does this group contain any elements of order 5? Does it contain any elements of order 2? Does it contain any elements of infinite order? Is it cyclic?
- (11) Classify all abelian groups of order 1080 up to isomorphism.
- (12) Let  $G$  be a group of order 12 whose center contains an element of order 4. Prove that  $G$  must be abelian. Does  $G$  have to be cyclic?

- (13) Prove that the intersections of two normal subgroups of a group  $G$  is a normal subgroup of  $G$ .
- (14) Consider the ring  $R = \mathbb{Z}/30\mathbb{Z}$ . Find all the units in  $R$ . Find all the zero divisors in  $R$ . Find all the maximal ideals in  $R$ . Find all the prime ideals in  $R$ .
- (15) Find all ring homomorphisms  $f : \mathbb{Z}/60\mathbb{Z} \rightarrow \mathbb{Z}/48\mathbb{Z}$ . Describe their kernels.
- (16) Let  $\mathbb{C}$  denote the field of complex numbers. Describe all the ideals of  $\mathbb{C}$ . Describe all the prime ideals of  $\mathbb{C}[x]$ . Describe all the maximal ideals of  $\mathbb{C}[x]$ .
- (17) Let  $F$  be a field. Let  $F[x, y]$  denote the ring of polynomials in two variables with coefficients in  $F$ . Is  $F[x, y]$  a principal ideal domain?
- (18) Consider the polynomial  $x^2 + x - 6$ . Find all the solutions of this polynomial in  $\mathbb{Z}/p\mathbb{Z}$  where  $p$  is a prime. How does your answer change if you instead find the solutions in  $\mathbb{Z}/12\mathbb{Z}$ ?
- (19) Are the rings  $\mathbb{Z}[\sqrt{2}]$  and  $\mathbb{Z}[\sqrt{7}]$  isomorphic rings? Prove or disprove.
- (20) Determine the number of elements in the ring  $\mathbb{Z}[i]/\langle 5 + i \rangle$ . Determine the characteristic of this ring.
- (21) Prove that the ring  $\mathbb{Q}(\sqrt{2})$  is isomorphic to  $\mathbb{Q}[x]/\langle x^2 - 2 \rangle$ .
- (22) Calculate  $109! \pmod{113}$ .
- (23) Prove that  $x^2 + 2$  is a prime ideal in  $\mathbb{Z}[x]$ . Is it maximal?
- (24) Is  $\mathbb{R}[x]/\langle x^3 + x + 1 \rangle$  a field? Is it an integral domain? How about  $\mathbb{R}[x]/\langle x^2 + 3 \rangle$ ?
- (25) Determine all the ring automorphisms of  $\mathbb{Q}(i)$ .
- (26) Are the rings  $\mathbb{R}[x, y]/\langle x - y \rangle$  and  $\mathbb{R}[x, y]/\langle x^2 - y^2 \rangle$  isomorphic? Prove your answer.
- (27) Prove that the number of elements in a finite field  $F$  has to be  $p^n$  where  $p$  is a prime number and  $n$  is a positive integer. Express the additive group  $(F, +)$  as a direct sum of cyclic groups.

In addition to these problems there are many good problems in the book. You can practice by solving the supplementary exercises on pages 275-277 and 339-340 and any other problem in the book that you have not solved during the semester.