1. Let X be a metric space.

a) **Define:** "$U \subset X$ is open".

b) **Prove:** X is separable $\iff$ there is a countable basis $\mathcal{B}$ for the metric topology on X.

2. Let $f\colon X \to Y$ be a map between metric spaces.

a) **Define:** "f is continuous" using the metric definition.

b) **Prove:** f is continuous $\iff$ $f(x_n) \to f(x)$ whenever $x_n \to x$ is a convergent sequence in X.

3. Let X be an infinite set. The *Zariski topology* on X is defined by the collection of subsets

$$\mathcal{T} = \{U = X - A \mid A \subset X, \text{ A is finite } \} \cup \{\emptyset\}$$

a) Show that $\mathcal{T}$ satisfies the axioms of a topology.

b) What is the closure of the set $A = \{1/n \mid n = 1, 2, \ldots\}$ in the Zariski topology on $\mathbb{R}$?

4. Let $K \subset \mathbb{R}^2$ be a compact subset of the plane with the metric topology. Prove that there exists a point $\xi \in K$ so that ξ is the furthest point from the origin $(0, 0)$ in K. ("Furthest" means the maximum distance from. You may assume that compact implies sequentially compact.)

5. Let $\mathcal{T}$ be a topology on a set X.

a) **Define:** "$(X, \mathcal{T})$ is connected".

b) Suppose that $f\colon X \to Y$ is a continuous onto map, where X and Y are topological spaces. Show that if X is connected, then Y is connected.

c) Suppose that $(X, \mathcal{T})$ is a connected topological space, and $f\colon X \to \mathbb{R}$ is a continuous map for the standard metric topology on $\mathbb{R}$. Prove that if there exists points $a, b \in X$ so that $f(a) < 0 < f(b)$, then there exists $c \in X$ such that $f(c) = 0$.

6. Show that a topological space X is *Hausdorff* if and only if the diagonal

$$\Delta(X) = \{(x, x) \mid x \in X\} \subset X \times X$$

is closed for the product topology. (Justify all of your claims!)