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SINGULARITIES ARE DETERMINED BY THE COHOMOLOGY OF THEIR COTANGENT COMPLEXES

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0. The cohomology of the cotangent complex characterizes analytic singularities (resp. analytic mapping germs) up to isomorphism. The present paper generalizes a result by J.N.Mather and S.S.T. Yau [6] that says that isolated hypersurface singularities are isomorphic iff their moduli algebras are isomorphic. This result turns out to be a special case of a general principle. [3] and [4] contain preliminary versions of the mentioned result. where it was shown e.g. for the $\mathcal{R}$-equivalence of functions and for the $\mathcal{K}$-equivalence of complete intersections that the isomorphism type of the module of infinitesimal deformations of first order $T^1(X_o,o)$ determines the singularity (X_o,o) up to isomorphism.

0.1. If the deformations of (X_o,o) are non-obstructed. then the above statement is always correct. Otherwise, the higher cohomology groups or all "infinitesimal families" of first order of (X_o,o) have to be taken into consideration.
We proceed as follows: We choose $n,\ p \in \mathbb{N}$ such that the singularity (X_o,o) can be embedded into $(\mathbb{A}^n,o)$ and is defined by p equations $\underline{f} \in J(n,p) := \underline{m}\ \mathbb{C}\{x_1,\dots,x_n\}^p$. The set of all possible representations of (X_o,o) in $J(n,p)$ corresponds to an orbit of the contact group $\mathcal{K}(n,p)$ in $J(n,p)$. In addition. we fix the integers $p_2,\dots,p_s \in \mathbb{N}$ such that the equations $\underline{f}$ have at most p_i relations of (i-1)-st degree. Now the generating relations can be represented by an element of a larger space $J(n,p_1,\dots,p_s)$,

and all these representations form an orbit of a generalized contact group $\mathcal{K}(n,p)$. To each of those elements of $J(n,p)$ a cotangent complex is assigned. The action of $\mathcal{K}(n,p)$ on $J(n,p)$ induces isomorphisms of the assigned cotangent complexes and their cohomology.

We have the "Mather-Yau equivalence":

> If $(n,p_1,\dots,p_s)$ is admissible with respect to (X_o,o), then the following statements are equivalent:
>
> (i) (X_o,o) is isomorphic to (X'_o,o).
>
> (ii) For $i\geq 1$ $T^i(X_o,o)$ is isomorphic to $T^i(X'_o,o)$ as $\mathbb{C}\{X_1,\dots,X_n\}$-module, where the isomorphism is induced by the action of $\mathcal{K}(n,p)$.
>
> (iii) $T^1_{(1)}(X_o,o)$ is isomorphic to $T^1_{(1)}(X'_o,o)$ as $\mathbb{C}\{(X_1,\dots,X_n\}$-module.

(Here $T^{\cdot}_{(r)}$ denotes the cohomology of the restricted cotangent complex).

0.2. Similar statements hold both for singularities over a fixed basis (S,o) (this generalizes the right-equivalence) and for mapping germs (this generalizes the right-left-equivalence). The proof involves transcendental methods. Simple examples show (cf.[6]) that the Mather-Yau equivalence fails over non-algebraically closed fields or over fields of characteristic $p\neq 0$.

0.3. The concept of infinitesimally trivial families will be an essential tool in proving the main theorem. We shall show that these families are trivial. For hypersurface singularities this was shown by K. Saito [8]. The statement is an analogue to the fact that the triviality of the Kodaira-Spencer map implies the local triviality of a deformation.

1. THE GENERALIZED CONTACT GROUP

1.0. Let $\mathcal{C}$ be the category of analytic germs of finite type over $\mathbb{C}$. A <u>singularity</u> over a base germ $S \in \mathrm{Ob}\,\mathcal{C}$ is a morphism $\varphi_o : X_o \longrightarrow S$. A <u>family of φ_o over $T \in \mathrm{Ob}\,\mathcal{C}$</u> is a pair consisting of a T-morphism $\varphi: X \longrightarrow S \times T$ (over T) and a fixed isomorphism

$$\begin{array}{ccc} X & \longrightarrow & S \times T \\ & \searrow \; T \; \swarrow & \end{array}$$

$X \times_T \{t_o\} \cong X_o$. If $X \longrightarrow T$ is flat, φ is called a <u>deformation of φ_o</u>. Now we describe the set of all possible coordinate representations of φ and the associated syzygies as an orbit of a generalized contact group. Together with S we fix a minimal embedding:

$S \hookrightarrow \mathbb{A}^q$, $O_S = \mathbb{C}\{U_1, \ldots, U_q\} / I(S) = \mathbb{C}\{u_1, \ldots, u_q\}$, $I(S) \subseteq (\underline{U})^2$.

Let $J_T(n,p) := (\underline{X})\, O_T\{X_1, \ldots, X_n\}^p$.

1.1. A <u>representative</u> of φ in $J_T(n,q+p)$ is a tuple

$F^* = (H_1, \ldots, H_q, F_1, \ldots, F_p) \in J_T(n,p+q)$, such that

$O_X \cong O_T\{\underline{X}\}/(F_1, \ldots, F_p) \cong O_{S \times T}\{\underline{X}\}/(u_1 - H_1, \ldots, u_q - H_q, \underline{F})$

and $\varphi^*(u_j) = H_j \mod (\underline{F}), j = 1, \ldots, q$.

This implies:

$G(\underline{U}) \in I(S) \Longrightarrow G(\underline{H}) \in (\underline{F})$,

$(\underline{H}, \underline{F})\big|_{T=t_o} = (\underline{h}, \underline{f}) \in J(n,q+p)$ represents φ_o .

<u>**Remark:**</u> φ_o has a representative in $J(n,p+q)$ iff

$n \geq \mathrm{emb\,dim}\,(X_o) = \dim_{\mathbb{C}}(\underline{m}\, O_{X_o} / \underline{m}^2)$ and

$p \geq i(X_o) := \dim_{\mathbb{C}}(I(X_o)/\underline{m}\, I(X_o))$ for a minimal embedding

$O_{X_o} = \mathbb{C}\{\underline{X}\} / I(X_o)$, $I(X_o) \subseteq \underline{m}^2$.

