LINE BUNDLES ON STACKS

HENRI GILLET

Vector Bundles

We begin by defining $\mathcal{V}_n$, the stack of rank n bundles. We assign to every scheme a groupoid $\mathcal{V}_n(S)$, the objects of which are the rank n bundles on S, and the morphisms of which are the *isomorphisms* between bundles. Given a morphism $f : S \to T$ of schemes, we have the usual pullback of vector bundles, corresponding to the Cartesian diagram:

$$\begin{tikzcd} f^*F \arrow[r] \arrow[d] & F \arrow[d, "p"] \\ S \arrow[r, "f"] & T \end{tikzcd}$$

This defines the base change functor $f^* : \mathcal{V}_n(S) \to \mathcal{V}_n(T)$.

Observe that a bundle over a scheme S determines a morphism of stacks $S \to \mathcal{V}_n$. This motivates the following definition:

Definition. A *rank n bundle* over a stack $\mathcal{X}$ is a morphism of stacks

$$\mathcal{X} \to \mathcal{V}_n.$$

For example, consider $\mathcal{X} = \mathcal{M}_{1,1}$, the stack of families of elliptic curves. The objects of $\mathcal{M}_{1,1}$ are the families $E \xrightarrow{p} S$ for which the fibers E_s are elliptic curves. A morphism from $\mathcal{M}_{1,1} \to \mathcal{V}_n$ means that for every scheme S, we have a morphism of groupoids $\mathcal{M}_{1,1}(S) \to \mathcal{V}_n(S)$, in other words, it is a rule which associates to a family of elliptic curves $E \to S$ a vector bundle $V(E) \to S$, and these morphisms should be compatible with base-change.

If $S = \operatorname{Spec} K$ is a point, it associates to a single elliptic curve a K-vector space of dimension n. This association must be functorial: an isomorphism between elliptic curves should give an isomorphism on vector bundles, it should behave well under pullbacks, i.e. given

$$\begin{tikzcd} f^*E \arrow[r] \arrow[d] & E \arrow[d] \\ T \arrow[r, "f"] & S \end{tikzcd}$$

Date: October 2, 2002.

Based on an excellent set of Lateχ notes taken by John Voight at the "Learning Stacks and Computational Methods through Problem Solving Workshop" at the University of Illinois, Urbana-Champaign, June 12–15, 2002.

we must have $V(f^*E) = f^*(V(E))$.

Example. Such an assignment compatible with these assignments takes $E \mapsto H^0(E, \Omega^1)$ on points, which gives us a 1-dimensional K-vector space. On the family $E \xrightarrow{p} S$, we associate $p_*\omega^1_{E/S}$, a rank 1 vector bundle over S.

What about $\mathcal{X} = \mathcal{V}_n$; what does it mean to give a vector bundle E on $\mathcal{V}_n$? We must give a morphism $\mathcal{V}_n \to \mathcal{V}_m$, which means in particular we must associate to every n-dimensional vector space an m-dimensional vector space in a functorial and "continuous" fashion, as above. Even the first property is fairly restrictive: given a morphism of stacks $\Phi : \mathcal{V}_n \to \mathcal{V}_m$, for every field K we obtain a functor between the groupoids $\mathcal{V}_n(K) \xrightarrow{\Phi_K} \mathcal{V}_m(K)$. Now, since any two n-dimensional K-vector spaces are isomorphic, the groupoid $\mathcal{V}_n(K)$is connected - *i.e.* the all n-dimensional vector spaces are isomorphic to K^n, and the groupoid is equivalent to the category with the single object K^n and morphisms $Gl_n(K)$. Therefore a functor $\mathcal{V}_n(K) \to \mathcal{V}_m(K)$ is determined, up to equivalence, by a vector space V associated to K^n, and a homomorphism $Gl_n(K) \to Aut(V)$. Thus, if we fix attention on the free module $K^n \mapsto \Phi(K^n)$, the image some m-dimensional vector space, we obtain a non-canonical isomorphism $GL_n(K) \to \mathrm{Aut}(\Phi(K^n)) \sim GL_m(K)$, i.e. we obtain a representation of $GL_n(K)$. If we insist that this association be functorial and behave well in families, then one can show that a general Φ corresponds to a representation of the group scheme GL_n over $\operatorname{Spec}\mathbb{Z}$. In other words, the representation is given by polynomial equations in the coefficients of the matrix with integer coefficients.

