Math 330 Questions

Q1

(a) Let σ be a permutation of order 12 of a finite set X. Suppose that σ can be written as a product of disjoint cycles, no two of which have the same length. What are the possible lengths of the cycles in this product?

(b) Suppose that σ is a permutation of order p, with p prime, of a finite set X. Let $X^\sigma := \{x \in X \mid \sigma(x) = x\}$. Prove that p divides the order of the complement $X \setminus X^\sigma$.

(c) If G is a group, and $\sigma : G \to G$ is an automorphism of G, prove that G^σ is a subgroup of G.

(d) Suppose that G is a group with $|G| = 27$, prove that it **cannot** have an automorphism $\sigma : G \to G$ of order 5.

Q2

(a) Prove that the intersection $H \cap K$ of two subgroups H, K of a group G is a subgroup of G.

(b) Given an example to show that the union $H \cup K$ need not be a subgroup.

(c) **Extra Credit:** Show that if H and K are proper subgroups, (*i.e.* not equal to G), then their union cannot be equal to G.

Q3

Show that H is a subgroup of S_n, the either **all** the elements of H are even permutations, or **exactly half** the members are even.

Q4

(a) Define what a *group homomorphism* is.

(b) Define the *kernel* of a groups homomorphism

(c) Define *Normal subgroup.*

(d) Prove that the kernel of a group homomorphism is a normal subgroup.

(e) Prove that the cosets G/N form a group with respect to multiplication, if $N \lhd G$ is a normal subgroup.

Q5

(a) Define *integral domain.*

(b) Define *field.*

(c) Prove that any finite integral domain is a field.

Q6 Is the homomorphic image of a principal ideal domain (PID) also a PID?

Q7 Show that $\mathbb{Z}[i]/ < 1 + i >$ is a field. How many elements does it have?