1.2. A resolution of the $O_{S \times T}$-algebra O_X by free $O_{S \times T}\{\underline{X}\}$-modules $0 \longrightarrow M_s \xrightarrow{l_s} \ldots \xrightarrow{l_1} M_o = O_{S \times T}\{\underline{X}\} \xrightarrow{l_o} O_X$ (1)

be representented as follows:

Let $\operatorname{rk} M_i = p_i$, $(p_o = 1)$, $\{e_1^{(i)},\ldots,e_{p_i}^{(i)}\}$ be a basis of M_i:

$$l_i(e_j^{(i)}) = \sum F_{jk}^{(i)} e_k^{(i-1)} \qquad (i > 1) \tag{1a}$$

and, without loss of generality, we choose only representations (1) where $\underline{F}^{(1)}$ has the form:

$$F_j^{(1)} = \begin{cases} U_j - H_j & j \leq q \\ F_{j-q} & \text{otherwise} \end{cases} \tag{1b}$$

and $\underline{F}^* = (\underline{H},\underline{F}) \in J_T(n,p_1)$ is a representative of φ .

Then (1) is completely described by

$$\underline{F}^{\cdot} = (\underline{F}^*,\underline{F}^{(2)},\ldots,\underline{F}^{(s)}) \in \underline{m}\left[O_T\{\underline{X}\}^{p_1} \oplus O_{S\times T}\{\underline{X}\}^{p_2p_1} \cdots \oplus O_{S\times T}\{X\}^{p_sp_{s-1}}\right]$$

$$=: J_T(n,p_1,\ldots,p_s),\ \underline{m} = \underline{m}_T \cdot (\underline{X}) \ .$$

By construction we have:

(i) $\underline{F}^{(r)} \cdot \underline{F}^{(r-1)} = 0$, $r = 2,\ldots,s$; $\underline{F}^{(r)}$ is a $p_r \times p_{r-1}$-matrix;

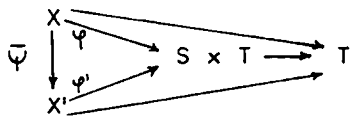
(ii) $F_1^{(r)},\ldots,F_{p_r}^{(r)}$ generate the (r-1)st syzygy module of $\underline{F}^{(1)}$.

(iii) $pr \geq \binom{p_1}{r-1}$.

Remark: 1.) Let n_r be the rank of the (r-1)-st syzygy module of a minimal representative of φ_o, then:
There is a representative of the described type for each resolution (1) of O_{X_o} in $J(n,p_1,\ldots,p_s)$ iff $n \geq \operatorname{emb} \dim X_o$, $p_1 \geq q + i(X_o) =: n_1$ and $p_i \geq n_i$, $(i \geq 2)$.
Such a tuple is called $(n,q,p_1,\ldots,p_s)$ is called admissible with respect to φ_o.

2.) If $n_r = \binom{n_1}{r-1}$ for $r \geq 1$, then φ_o is called non-obstructed. This is the case iff O_{X_o} is a complete intersection and O_S is smooth.

1.3. Let two families φ and φ' of φ_o over T, and two representatives $\underline{F}^{\cdot}$ and $\underline{F}'$ of resolutions of the same type $(n,q,\dot{p})$ be given. A T-isomorphism $\bar{\Psi}$ from φ to φ':



can be lifted to a T-isomorphism Ψ of $T \times \mathbb{A}^n$ such that

$$\Psi^*(F'_1,\ldots,F'_s) = (F_1,\ldots,F_s) \qquad (2)$$

and

$$\Psi^*(H_j') = H_j \mod (F_1,\ldots,F_s) \quad .$$

It is possible to extend Ψ^* to an isomorphism of the free resolutions:

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & M_s & \xrightarrow{l_s} & \ldots & \longrightarrow & M_1 & \longrightarrow & O_{S\times T}\{\underline{X}\} & \longrightarrow & O_X \\
 & & A_s\uparrow & & & & \uparrow A_1 & & \uparrow\Psi^* & & \uparrow\Psi^* \\
0 & \longrightarrow & M_s & \xrightarrow{l'_s} & \ldots & \longrightarrow & M_1 & \longrightarrow & O_{S\times T}\{\underline{X}\} & \longrightarrow & O_{X'}
\end{array} \quad .$$

We identify A_r with the associated matrix

$$(a_{ij}^{(r)}) \in Gl_{p_r}(O_{S\times T}\{\underline{X}\}) =: G_{p_r} \quad ,$$

$$A_r(e_i^{(r)}) = \sum_j a_{ij}^{(r)} e_j^{(r)} \quad , \qquad r = 2,\ldots,s$$

and, because of (2), A_1 belongs to

$$\tilde{G}_{p_1}(q) = \left\{ \left(\begin{array}{c|c} E_q & \ddots \\ \hline 0 & \ddots \end{array} \right) \right\} \subset Gl_{p_1}(O_T\{X\}) \quad .$$

We obtain:

$$\left.\begin{array}{ll} F^{*} = A_1^{-1} \; \underline{F}'^{*}(\psi^{*}) & \text{(written as column vector)} \\ \underline{F}^{(r)} = A_r^{-1} \; \underline{F}'^{(r)}(\psi^{*}) \cdot A_{r-1} \,, & r \geq 2 \end{array}\right\} \qquad (3)$$

1.4. The <u>generalized contact group</u> $\mathcal{K}_T = \mathcal{K}_T(n,q,p_1,\ldots,p_s)$ is defined as the semi-direct product of the groups

$$\mathrm{Aut}_T(T \times \mathbb{A}^n) \times \widetilde{G}_{p_1}(q) \times G_{p_2} \times \ldots \times G_{p_s}.$$

$\mathcal{K}_T$ is acting on $J_T(n,\underline{p})$ according to the formulas (3).

