

## Math 330 MT 1 Solutions

Q1 a) The triple  $(G, *, e)$  is a group if:

i)  $\forall x, y, z \in G \quad (x * y) * z = x * (y * z)$   
( $*$  is Associative)

ii)  $\forall x \in G \quad e * x = x$  (e is a left identity)

iii)  $\forall x \in G \quad \exists y \quad y * x = e$  (existence of left inverses)

b) If  $x \in G$  and  $x^2 = x$  then  $x = e$ .

Proof Suppose  $x^2 = x$

$$\Rightarrow xx = x$$

by Axiom iii)  $\exists y, yx = e$

$$\Rightarrow \text{so } xx = x$$

$$\Rightarrow y(xx) = yx$$

$$\Rightarrow (yx)x = yx$$

$$\Rightarrow ex = e$$

$$\Rightarrow x = e$$

by Axiom i)

by the choice of  $y$

by axiom ii)

1 (i) Every left inverse is a right inverse.

i.e.

$$\forall x, y \in G \quad yx = e \Rightarrow xy = e.$$

Proof Suppose  $x, y \in G$  and  $yx = e$

then

$$\begin{aligned}(xy)^2 &= (xy)(xy) \\ &= x(y(xy)) \text{ by axiom i)} \\ &= x(yx)y \text{ by axiom i)} \\ &= x(ey) \text{ by assumption} \\ &= xy \text{ by axiom ii)}\end{aligned}$$

Hence  $xy = e$  by part b)

ii) ~~Suppose~~ Inverses are unique:  $\forall x, y, z \in G, yx = e$   
&  $zx = e$   
 $\Rightarrow y = z.$   
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Proof, Suppose  $yx = e$  &  $zx = e$

Then

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by part i)  $xy = xz (= e)$

hence by the assumption that  $yx = e$ :

$$y(xy) = y(xz)$$

$$\Rightarrow (yx)y = (yx)z$$

$$\Rightarrow ey = ez$$

$\Rightarrow y = z$  by axiom iii)

~~by assumption~~  
~~by part i)~~

Q2 Suppose  $G$  is a group in which  
 $x \neq e \Rightarrow x$  has order 2. Show  $G$  is abelian,  
Then i.e.  $\forall x, y \in G, xy = yx$ .

Proof Suppose  $\forall x \in G, x^2 = e$ .

Then  $\forall x, y \in G,$   
 $(x^2 y)^2 = e, x^2 = e, y^2 = e$

$$\Rightarrow (xy)(xy) = e$$

$$\Rightarrow x(xy)(xy) = x$$

$$\Rightarrow x^2 yxy = x$$

$$\Rightarrow yxy = x$$

since  $x^2 = e$

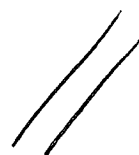
$$\Rightarrow y(yxy) = yx$$

$$\Rightarrow y^2(xy) = yx$$

$$\Rightarrow exy = yx$$

since  $y^2 = e$

$$\Rightarrow xy = yx$$



Q3

$\alpha \in S_8$  is the permutation

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 4 & 6 & 5 & 7 & 3 & 2 & 8 & 1 \end{bmatrix}$$

a)  $\alpha = (1478)(26)(35)$

b) The order of a permutation is the lcm of the length of the cycles in a disjoint cycle decomposition of  $\sigma$ .

So  $\text{order}(\alpha) = \text{lcm}(4, 2, 2) = 4$ .

c) The sign of  $\alpha$  is  $(-1)^{8-3} = -1$

(In general the order of a permutation in  $S_n$  is  $(-1)^{n-r}$  where  $r$  is the no of cycles in a complete cycle decomposition).

Q4 Suppose  $G$  is a group with an even no of elements. Prove that  $G$  has an element of order 2.

Proof Suppose  $\#(G) = 2n$  for  $n \in \mathbb{N}$ .

We can write  $G$  as the disjoint union of three sets:

$$G = \{e\} \cup A \cup B$$

$$\text{where } A = \{x \in G \mid \text{order}(x) = 2\}$$

$$B = \{x \in G \mid \text{order}(x) > 2\}.$$

Claim  $B$  has an even no of elements

Proof of claim.  $x \in B \iff x \neq x^{-1}$ . Furthermore,

if  $y = x^{-1}$ , then  $x = y^{-1}$ , so if  $x \in B$  then  $x^{-1} \in B$

and  $x \neq x^{-1}$ . So  $B$  is the disjoint union of subsets of the form  $\{x, y\}$  with  $y = x^{-1}$  &  $y \neq x$ .

Hence  $B$  has an even no of elements. //

It follows that  $\{e\} \cup A$  has an even ~~no~~  $2k$  of elements, and  $2k > 1$ . Hence  $A$  has  $2k-1 > 0$

elements, and so  $G$  has an element of order 2.