Local Connectivity of the Boundary of a Coxeter System

Brad Groff

Department of Mathematics
University of Florida

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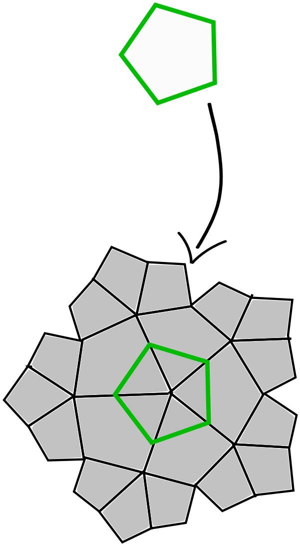
The Basics

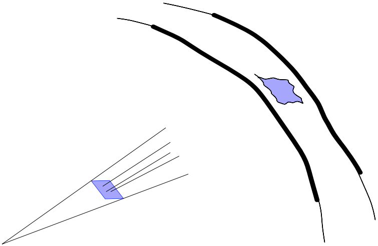
Assume that G is a right-angled Coxeter group.

- The algebraic information of G is contained in the nerve K, a simplicial complex with one n-simplex for each cardinality n subset of S which generates a finite subgroup.
- In this setting, K is a flag complex.
- G acts geometrically on a particular CAT(0) complex called the Davis complex X of G.

- X carries a natural structure of a cubical complex.
- With this structure, the Cayley graph of G can be embedded into X such that $vert(X) = vert(\Gamma_S(G))$.
- The link of every vertex of X is K.



The *Boundary* ∂X of a Coxeter system is the set of geodesic rays from a fixed basepoint in the associated Davis complex X, topologized via the shadows of open sets. The shadows of $St(v)$ for vertices v of X form a basis.



Statement of Main Result.

Theorem

If K is the nerve of the right-angled Coxeter system (G, S) and has the following properties for every vertex v

1. *K is a graph*
2. *K is connected*
3. *$K \setminus v$ is connected*
4. *$K \setminus St(v)$ is connected*

then the Davis complex associated to (G, S) has locally connected boundary.

Related Known Facts

(2) and (3) imply that ∂X is connected. (1) is almost certainly unnecessary and quite restrictive.

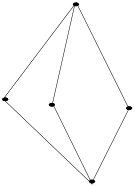
Remark

(2), (3) and (4) cannot be omitted.

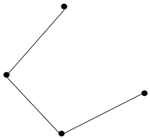
Example

- Let G have nerve (a), then G satisfies (1), (2) and (3) but ∂G is the suspension of a Cantor set.
- Let H have nerve (b), then H satisfies (1), (2) and (4) but H is infinitely ended and ∂H is not locally connected.

(a)



(b)



Sketch of Proof

1. Select a set in ∂X and produce a cubical complex approximately beneath it
2. Describe the appropriate notion of Morse function to work within this set
3. Enumerate the possible types of critical point and show that they pose no threats to connectivity
4. Converge to the boundary

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Useful Facts about K

With the conditions that K, $K \setminus v$ and $K \setminus St(v)$ are all connected, the nerve has a particular form.

- The link of each vertex in K is a discrete set of points
- No vertex is connected to fewer than 2 other vertices
- There are no 3-cycles

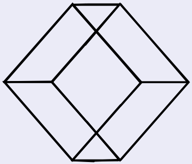


Figure: 1-skeleton of a cube

The embedding of K induces connectivity in some important regions of X.

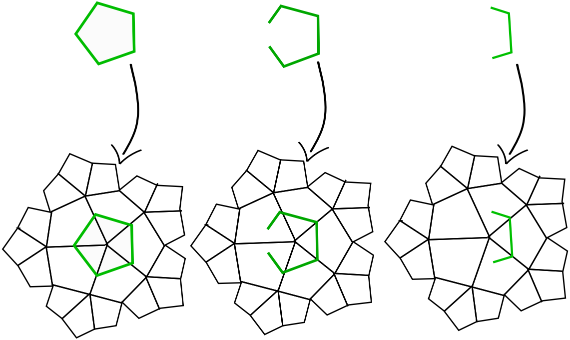


Figure: The three connected subcomplexes of K embedded into X

Cellular Approximation of a Shadow

- We choose a basic neighborhood W of ∂X, so W is the set of all geodesic rays through V and a basepoint for some set $V \subset X$.
- Let U be the shadow of V.
- U isn't quite nice enough.
- We approximate U by a nicer cubical complex, call this CU.
- The inclusion condition for each cell is a '2-vertex' rule. If two vertices of a square are in U, we include the square in CU.

U routinely includes odd portions of some cells while excluding important portions of other cells, a property which *CU* avoids. The lines in blue correspond to different levels of combinatorial spheres in *X*.