Remark: 1.3. implies that the set of all representations of resolutions of φ of type $(n,q,p_1,\ldots,p_s)$ is a $\mathcal{K}_T$-orbit in $J_T(n,p)$.

Example: With one exception we consider only the "absolute" contact group $\mathcal{K} = \mathcal{K}_{\{t_o\}}$. For s = 1 $\mathcal{K}\,(n,q,p)$ contains a lot of known groups:

(i) $\mathcal{K}\,(n,q,q) = R(n)$ is the group of right-equivalences on $J(n,q)$.

(ii) $\mathcal{K}(n,o,p)$ is a subgroup of $\mathcal{K}(n,p)$, the contact group in the sense of J.N. Mather [5], but the orbits of both groups coincide in $J(n,p)$.

(iii) Let $S = \mathbb{C}^1$, $q = 1$: $\mathcal{K}(n,1,p)$ defines a generalized right-equivalence of functions on singularities. For hypersurfaces this equivalence was introduced by A. Dimca in [1], where the simple $\mathcal{K}(n,1,2)$-orbits have been classified as well.

2. INFINITESIMAL DEFORMATIONS

Infinitesimal deformations of first order are classified by the first cohomology group of the cotangent complex. Now we give a short description of Palamodov's construction of the Tjurina-resolvent [7]:

<u>2.1. The Tjurina-resolvent:</u>

Let $\varphi_o^* : 0_S \longrightarrow 0_{X_o}$ be a local homomorphism of analytic algebras. (R,l) is called a Tjurina-resolvent of φ_o^* if the following conditions are satisfied:

(i) $R = R_o \lceil v_1,\dots,v_n \rceil$ is a free graded anticommutative R_o-algebra, $\deg v_i = |v_i| < 0$, $v_i \cdot v_j = (-1)^{|v_i||v_j|} v_j v_i$.

(ii) $R_o = 0_S \{X_1,\dots,X_n\}$ is a free 0_S-algebra and

$$\begin{array}{ccccccccc} \dots R_k & \xrightarrow{l_k} & R_{k+1} & \longrightarrow & \dots & \longrightarrow & R_o & \xrightarrow{l_o} & 0_{X_o} \longrightarrow 0 \\ & & & & & & \uparrow & \nearrow \varphi_o^* & \\ & & & & & & 0_S & & \end{array}$$

is a resolution of 0_{X_o} by free R_o-modules.

(iii) $l : R \longrightarrow R$ is a derivation of degree 1, i.e. $l(a \cdot b) = l(a)b + (-1)^{|a|} a l(b)$.

The Tjurina-resolvent is uniquely determined up to homotopy. R has at least $n_r - \binom{n_1}{r-1}$ generators of degree $-r$, $r > 1$, and n_1 generators of degree -1, respectively, (cf. 1.2.). To each representative $F^\cdot$ of a resolution (1) of φ_o^* a Tjurina-resovent is assigned:

$$R := 0_S\{X_1,\dots,X_n\} \lceil v_1,\dots,v_N \rceil \ , \quad N = \sum_{i=1}^{s} p_i p_{i-1} \quad .$$

The v_i's are associated with the generators of M_r and get the degree $-r$, $r = 1,\dots,s$. Due to (1a) and (1b) l is defined on the generators by the corresponding map l of the $e^{(r)}$. The resolvent (R,l) constructed in this way is not minimal.

<u>**Example:**</u> Let $\varphi_o^* : \mathbb{C}\{u_1,\dots,u_q\} \longrightarrow 0_{X_o}$, $0_{X_o} = \mathbb{C}\{X_1,\dots,X_n\} / (f_1,\dots,f_p)$ be a complete intersection, $n = \text{emb dim } X_o$, $p = i(X_o)$. Then φ_o^* is non-obstructed and a minimal resolvent is obtained by:

$$R = \mathbb{C}\{u,X\} \lceil v_1,\dots,v_{p+q} \rceil \ , \ \deg v_i = -1 \quad ,$$

$$I(v_i) = f_i^{(1)} = \begin{cases} U_i - h_i \ , & i = 1,\dots,q;\ h_i \text{ a representative of } \varphi_o^*(U_i) \\ f_{i-q} \ , & \text{otherwise} \end{cases} .$$

2.2. The cotangent complex

Let $\tilde{R} := \mathrm{Der}_{O_S}(R,R)$; $\tilde{R}$ is a graded O_S-module and

$$\tilde{R}_k := \left\{ \delta : R \longrightarrow R \;\middle|\; \begin{array}{l} \delta \text{ be } O_S\text{-linear}, \ \delta(R_l) \subset R_{l+k}, \\ \delta(ab) = \delta(a)b + (-1)^{k|a|}\, a\delta(b) \end{array} \right\} .$$

Every $\delta \in \tilde{R}_k$ is uniquely determined by the images of the v_i, $|v_i| = \alpha = -k$, $\delta(v_i) \in R_{d+k}$ and by $\delta|_{R_0} = \sum_{i=1}^{n} r_i \frac{\partial}{\partial X_i}$, $r_i \in R_k$ for $k \leq 0$.

Therefore, $\tilde{R}_k$ is a free R_0-module:

$$\tilde{R}_k = R_k^n + \bigoplus_{\alpha < 0} R_{\alpha+k}^{n(\alpha)} \quad \text{and} \quad n(\alpha) = \#\{ v_i \,|\, |v_i| = \alpha \} .$$

By $[\delta, \delta'] := \delta\delta' - (-1)^{|\delta||\delta'|}\delta'\delta$ $\tilde{R}$ is a graded Lie-algebra and a cotangent complex by $d(\delta) := [\delta, I] = \delta I - (-1)^{|\delta|} I\delta$. The cohomology of the cotangent complex $T^i = \ker d_i / \mathrm{Im}\, d_{i-1}$ is independent of the choice of a resolvent and has the structure of an R_0-module. By direct calculations we obtain:

(i) $T^i(\varphi_o^*) = 0$ for $i < 0$,

(ii) $T^o(\varphi_o^*) = \mathrm{Der}_{O_S}(O_{X_o}, O_{X_o})$,

(iii) If φ_o^* is non-obstructed, then $T^i(\varphi_o^*) = 0$ for $i > 1$ (cf. the following example),

(iv) $T^1(\varphi_o^*)$ classifies the infinitesimal deformations of first order of φ_o^* up to isomorphy: $[\delta] \in T^1$ is uniquely determined by

$$\delta(v_i) = f_i^* = \begin{cases} \tilde{h}_i & i \leq q \quad \text{for} \quad |v_i| = -1 \\ \tilde{f}_{i-q} \ , & \text{otherwise} \end{cases}$$

then φ_δ is represnted by $(h + \varepsilon\tilde{h}\ ,\ f + \varepsilon\tilde{f}) \in J_{\mathbb{C}[\varepsilon]}(n, p_1)$.

