Improper Integrals

The integrals considered so far $\int_a^b f(x) \, dx$ assume implicitly that $a$ and $b$ are finite numbers and that the function $f(x)$ is nicely behaved on the interval. Improper integrals arise when

- The function $f(x)$ blows up (goes to $\pm\infty$) at one of the endpoints, or
- One of the end points $a$ and/or $b$ is infinite,
- A combination of both of the above

Examples.

- Find the total area under the curve $y = xe^{-x}$, $0 \leq x < \infty$.

The integral to calculate is

$$\int_0^\infty xe^{-x} \, dx.$$ 

Since $\int xe^{-x} \, dx = -xe^{-x} - e^{-x}$, the area out to $b$ is $\int_0^b te^{-t} \, dt = (-te^{-t} - e^{-t})|_{t=0}^{t=b} = -be^{-b} - e^{-b} + 1$, which tends to 1 as $b$ tends to $\infty$. Thus the area is finite and should be said to be 1. We say the improper integral $\int_0^\infty xe^{-x} \, dx$ converges to the value 1.

We are really calculating $\int_0^\infty xe^{-x} \, dx = (-xe^{-x} - e^{-x})|_0^\infty$ and interpreting the expression $(-xe^{-x} - e^{-x})$ at $x = \infty$ as 0 in the sense of limits.

- Find the total area under the curve $y = \frac{1}{x \ln(x)}$, $e \leq x < \infty$.

The integral to calculate is

$$\int_e^\infty \frac{1}{x \ln(x)} \, dx.$$ 

Since $\int \frac{1}{x \ln(x)} \, dx = \ln(\ln(x)) + C$, $\int_e^\infty \frac{1}{x \ln(x)} \, dx = \ln(\ln(x))|_e^\infty$, and $\ln(\ln(\infty))$ is to be interpreted as $\infty$ in the sense of limits.

We thus say that the improper integral diverges [to $\infty$] and the total area is infinite. Note that

$$\int_1^\infty \frac{1}{x \ln(x)} \, dx$$

is improper for an additional reason - at the initial end point $x = 1$, $\ln(1) = 0$.

- The improper integral $\int_0^1 \frac{1}{\sqrt{1 - x^2}} \, dx$

converges because $\arcsin(x)|_{x=1}^{x=0}$ can be evaluated because $\arcsin(1) = \frac{\pi}{2}$ in the sense of limits. The integral is improper since the integrand blows up near the right hand end point.

The Method

- In each of the examples, we took the lim as $x \to$ singular point of the anti-derivative function created after the integration.

\[1\text{This note was written by J. Lewis, and slightly revised by S. Hurder 1/23/2001}\]
Comparison Tests

For non-negative functions $f(x)$, the improper integrals $\int_{-\infty}^{\infty} f(x) \, dx$ converges if and only if the approximating integrals $\int_{a}^{b} f(t) \, dt$ are bounded as $b \to \infty$. This is because $\int_{a}^{b} f(t) \, dt = F(b) - F(\cdot)$ and $F(b)$ is increasing and $F(b)$ has a limit if and only if $F(b)$ is bounded.

A similar statement can be made for integrals of the form $\int_{a}^{\infty} f(x) \, dx$ ($f(x)$ blows up at $a$) or $\int_{-\infty}^{b} f(x) \, dx$ ($f(x)$ blows up at $b$).

- If $0 \leq f(x) \leq g(x)$, then $0 \leq \int f(x) \, dx \leq \int g(x) \, dx$

- If $0 \leq f(x) \leq g(x)$, and $\int g(x) \, dx$ converges, then $\int f(x) \, dx$ converges also.

- If $0 \leq f(x) \leq g(x)$, and $\int f(x) \, dx$ diverges, then $\int g(x) \, dx$ diverges also.

$p$-tests for improper integrals:

- $\int_{a}^{\infty} \frac{1}{x^p} \, dx \begin{cases} \text{converges if } & p > 1, \\ \text{diverges if } & p \leq 1. \end{cases}$

- $\int_{0}^{b} \frac{1}{x^p} \, dx \begin{cases} \text{converges if } & p < 1, \\ \text{diverges if } & p \geq 1. \end{cases}$

- $\int_{0}^{\infty} \frac{1}{x^p} \, dx$ diverges for all $p$.

Examples

- $\int_{1}^{\infty} \frac{\cos^2(\phi)}{\phi^2} \, d\phi$ converges by comparison with $\int_{1}^{\infty} \frac{1}{\phi^p} \, d\phi$ or simply by the $p$ - test with $p = 2$.

- $\int_{1}^{\infty} \frac{\cos^2(\phi)}{\phi} \, d\phi$ diverges, but cannot be handled directly by the comparison with $\int_{1}^{\infty} \frac{1}{\phi} \, d\phi$.

- $\int_{0}^{\infty} e^{-x^2} \, dx$ The integral converges since $e^{-x^2} \ll e^{-x}$ for large $x$, and $\int_{-\infty}^{\infty} e^{-x} \, dx$ converges.

- $\int_{-4}^{3} \frac{1}{x^2} \, dx$ DIVERGES.