Math 330 MT 1 Solutions

Q1 a) The triple $(G, *, e)$ is a group if:

i) $\forall x, y, z \in G \quad (x * y) * z = x * (y * z)$ ($*$ is Associative)

ii) $\forall x \in G \quad e * x = x$ (e is a left identity)

iii) $\forall x \in G \quad \exists y \quad yx = e$ (existence of left inverses)

b) If $x \in G$ and $x^2 = x$ then $x = e$.

Proof Suppose $x^2 = x$

$\Rightarrow \quad xx = x$

by Axiom iii) $\exists y,\ yx = e$

$\Rightarrow$ So $\quad xx = x$

$\Rightarrow \quad y(xx) = yx$

$\Rightarrow \quad (yx)x = yx$ by Axiom i)

$\Rightarrow \quad ex = e$ by the choice of y

$\Rightarrow \quad x = e$ by axiom ii)

I c) i) Every left inverse is a right inverse.

<u>i.e.</u> $\forall x,y \in G \quad yx = e \Rightarrow xy = e.$

<u>Proof</u> Suppose $x,y \in G$ and $yx = e$

then
$$\begin{aligned}(xy)^2 &= (xy)(xy)\\ &= x(y(xy)) && \text{by axiom i)}\\ &= x((yx)y) && \text{by axiom i)}\\ &= x(ey) && \text{by assumption}\\ &= xy && \text{by axiom ii)}\end{aligned}$$

Hence $xy = e$ by part b)

ii) ~~Suppose~~ Inverses are unique: $\forall x,y,z \in G$, $yx = e$ & $zx = e \Rightarrow y = z$.

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<u>Proof</u>, Suppose $yx = e$ & $zx = e$

Then by part i) $xy = xz \; (= e)$

hence by the assumption that $yx = e$:

$$y(xy) = y(xz)$$
$$\Rightarrow (yx)y = (yx)z$$
$$\Rightarrow ey = ez$$
$$\Rightarrow y = z \quad \text{by axiom iii)}$$

Q2 Suppose G is a group in which $x \neq e \Rightarrow x$ has order 2. Show G is abelian, ~~Then~~ i.e. $\forall x, y \in G$, $xy = yx$.

Proof Suppose $\forall x \in G$, $x^2 = e$.

Then $\forall x, y \in G$,

$$(x*y)^2 = e, \quad x^2 = e, \quad y^2 = e$$

$$\Rightarrow \quad (xy)(xy) = e$$

$$\Rightarrow \quad x(xy)(xy) = x$$

$$\Rightarrow \quad x^2 yxy = x$$

$$\Rightarrow \quad yxy = x \qquad \text{since } x^2 = e$$

$$\Rightarrow \quad y(yxy) = yx$$

$$\Rightarrow \quad y^2(xy) = yx$$

$$\Rightarrow \quad exy = yx \qquad \text{since } y^2 = e$$

$$\Rightarrow \quad xy = yx \quad //$$

Q3 $\alpha \in S_8$ is the permutation

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 4 & 6 & 5 & 7 & 3 & 2 & 8 & 1 \end{bmatrix}$$

a) $\alpha = (1\,4\,7\,8)(2\,6)(3\,5)$

b) The order of a permutation is the lcm of the length of the cycles in a disjoint cycle decomposition of σ.
So order $(\alpha) =$ lcm $(4, 2, 2) = 4$.

c) The sign of α is $(-1)^{8-3} = -1$
(In general the order of a permutation in S_n is $(-1)^{n-r}$ where r is the no of cycles in a complete cycle decomposition).

Q4 Suppose G is a group with an even nº of elements. Prove that g has an element of order 2.

<u>Proof</u> Suppose $\#(G) = 2n$ for $n \in \mathbb{N}$.

We can write G as the disjoint union of three sets:

$$G = \{e\} \cup A \cup B$$

where $A = \{x \in G \mid \text{order}(x) = 2\}$
$B = \{x \in G \mid \text{order}(x) > 2\}$.

<u>Claim</u> B has an even nº of elements

<u>Proof of claim.</u> $x \in B \iff x \neq x^{-1}$. Furthermore, if $y = x^{-1}$, then $x = y^{-1}$, so if $x \in B$ then $x^{-1} \in B$ and $x \neq x^{-1}$. So B is the disjoint union of subsets of the form $\{x, y\}$ with $y = x^{-1}$ & $y \neq x$.
Hence B has an even nº of elements. //

It follows that $\{e\} \cup A$ has an even $2k$ of elements, and $2k > 1$. Hence A has $2k-1 > 0$ elements, and so G has an element of order 2.