2.3. Example (continued):

$\widetilde{R}_i = 0$ for $i \geq 2$, then $\delta \in R_i$ is determined by $\delta(v_i) \in R_{i-1} = 0$, and thus $T^i = 0$ for $i \geq 2$ and $\ker d_1 \cdot = \widetilde{R}_1$.

Let $\delta \in \widetilde{R}_o$ be given by $\delta|_{R_o} = \sum g_i \frac{\partial}{\partial X_i}$ and $\delta(v_i) = \sum g_{ij} v_j$; $g_i, g_{ij} \in R_o$, then $d(\delta)$ is determined by $d\delta(v_i)$:

$$d\delta(v_i) = \delta(f_i^{(1)}) - I(\sum_j g_{ij} v_j)$$

$$= \sum_k g_k \frac{\partial f_i^{(1)}}{\partial X_k} - \sum_j g_{ij} f_j^{(1)} .$$

This means: $\operatorname{Im} d_o := \sum R_o \frac{\partial}{\partial X_i} f^{(1)} + (f^{(1)})\widetilde{R}_1$, $\widetilde{R}_1 = R_o^{p_1}$

and thus

$$T^1(\varphi_o) = \mathbb{C}\{X\}^{p+q}/\ t(h,f) = O_{X_o}^{p+q}/\ \bar{t}(h,f) , \tag{4}$$

$$t(h,f) = \sum_i \mathbb{C}\{X\} \frac{\partial f^*}{\partial X_i} + (f)\ \mathbb{C}\{X\}^{p+q}$$

$$\bar{t}(h,f) = t(h,f)\ O_{X_o}^{p+q} .$$

2.4. The restricted cotangent complex

If we consider only the syzygies up to order k, we obtain the k-th restricted resolvent:

$R_{(k)} := R_o \oplus R_{-1} \oplus \dots \oplus R_{-k} = R/I_k$. Let $\widetilde{R}_{(k)}$ be the k-th associated cotangent complex and let $T^{\cdot}_{(k)}$ be the k-th restricted cohomology. By construction: $T^r_{(k)} = 0$ for $r > k$ and $T^k_{(k)} = \widetilde{R}_k / \operatorname{Im} d_{k-1} + I_k\ \widetilde{R}_k$. Similar to example 2.3 we have:

$T^1_{(1)}(\varphi_o^*) = O_{X_o}^{p_1} / \bar{t}(h,f)$ if $f^* = (h,f) \in J(n,p_1)$ represents φ_o^*.

The O_S-module structure of $T^1_{(1)}$ is given by $U_i\ \xi := h_i\ \xi$, $\xi \in T^1_{(1)}$.

Remark:

- $T^1_{(1)}$ classifies all infinitesimal families of φ_o of first order up to isomorphy.
- T^1 is an O_S-submodule of $T^1_{(1)}$, cf. 2.2 (iv).

2.5. The action of the generalized contact group

Let $(n,q,p_1,\dots,p_s)$ be admissible with respect to φ_o, then $T^{\cdot}(\varphi_o)$ has a natural $O_S\{X\}$-module structure.

Let $f^{\cdot} = (f^*, f^{(2)}, \dots, f^{(s)})$ be a representative of given type of a resolution of φ_o. Any other representative $f'^{\cdot}$ has the form $k \cdot f^{\cdot}$, $k \in \mathcal{K}(n,q,p_1,\dots,p_s)$. k induces an isomorphism ξ of the associated resolvents $R \cong R'$, which is compatible with I:
$k = (\Psi, A_1, \dots, A_s)$

$$\xi : R = O_S\{X\}\Gamma\, v_1,\dots,v_N\,] \longrightarrow R' = O_S\{X'\}\Gamma\, v'_1,\dots,v'_N\,]$$

$$X_i \longmapsto \Psi^*(X_i)$$

$$v_\alpha \longmapsto A_i \begin{pmatrix} v'_{\beta_1} \\ \vdots \\ v'_{\beta_r} \end{pmatrix}, \quad |v'_\beta| = |v_\alpha| = -i$$

and ξ induces an isomorphism $\tilde{\xi}$ of the cotangent complex because ξ is compatible with I.

$$\tilde{\xi} : \tilde{R} \longrightarrow \tilde{R}, \quad \tilde{\xi}(\delta) := \xi\,\delta\,\xi^{-1}$$

Analogously, $k_{(r)} = (\Psi, A_1, \dots, A_r) \in (n,q,p_1,\dots,p_r)$ induces an isomorphism of the restricted cotangent complex.

Restricted to elements of degree i $\tilde{\xi}$ is an $O_S\{X\}$-module isomorphism and we get $O_S\{X\}$-module isomorphisms $T^i(\varphi_o) \longrightarrow T^i(\varphi'_o)$, φ'_o is represented by f'^*.

3. INFINITESIMALLY TRIVIAL FAMILIES

The notion of infinitesimally trivial families is an effective tool to prove the triviality of a family. Roughly spoken, a family is called infinitesimally trivial if the lifting of a family to each tangent direction of the basis T is trivial, i.e. isomorphic to $\varphi \times id_I$, $I = \mathrm{Spec}\ \mathbb{C}[\varepsilon]$, $\varepsilon^2 = 0$. If T is smooth, then every infinitesimally trivial family is trivial, i.e. $\varphi \cong \varphi_o \times id_T$.
In other words, we obtain a local version of the theorem on the Kodaira-Spencer map associated to a deformation of a complex space: A deformation is trivial iff the Kodaira-Spencer map vanishes.