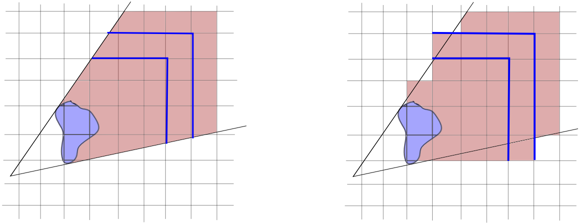


Figure: U (left) compared with CU

PL Morse Theory

Bestvina and Brady developed a Morse theory for affine polytope complexes to investigate finiteness properties of groups. We adapt their theory.

Definition

A Morse function $f : X \to \mathbb{R}$ is one which is

- affine - the restriction of f to any cell can be realized by an affine function on a polytope in $\mathbb{R}^m$
- has no horizontal cells - f is only constant on vertices
- separates vertices - $f(v_1) = f(v_2)$ iff $v_1 = v_2$.

Remark

Consequently, we have that

- Critical points only occur at vertices
- Vertices uniquely attain both the max and min values in their cells

Additional Structure

This is not enough for our purposes, so we make the additional requirements that

- f separates combinatorial spheres
- In a given combinatorial sphere, critical points that are closer to the boundary of CU have lower values than other critical points in the same sphere.

We apply this Morse theory to the complex *CU* and check to see that level sets are connected after passing each critical point.

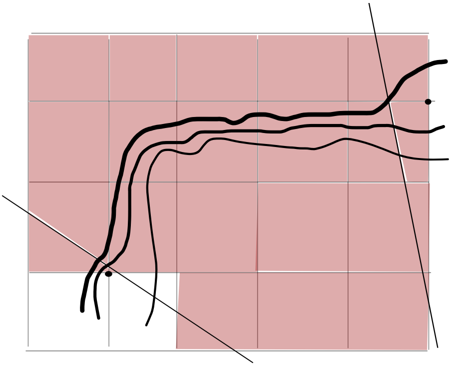
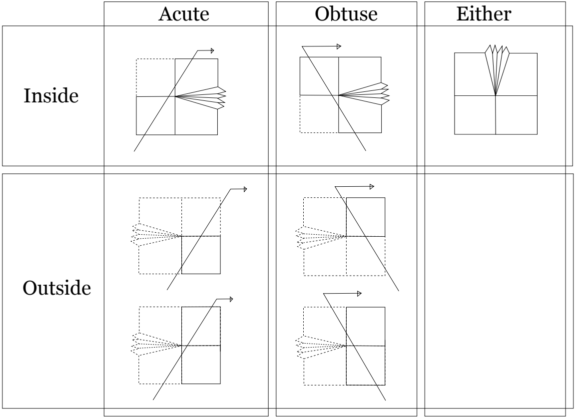


Figure: Level sets passing critical points.

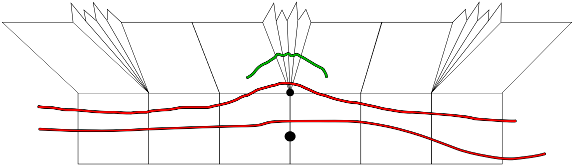
There are only 7 types of critical points, determined by only 3 criterion.

- whether v is contained in U
- whether all cells containing v are contained in U
- the angle at which ∂U intersects $St(v)$



The Easiest Case

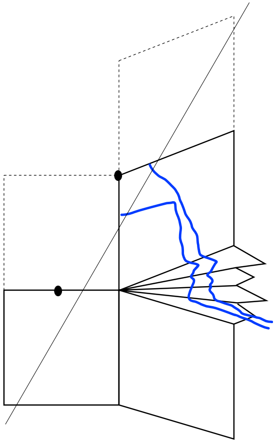
The entire subcomplex $St(v)$ is contained in CU.



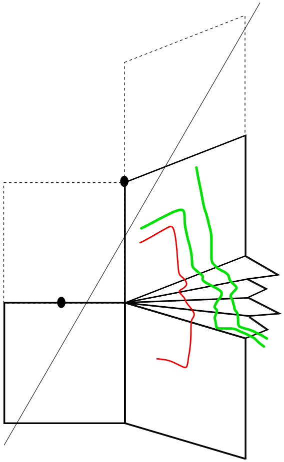
Here the red lines are increasing level sets, and the green represents $K \setminus St(v)$ embedded in X, v is the lower point. Because this is connected by hypothesis, we can see that passing the critical point does not affect connectivity.

A Different Type of Critical Point

If U intersects $St(v)$ in an acute angle and CU does not contain all of $St(v)$, then we need to pass through a more troublesome critical point.



However, the previous diagram looks almost exactly like the following diagram, in which the red represents $K \setminus St(v)$, which is assumed to be connected. Hence, the level set is connected after passing the critical point.



The level sets of f converge to W in the boundary. Since they are all connected, W is connected, establishing the result.

Thank you!