Math 215 Sample Final Exam

2 Hours. Show all work – unsupported answers will not receive credit.
Complete, correct, answers to 8 questions get 100%. (There will be more questions on the actual final)

Q1 a) Prove by induction that $\sum_{i=1}^{n} i(i+1) = \frac{n(n+1)(n+2)}{3}$

b) Assuming the well ordering principal – i,e, that every non-empty subset of $\mathbb{N}$ has a least element, prove the principal of mathematical induction.

Q2 a) Compute the truth table for $(S \wedge \sim T) \Rightarrow (T \wedge \sim S)$

b) Let $f : \mathbb{N} \to \mathbb{R}$ be the function $f(n) = 1/n$. Which of the statements
$\forall n \in \mathbb{N}, \exists j \in \mathbb{R}, j > 0, \forall e \in \mathbb{R}, (j > e > 0) \Rightarrow (f(n) < e)$ and
$\forall e \in \mathbb{R}, (e > 0) \Rightarrow (\exists k \in \mathbb{N}, \forall n > k, f(n) < e)$ is true.

Q3 a) If S is a set, define the power set $\mathcal{P}(S)$ of S.

b) Let S be a finite set. Write #(S) for the cardinality of S. Prove that the power set $\mathcal{P}(S)$ of S has $2^{\#(S)}$ elements.

Q4 Recall that two numbers m and n are coprime (i.e. have g.c.d 1) if and only if $\exists x \in \mathbb{Z}, \exists y \in \mathbb{Z}, mx + ny = 1$. Recall that a number p is prime if it has no divisors other than 1 and itself. Prove

a) A number n and a prime number p are coprime if and only if p does not divide n.
b) That p is a prime, and if a and b are natural numbers, then $p \mid ab \Rightarrow (p \mid a) \vee (p \mid b)$

Q5 a) State the axioms for a group.

b) Prove that if $G = (G, \bullet, e)$ is a group, then
$\forall g \in G, \forall h \in G, \forall k \in G, (gh = gk) \Rightarrow (h = k)$

Q6 Let X and Y be sets. Define the Cartesian product $X \times Y$, define what is a meant by function $f : X \to Y$. Define what is meant by saying that such a function is *injective*, *surjective, bijective*. Prove that if W,X,Y,Z are all sets, $f : X \to Y$ is a function, and $g, h : Y \to Z$ and $a, b : W \to X$, are all functions, then:

a) f injective and $fa = fb \Rightarrow a = b$
b) f surjective and $gf = hf \Rightarrow g = h$

Q7 Define <u>equivalence relation</u> on a set X. Prove that congruence modulo n (i.e. $a \equiv b \pmod{n} \Leftrightarrow n \mid (a - b)$) is an equivalence relation on $\mathbb{Z}$.

Q8 Prove that the rational numbers and the natural numbers have the same cardinality. OR
Prove that the set of all sequences of 0's and 1's (*i.e.* the set of all functions from $\mathbb{N}$ to the set {0,1}) is uncountable