3.1. The Kodaira-Spencer map

The Kodaira-Spencer map of a deformation $\varphi : X \longrightarrow S \times T$ of a singularity φ_o is a map $\vartheta\varphi$: Der $0_T \longrightarrow T^1_{(1)}(\varphi)$ defined as follows:
For every derivation δ_a of 0_T there is a morphism $t_a : I \longrightarrow T$ which can be lifted to a T-morphism
$\tilde{t}_a : T \times I \longrightarrow T \times t_a(I) \longrightarrow T \times T \xrightarrow{pr_1} T.$
$\vartheta\varphi$ assigns to δ_a the class in $T^1_{(1)}(\varphi)$ of the family of φ over I $\varphi_a := \varphi \times_T \tilde{t}_a$ induced from φ by $\tilde{t}_a$.

Definition: A family φ of φ_o over T is called infinitesimally trivial if the Kodeira-Spencer map $\vartheta\varphi$ vanishes identically or, in other word, if for every tangent vector a of T the family φ_a induced by φ over I is trivial.

Example: Let T be smooth, $0_T = \mathbb{C}\{T_1,\dots,T_r\}$, $t_a^*(T_i) = \varepsilon a_i$, $a_i \in \mathbb{C}$ be a tangent vector of T, then $\tilde{t}^*(h(T)) = h + \sum \varepsilon a_i \frac{\partial h}{\partial T_i}$ is the restricted Taylor expansion of h in direction a. Let X_o be an arbitrary singularity, $0_{X_o} = \mathbb{C}\{X_1,\dots,X_n\} / (f_1,\dots,f_p)$ and let $\varphi : X \longrightarrow T = \mathbb{C}^r_{,o}$ be a family of X_o, $0_X = \mathbb{C}\{T,X\} / (F)$.
Then φ is infinitesimally trivial iff $\tilde{t}_a^*(F) = F + \varepsilon \sum a_i \frac{\partial}{\partial T_i} F$ is a trivial deformation of F iff $\frac{\partial F}{\partial T_i} \in t(F) \subseteq \mathbb{C}\{T,X\}^p$.
This example can be generalized:

Lemma: Let φ be a family of φ_o over $T = \mathbb{C}^r_{,o}$ and let $\tilde{F}^* = F^* + \varepsilon F'^*$ be a representative of φ_a, $F^* = (H,F)$ be a representative of φ. Then φ_a is trivial iff $F'^* \in t(H,F)$. (5)

Proof: Let $F'^* \in t(H,F)$ have the form
$F'^* = \sum a_i \frac{\partial}{\partial X_i} F^* + (b_{kj}) F$, a_i, $b_{kj} \in \mathbb{C}\{T,X\}$, then (ψ, A) determines an element k from $\mathcal{K}_I(r+n,q,p+q)$, where
$\psi^* : X_i \longmapsto X_i + a_i\varepsilon$, $T_j \longmapsto T_j$ and $A = E_{p+q} + (0 \mid b_{kj}) \cdot \varepsilon$.

It holds that $kF^* = \tilde{F}^*$.
On the other hand, let $F^* = (H,F) \in J(r+n,q,p+q)$ be an arbitrary representative of φ and let $\tilde{F}^* \in J_1(r+n,q,p+q)$ be a representative of φ_a. If φ_a is trivial, then there exists $k = (\Psi, A) \in \mathcal{K}_1(r+n,q,p+q)$ such that $kF^* = \tilde{F}^*$. Without loss of generality, let $k\big|_{\varepsilon=0} = 1_{\mathcal{K}}$. Then k is given by

$$\Psi^* : X_i \longmapsto X_i + \varepsilon a_i \ , \quad A = E_{p+q} + \varepsilon\,(0 \mid b_{kj}) \quad \text{and}$$

$$kF^* = F^* + \varepsilon\,(\sum a_i \frac{\partial}{\partial X_i} F^* + (b_{kj})F) = \tilde{F}^* \ .$$

Comparing the coefficients of the last equation we obtain (5).

3.2. The following proposition is a generalization of Lemma 3.5 of [8] :

Proposition: Let φ be a family of φ_o over T, T be smooth, then φ is trivial iff φ is infinitesimally trivial iff the Kodaira-Spencer map $\vartheta\varphi$ vanishes identically.

Proof: Let F^* be a representative of φ. If φ is infinitesimally trivial, then, by Lemma 3.1:

$$\frac{\partial}{\partial T_i} F^* \in t(H,F) \quad \text{for} \quad i = 1,\dots,r, \quad r = \dim T.$$

Induction on r:

$$r = 1 : \quad \text{Let} \quad \frac{\partial}{\partial T} F^* + \sum_i a_i \frac{\partial}{\partial X_i} F^* = \sum_{k=1}^{p} (b_{jk}(T,X))F_k \ .$$

The vector field $\frac{\partial}{\partial T} + \sum a_i(X,T)\frac{\partial}{\partial X_i}$ has an integral curve through (T,X) given by $(V_o,\dots,V_n) \in \mathbb{C}\{T,X,y\}^{n+1}$:

$$\frac{\partial V_i}{\partial y} = a_i(V) \ , \quad V_i(T,X,0) = X_i \ , \quad i = 1,\dots,n \ ,$$

$$\frac{\partial V_i}{\partial y} = 1 \quad , \quad V_o(T,X,0) = T \quad .$$

Then $F_1^*(V),\dots,F_{p+q}^*(V)$ are the solutions of the differential equation

$$\frac{\partial Z_j}{\partial y} = \sum_{k=1}^{p} b_{jk}(V)Z_k \ , \quad j = 1,\ldots,p+q \tag{6}$$

for the initial value $Z(T,X,0) = F^*$.

