Math 330: Abstract Algebra
Sample Final Exam

Look also at the problems on the midterm and sample midterm

1) Define the following concepts:
a) H is a normal subgroup of G and K is the factor group G/H.
b) $I \subseteq R$ is a maximal ideal;
c) $a \in R$ is irreducible.
d) $\phi : R \to S$ is a ring homomorphism.

2) Decide if the following statements are TRUE or FALSE. If FALSE give a counterexample or an explanation why the statement is false.
a) If R is a commutative ring and $I \subseteq R$ is an ideal, then R/I is commutative.
b) If G is a finite Abelian group and n divides $|G|$, then G has an element of order n.
c) If F is a field, then $F[X,Y]$ is a principal ideal domain.
d) There is a field F with $|F| = 81$.
e) If G is a group, H is a subgroup of G and H is Abelian, then H is a normal subgroup of G.
f) If R is a ring and $ab = ac$, then $b = c$.
g) If F is a field, $I \subseteq F$ is an ideal, $a \in F \setminus \{0\}$ and $a \in I$, then $I = F$.

3) State the following theorem:
a) LaGrange's Theorem
b) The Fundamental Theorem of Field Theory

4) a) Find all Abelian groups (up to isomorphism) of order 16.
b) Which of these groups is isomorphic to $\mathbb{Z}_4 \times \mathbb{Z}_{24}/\langle(2,4)\rangle$?

5) a) Show that $p(X) = X^3 + X + 1$ is irreducible in $\mathbb{Z}_2[X]$.
b) Suppose F is an exension field of $\mathbb{Z}_2$, $\alpha \in F$ and $p(\alpha) = 0$. Find $a, b, c \in \mathbb{Z}_2$ such that

$$a\alpha^2 + b\alpha + c = (\alpha^2 + 1)(\alpha + 1).$$

6) Find a noncylic subgroup of order 4 in $\mathbb{Z}_4 \oplus \mathbb{Z}_{10}$.

7) Show that the homomorphic image of a principal ideal domain is a prinipal ideal domain.

8) Suppose R is a commutative ring and $I \subset R$ an ideal. Let

$$\sqrt{I} = \{a \in R : a^n \in I\}$$

for some n.
a) Show that $\sqrt{I}$ is an ideal.
b) Show that the factor ring $R/\sqrt{I}$ has no nonzero nilpotent elements (recall that $a \in R$ is *nilpotent* if $a^n = 0$ for some $n = 1, 2, \ldots$).