As examples, we have exterior powers, symmetric powers, etc. In particular, to any n-dimensional vector bundle $E \to S$, we can associate $\bigwedge^n E \to S$, a line bundle.

Recall that if G is a discrete group then BG/S is the stack of sheaves of G-torsors in the étale topology over some fixed base S. If we consider the torsors which are constant - *i.e.* which are just G-sets, then the functor associates to every G-set X a vector bundle E_X on S, and to every isomorphism of G-sets an isomorphism of bundles. Since the groupoid of G-sets is connected this is equivalent to giving a bundle E_G and an action of G on this bundle - *i.e.* to a representation of G. If BG is over an affine $S = \operatorname{Spec} R$, then this representation is into $\mathrm{Aut}(V)$ for some free module V over R.

In fact this determines the functor on the whole of BG. Since this is a morphism of stacks, it respects the sheaf property of the stacks involved. Thus given E_G as above, and a sheaf T of G-sets which is only locally isomorphic to G, we get a bundle E_T, which is locally isomorphic to E_X.

Vector Bundles and Charts

If you open any book on differential topology, a vector bundle is described, given a covering of a manifold by open charts, by giving, for each open chart, a trivialization of the restriction of the bundle to the chart, such that on the overlap of two charts there is a continuous map to GL_n which gives the transition matrices between the two trivialization. This may seem to be absent from the description above, but this is only because we need to rethink what we mean by charts.

Let $\mathcal{X}$ be a stack. Although there are many different ways of understanding charts, with corresponding assumptions on them, let us assume that there exists a "faithfully flat" representable morphism $\varepsilon : X_0 \to \mathcal{X}$ with X_0 a scheme. To say that ε is representable means that for any scheme S, in the fiber product diagram of stacks,

$$\begin{tikzcd} X_0 \times_{\mathcal{X}} S \arrow[r, "\varepsilon_S"] \arrow[d] & S \arrow[d] \\ X_0 \arrow[r, "\varepsilon"] & \mathcal{X} \end{tikzcd}$$

the stack $X_0 \times_{\mathcal{X}} S$ is a scheme. To be faithfully flat means that it as a map of schemes, ε_S is always flat and surjective.

In particular, we may take $S = X_0$, and we obtain $X_1 = X_0 \times_{\mathcal{X}} X_0 \to X_0$ with two maps to X_0, and a diagonal map $X_0 \xrightarrow{\Delta} X_1$. This gives a "groupoid presentation" of $\mathcal{X}$.

$$\begin{tikzcd} X_0 \arrow[dr, "\Delta"] & & \\ & X_1 = X_0 \times_{\mathcal{X}} X_0 \arrow[r, "t"] \arrow[d, "s"] & X_0 \arrow[d] \\ & X_0 \arrow[r, "\varepsilon"] & \mathcal{X} \end{tikzcd}$$

Observe that if $E_0 \to X_0$ is a vector bundle, we obtain an isomorphism $s^*E_0 \simeq t^*E_0$, since it doesn't matter which way around the diagram we go, and this isomorphism is a "transition function".

For instance, if you have a vector bundle $F \to M$ over a manifold, then given charts in the usual sense, and local trivializations of the bundle $U_\alpha \xrightarrow{\phi_\alpha} M$, we obtain

$$\begin{tikzcd} (\coprod U_\alpha \cap U_\beta) \times \mathbb{C}^n \arrow[r, "g_{\alpha\beta}"] \arrow[d] & \coprod U_\alpha \times \mathbb{C}^n \arrow[r] \arrow[d] & F \arrow[d] \\ \coprod U_\alpha \cap U_\beta \arrow[r] & \coprod U_\alpha \arrow[r, "\varepsilon"] & M \end{tikzcd}$$

Returning to the example of the finite group G, if we view a point as a trivial G-torsor (viewing G itself as a G-torsor), then the fiber product is

$$\begin{tikzcd} G \arrow[r, "t"] \arrow[d, "s"] & \{\cdot\} \arrow[d] \\ \{\cdot\} \arrow[r] & BG \end{tikzcd}$$

since the only isomorphisms of G as a G-torsor are multiplications by elements on G. The map $s^*E \xrightarrow{\theta_g} t^*E$ obtained above must then satisfy $\theta_g\theta_h = \theta_{gh}$, which encodes the information about the representation of G discussed earlier.