We give the following formulation:

$$F^*(V) - \binom{H}{0} = (S_{jk})F^* \ , \quad j = 1,\ldots,p+q \quad k = 1,\ldots,p$$

and

$$S_{jk}\Big|_{y=0} = \begin{cases} \delta_{j-qk} & j > q \\ 0 & \text{otherwise} \end{cases} .$$

Inserting this into (6) we obtain:

$$\frac{\partial S_{jl}}{\partial y} F^* = (b_{jk}(V))(S_{kl})F^* \qquad j = 1,\ldots,p+q \quad l,k = 1,\ldots,p \ . \tag{7}$$

Hence, we get the solution of (7) by the differential equation

$$\frac{\partial W_{jl}}{\partial y} = \sum_{k} b_{jk}(V)W_{kl} \qquad \text{with the initial value}$$

$$W_{jl}\Big|_{y=0} = \begin{cases} \delta_{j-ql} & j > q \\ 0 & \text{otherwise} \end{cases} .$$

Let Ψ^* be defined by $X_i \longmapsto V_i(0,X,T)$ and

$A = \left(\frac{E_q}{0} \Big| S_{jk}(0,X,T) \right)$, then $k = (\Psi, A) \in \mathcal{H}_T(n,q,p+q)$ and

$k \cdot (H,F) = (h,f)$, $(h,f) = (H,F)\Big|_{T=0}$ represents φ_o , hence

$\varphi_o \times id_T \cong \varphi$.

Induction step:

Let $T' = \operatorname{Spec} \mathbb{C}\{T_1,\ldots,T_{r-1}\}$, $T_o = \operatorname{Spec} \mathbb{C}\{T_r\}$,

$T = T' \times T_o$, $F'^* = F^*\big|_{T_r=0}$. $F'^* \widetilde{\mathcal{H}} \ F^*$ by the basis of the

induction and $F'^* \widetilde{\mathcal{H}} \ f^*$ by the induction hypothesis.

q.e.d.

Corollary: Let $\Phi : \mathfrak{X} \longrightarrow S \times \mathcal{T}$ be a family of S-germs over a smooth connected curve $\mathcal{T}$. If the germ of Φ is infinitesimally trivial at any point $t \in \mathcal{T}$ of the curye, then Φ is a $\mathcal{T}$-trivial family.

Proof: The trivialization of each germ Φ_t has a representative defined in the neighbourhood $U(t) \subset \mathcal{T}$ of t. Pasting together the local trivialization then yields a global one.

4. THE GENERALIZED MATHER-YAU EQUIVALENCE

Proposition: Let $\varphi_o : X_o \longrightarrow S$ be a singularity and $(n,q,p_1,\ldots,p_s)$ be admissible, then the following statements are equivalent:

(i) φ_o and φ'_o are isomorphic in $\mathcal{C}_S$.

(ii) For all $r \geq r$: $T^r(\varphi_o) \cong T^r(\varphi'_o)$ as $O_S\{X_1,\ldots,X_n\}$-module, where the isomorphism is induced by $\mathcal{K}(n,q,p_1,\ldots,p_r)$.

(iii) $T^1_{(1)}(\varphi_o) \cong T^1_{(1)}(\varphi'_o)$ as $O_S\{X\}$-module, where the isomorphism is induced by $\mathcal{K}(n,q,p_1)$.

4.1. Proof:

(i) $\longrightarrow$ (ii) by 1.3 and 2.5.

(ii) $\longrightarrow$ (iii) by 2.3.

(iii) $\longrightarrow$ (i) We choose representatives f^* and f'^*, respectively, in $J(n,p_1)$.

The isomorphism in (iii) is induced by some $k \in \mathcal{K}(n,q,p_1)$. We replace f^* by $k \cdot f^*$, then $t(h,f) = t(h',f')$.

$F^* := (1-T)f^* + Tf'^*$ represents a family Φ of S-germs over $\mathbb{A}^1$, such that the germ Φ_t of Φ at $t = 0$ (resp. $t=1$) defines a family of φ_o (resp. φ'_o).

Lemma: Up to a finite number of values $t \neq t_1, \dots, t_Q$ we have $t(h_t, f_t) = t(h,f)$.

Proof: We have $t(h_t, f_t) \subseteq t(h,f) = t(h', f')$. $t(h_t, f_t)$ is a finitely generated submodule of $\mathbb{C}\{X\}^{p_1}$ with a system of generators $m_1(t), \dots, m_M(t)$ and $m_j(t) = \sum c_{jk}(t) m_k(0)$. Up to a finite number of values that are the zeros of det $c_{jk}(T) \in \mathbb{C}[T]$ $c_{jk}(t)$ is a regular matrix.
Let $U = \mathbb{C} - \{t_1, \dots, t_Q\}$, at every point of U the germ Φ_t is infinitesimally trivial, because

$\frac{\partial}{\partial T} F_t^* = f'^* - f^* \in t(h,f) \subseteq t(H_t, F_t)$. This statement is obvious for the last p components F_t of F_t^*. As the O_S-module structures of $T^1_{(1)}(\varphi_o)$ and $T^1_{(1)}(\varphi_o')$ coincide, we have $(h'_j - h_j)e_k \in t(h,f)$. Now U is connected and by the corollary $\Phi|_U$ is a trivial family over U, hence, $F_o^*|_{T=0}$ and $F_1^*|_{T=0}$ represent isomorphic S-germs.

4.2. Interpretation of special cases

Due to the proposition it is sufficient to know the cohomology $T^\cdot(\varphi_o)$ to determine the singularity up to isomorphism. But, it is difficult to decide whether an isomorphism of $T^\cdot$ is induced by the contact group or not. If we consider non-obstructed singularities, then the statements (ii) and (iii) coincide (cf. 2.3):

(a) Let $S = \{0\}$ and X_o be a complete intersection: Any $\mathbb{C}\{X\}$ - module isomorphism of $T^1(\varphi_o)$ has a lift to $\mathbb{C}\{X\}^{p_1}$ (cf. (4)). Any such isomorphism is given by an isomorphism ψ^* of $\mathbb{C}\{X\}$ and a matrix $A \in GL_{p_1}(\mathbb{C}\{X\})$, i.e. induced by the contact group $\mathcal{K}(n,0,p_1)$. Hence, we obtain a generalization of the Mather-Yau result for isolated hypersurface singularities to the case of complete intersections with arbitrary singularity.

