Math 330 Mid Term, March 3 2006

Q1 Let G be a non empty set, together with a binary operation $\cdot$, which is to be written as multiplication $(a, b) \mapsto ab$, for $a, b \in G$.

a. What three properties of the pair $(G, \cdot)$ must be satisfied for it to be a group?

b. Prove that for each element a in a group G, there is a *unique* element b such that $ab = ba = e$.

Q2 State the following theorems:

a. The Orbit-Stabilizer theorem.

b. Lagrange's theorem.

Q3 Let $\alpha \in S_8$ be the permutation

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 4 & 6 & 5 & 7 & 3 & 2 & 8 & 1 \end{bmatrix}$$

a. What is the cycle decomposition of α?

b. What is $|\alpha|$?

Q4 Prove that any group of even order has an element of order 2.

Q5 Let G be a group.

a. Define the center $Z(G)$ of G.

b. Prove that $Z(G)$ is a subgroup of G

Q6 Prove that if m and n are both positive integers, then $\mathbb{Z}_m \times \mathbb{Z}_n$ is cyclic if and only if $\gcd(m, n) = 1$