Picard Group of $\mathcal{M}_{1,1}$

Given two line bundles on a stack, one can take the tensor product by taking it on the line bundles over the schemes themselves. The Picard group corresponds to isomorphism classes of these line bundles with this group operation.

¿From now on, lets work over a field of characteristic not 2 or 3. Any elliptic curve over this ring can be embedded with an affine Weierstrass equation $y^2 = x^3 + Ax + B$. Two elliptic curves are isomorphic (over $\mathbb{C}$, say) if and only if their j-invariants are equal, where

$$j(E) = 1728\frac{(4A)^3}{16(4A^3 + 27B^2)}.$$

The only nontrivial automorphism for a general curve is sending $y \mapsto -y$, of order 2; for j-invariants $j = 0$ one has the automorphism $(x, y) \mapsto (-x, iy)$ and for $j = 1728$ one has the automorphism $(x, y) \mapsto (\omega x, -y)$ of order 6, where ω is a primitive cube root of unity.

¿From these two elliptic curves with non-trivial automorphisms, we obtain maps $B\mu_6 \to \mathcal{M}_{1,1}$ and $B\mu_4 \to \mathcal{M}_{1,1}$. Given a line bundle $\mathcal{L}$ on $\mathcal{M}_{1,1}$, we can pull these back to line bundles on $B\mu_6$ and $B\mu_4$ (recall that μ_n is the group of n-th roots of 1), getting one dimensional representations of these two groups. But a one-dimensional representation of a group G over a ring R, is nothing more than a homomorphism from the group to R^*, indeed for G finite of order n to the group μ_n of n-th roots of 1. Hence every line bundle over $B\mu_6$ determines a homomorphism $\mu_6 \to \mu_6$, *i.e.* an element of $\mathbb{Z}/6\mathbb{Z}$. Thus we get a homomorphism $\mathrm{Pic}(\mathcal{M}_{1,1}) \to \mathbb{Z}/6\mathbb{Z}$ corresponding to resrtiction to j-invariant 1728, and similarily, for j-invariant 0, we get a homomorphism $\mathrm{Pic}(\mathcal{M}_{1,1}) \to \mathbb{Z}/4\mathbb{Z}$, combining these (and composing with the map $\mathbb{Z}/6\mathbb{Z} \to \mathbb{Z}/63\mathbb{Z}$), we get map $\mathrm{Pic}(\mathcal{M}_{1,1}) \to \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z} \simeq \mathbb{Z}/12\mathbb{Z}$. In fact, one can show that this is an isomorphism as follows.

Consider the line bundle $E \to H^0(E, \Omega^1)$. This really has order 12, which one can see by writing down the generator dx/y and checking how the roots of unity act on it: hence the above map is surjective. To show that it is injective, one can show that for any line bundle $\mathcal{L}$ on $\mathrm{Pic}(\mathcal{M}_{1,1})$, we have $\mathcal{L}^{\otimes 12}$ trivial since we can pull it back from the j-line $\mathbb{A}^1$, and there are no nontrivial line bundles on $\mathbb{A}^1$. This completes the proof.

In the usual construction of the coarse moduli space of elliptic curves, recall that an elliptic curve corresponds to an equivalence classes of lattices (modulo the action of $SL_2(\mathbb{Z})$). The set of such equivalence classes is the quotient $\mathfrak{h}/SL_2(\mathbb{Z})$ of the upper half plane by the action of $SL_2(\mathbb{Z})$ where we glue together the boundaries of the classical fundamental domain to obtain a sphere with one point removed (corresponding to $j = \infty$, a singular curve), and where there are two points associated to the elliptic curves with groups of automorphisms of order 4 and 6. This is an orbifold representation of the moduli space.

To conclude, do the following exercise: explain why the 12 in $\zeta(-1) = -1/12$, is the same 12 of that of the Picard group $\mathrm{Pic}(\mathcal{M}_{1,1})$. Hint - it has to do with $\mathfrak{h}/SL_2(\mathbb{Z})$.

The discussion above is based, in large part, on the paper by David Mumford, entitled "Picard groups of moduli problems". Which appears in pp. 33–81 of the 1965 book *Arithmetical Algebraic Geometry* published by Harper & Row, New York (MR 34 #1327).