(b) Let $S = \mathbb{A}^q, o$ be smooth and let $X_o = \mathbb{A}^n, o$ be smooth, then an isomorphism of $T^1(\varphi_o)$ is induced by $\mathcal{K}(n,q,q) = R(n)$ iff it is induced by a ring-isomorphism ψ^* of $\mathbb{C}\{X\}$. This corresponds

to the right-equivalence of q-tuples of functions on $\mathbb{C}^n, o$.
For $q = 1$ the Mather-Yau equivalence implies:
Two functions $f, f' : \mathbb{C}^n, o \longrightarrow \mathbb{C}^1, o$ are right-equivalent iff the local algebras $Q_f = \mathbb{C}\{X\} / (\frac{\partial f}{\partial X_1}, \dots, \frac{\partial f}{\partial X_n})$ and $Q_{f'}$ are isomorphic as $\mathbb{C}\{T\}$-algebras.
For functions with isolated critical points at 0 this was shown by J. Scherk [9], who used the methods of [6].

(c) Let S be smooth and X_o be a complete intersection: The isomorphy of such singularities corresponds to a "right-equivalence" of q-tuples of functions on complete intersections. As in (a) an isomorphism of T^1 lifts to an isomorphism of $\mathbb{C}\{X\}^{p_1}$ given by Ψ and A. A sufficient condition, that (Ψ, A) on T^1 induces an $O_S\{X\}$-isomorphism, is that A is an element of $\widetilde{G}_{p_1}(q)$, i.e. is induced from an element of $\mathcal{K}(n,q,p_1)$.
For obstructed singularities we conclude the following:

(d) Let $S = \{0\}$, X_o be arbitrary:
$T^r(\varphi_o)$ is a submodule of $\widetilde{R}_r / \operatorname{Im} d_{r-1}$, $\widetilde{R}_r$ a free $\mathbb{C}\{X\}$-module.
An isomorphism $T^r(\varphi_o) \cong T^r(\varphi_o')$ is induced by $\mathcal{K}$ if it can be extended to a $\mathbb{C}\{X\}$-module isomorphism of $\widetilde{R}_r$. Analogous to (a) each of these isomorphisms is admissible.

(e) In the general case $T^r(\varphi_o)$, $(r > 1)$, is a submodule of $\widetilde{R}_r / \operatorname{Im} d_{r-1}$, $\widetilde{R}_r$ a free $O_S\{X\}$-module. An isomorphism is induced from the contact group if it can be extended to an $O_S\{X\}$-isomorphism of $\widetilde{R}_r$. For $T^1 \subset T^1_{(1)} = \mathbb{C}\{X\}^{p_1} / t(h,f)$ we have in addition (cf. (c)):
An isomorphism of $T^1_{(1)}$ is admissible if it has a lift to $\mathbb{C}\{X\}^{p_1}$, given by (Ψ, A) and $A \in \widetilde{G}_{p_1}(q)$. It is still an open question whether the condition $A \in \widetilde{G}_{p_1}(q)$ is really necessary, i.e. whether any $O_S\{X\}$-module isomorphism of $T^1_{(1)}$ is induced from $\mathcal{K}$.

5. THE MATHER-YAU EQUIVALENCE FOR MAPPING GERMS

For mapping germs and their deformations the present theory can be established in a rather analogous way. The main points will shortly be described here. Thereby we shall ignore the higher cohomology groups and equivalence will be given only for the restricted cohomology $T^1_{(1)}$. The theory of the cotangent complex of a morphism between analytic spaces has been developped by H. Flenner [2].

5.1. The group of the generalized right-left equivalences

In contrast to families of singularities φ_o over S, where the base germ S is fixed, we now also deform the image germ. By a family η of a mapping germ $\eta_o : X_o \longrightarrow S_o \in \mathcal{C}$ over T we mean a T-morphism $\eta : X \longrightarrow S$ (over T) with an isomorphism $\eta|_{t=0} \cong \eta_o$.

A family η can be represented by
$F^* = (H,F,G) \in \tilde{J}_T := J_T(n,p+q) \times J_T(q,r)$, such that

$$O_X \cong O_T\{X_1,\dots,X_n\} / (F_1,\dots,F_p)$$
$$O_S \cong O_T\{U_1,\dots,U_q\} / (G_1,\dots,G_r)$$
$$\eta^*(U_j) = H_j \bmod (F) \qquad j = 1,\dots,q .$$

A tuple (n,q,p,r) is called admissible with respect to η_o if η_o has a representative in $\tilde{J}$.
This is the case iff

$$n \geq \text{emb dim } X_o,$$
$$q \geq \text{emb dim } S_o,$$
$$p \geq i(X_o) \quad \text{and} \quad r \geq i(S_o) .$$

The set of all representatives of fixed type forms an orbit of a group $\mathcal{A}_T(n,p,q,r)$ in $\tilde{J}_T$. $\mathcal{A}_T$ is the semidirect product of $\mathcal{K}_T(n,q,p+q)$ and $\mathcal{K}_T(q,0,r)$.
$\mathcal{A}_T$ acts on $\tilde{J}_T$ as follows: Let $a \in \mathcal{A}_T$ be represented by

$$(\Psi_1 , \left(\begin{array}{c|c} E_q & A_1 \\ \hline 0 & A_2 \end{array}\right)) \in \mathcal{K}_T(n,q,p+q) \quad \text{and} \quad (\Psi_2, A_3) \in \mathcal{K}_T(q,0,r);$$

then $a F^* = \tilde{F}^*$ and

$$\tilde{F} := A_2^{-1} \Psi_1^* (F)$$

$$\tilde{G} := A_3^{-1} \Psi_2^* (G)$$

$$\tilde{H} := \Psi_1^* (H \cdot \Psi_2^{*-1}) - A_1 A_2^{-1} \Psi_1^* (F) \quad .$$

If $T = \{0\}$ and $p = r = 0$ (i.e. X_o and S_o are smooth), then $\mathcal{A} = \mathcal{R}(n) \times \mathcal{L}(n)$ and the action of $\mathcal{A}$ on $J(n,q)$ is just the right-left-equivalence in the sense of Mather ([5]).

5.2. Infinitesimal families

Infinitesimal families η' of η_o on $I = \operatorname{spec} \mathbb{C}[\varepsilon]$ have representatives of the form $f^* + \varepsilon f'^*$, $f'^* = (h', f', g') \in \mathbb{C}\{X\}^{p+q} \oplus \mathbb{C}\{U\}^r$.

A family η' is trivial iff $f'^* \in t(h,f,g)$. $t(h,f,g)$ is a submodule of $\mathbb{C}\{X\}^{q+p} \oplus \mathbb{C}\{U\}^r$, which is composed of

$t_1 = t(h,f) = \sum \mathbb{C}\{X\} \frac{\partial}{\partial X_i} \binom{h}{f} + \sum \mathbb{C}\{X\}^{p+q} f_j$, a $\mathbb{C}\{X\}$-submodule of $\mathbb{C}\{X\}^{p+q}$,

$t_2 = \sum \mathbb{C}\{U\}^r g_j$, a $\mathbb{C}\{U\}$-submodule of $\mathbb{C}\{U\}^r$

and

$t_3 = \left\{ \sum_{i=1}^{q} \lambda_i \frac{\partial}{\partial U_i} f^* + \lambda(h) \;\middle|\; \lambda = (\lambda_1, \dots, \lambda_q, 0)^t, \ \lambda_i \in \mathbb{C}\{U\} \right\}$,

a $\mathbb{C}\{U\}$-submodule of $\mathbb{C}\{X\}^q \oplus \mathbb{C}\{U\}^r$.

$T^1_{(1)}(\eta_o) := \mathbb{C}\{X\}^{p+q} \oplus \mathbb{C}\{U\}^r / t(h,f,g)$ classifies all infinitesimal families of first order of η_o up to homotopy.

The structure of $T^1_{(1)}$ is richer than that of a $\mathbb{C}\{U\}$-module, due to the subdivision into the three components. The action of $\mathcal{A}$ on $\mathcal{J}$ induces $\mathbb{C}\{U\}$-module isomorphisms of $T^1_{(1)}(\eta_o)$, which preserves this finer structure.

5.3. Infinitesimal triviality

The notion of infinitesimal triviality (cf. Definition 3.1) can also be applied to families of mapping germs and we get the same proposition.

Proposition: Let T be smooth and let η be a family of η_o over T. η is infinitesimally trivial iff η is trivial.

Proof: The proof is almost the same as in 3.2. Without loss of generality let dim T = 1 and let F^* be a representative of η. The infinitesimal triviality of η implies:

$$\frac{\partial}{\partial T} F^* + \sum_{i=1}^{n} a_i(T,X) \frac{\partial}{\partial X_i} F^* + \sum_{j=1}^{q} c_j(T,U) \frac{\partial}{\partial U_i} F^* = BF^* + c(T,H)$$

where B is a (q+p+r,q+p+r)-matrix of the form

$$\begin{pmatrix} 0 & B_1(T,X) & 0 \\ 0 & B_2(T,X) & 0 \\ 0 & 0 & B_3(T,U) \end{pmatrix}$$

and $c = (c_1, \ldots, c_q, 0, \ldots, 0)$.

We obtain integral curves

$V_o = T + y,\ V_1, \ldots, V_n$ of $\frac{\partial}{\partial T} + \sum a_i \frac{\partial}{\partial X_i}$, $V_i \in \mathbb{C}\{T,X,y\}$

and $V_o, V_{n+1}, \ldots, V_{n+q}$ of $\frac{\partial}{\partial T} + \sum c_j \frac{\partial}{\partial U_j}$, V_{n+j} $\mathbb{C}\{T,U,y\}$.

Then $F^*(V)$ is the solution of the differential equation

$\frac{\partial Z}{\partial y} = B(V) \cdot Z + c(Z)$.

Here formulation (7) has the form:

$$F^*(V) - \binom{H}{0} - c(H) = S \cdot F^*,$$

$$S = \begin{pmatrix} 0 & S_1 & 0 \\ 0 & S_2 & 0 \\ 0 & 0 & S_3 \end{pmatrix}$$

and S fulfills the differential equation $\frac{\partial W}{\partial y} = B(V) \cdot W$.

Hence we obtain $a \in \mathcal{A}_T$ by

$$\Psi_1^* : X_i \longmapsto V_i(0,X,T)$$

$$\Psi_2^* : U_j \longmapsto V_{n+j}(0,U,T)$$

$$A_i := S_i \Big|_{T := 0,\ y := T}$$

and then $\Psi_1^* F = A_2 f$

$$\Psi_2^* G = A_3 g$$

$$\Psi_1^* H = h\Psi_2^* + A_1 f = h \cdot \Psi_2^* + A_1 A_2^{-1} \Psi_1^* F$$

hence,

$$aF^* = f^{**} .$$

q.e.d.

<u>5.4. Proposition:</u> Let $\eta_o : X_o \longrightarrow S_o$ be a mapping germ and let (n,p,q,r) be admissible with respect to η_o. Then the following statements are equivalent:

(i) η_o is isomorphic to η_o

(ii) $T^1_{(1)}(\eta_o) \cong T^1_{(1)}(\eta'_o)$ as $\mathbb{C}\{U\}$-modules induced by an element of $\mathcal{A}(n,p,q,r)$.

The proof is a repetition of the arguments of 4.1.

<u>Remark:</u> If X_o and S_o are smooth, then an isomorphism $T^1_{(1)}(\eta_o) = T^1_{(1)}(\eta'_o)$ is admissible iff it is induced by a pair of ring-isomorphisms of $\mathbb{C}\{X\}$ and $\mathbb{C}\{U\}$.

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