1. Let X be a metric space.

a) **Define:** "$U \subset X$ is open".

Solution: A set U is open if for every $x \in U$, there exists $\epsilon > 0$ so that the ball $B(x, \epsilon) \subset U$.

b) **Prove:** X is separable $\Longleftrightarrow$ there is a countable basis $\mathcal{B}$ for the metric topology on X.

Solution: Let $\mathcal{S} = \{x_1, x_2, \ldots\} \subset X$ be a countable dense subset. Then set collection $\{B(x_m, r) \mid m \in \mathbb{N}\ , r \in \mathbb{Q}\ , r > 0\}$ is a countable basis for the metric topology on X.

Conversely, let $\mathcal{B} = \{B_n \mid n \in \mathbb{N}\}$ be a countable basis for the topology on X. For each n choose $x_n \in B_n$. Then we claim $\mathcal{S} = \{x_1, x_2, \ldots\}$ is a countable dense subset. For, given any open set $U \subset X$ and $y \in U$ there exists $B_n \subset U$ with $y \in B_n$. But then $x_n \in U$ also, so $\mathcal{S}$ is dense.

2. Let $f\colon X \to Y$ be a map between metric spaces.

a) **Define:** "f is continuous" using the metric definition.

Solution: $f\colon X \to Y$ is continuous if given any $x \in X$ and $\epsilon > 0$, there exists $\delta > 0$ such that $d(x, y) < \delta$ implies that $d(f(x), f(y)) < \epsilon$. That is, $f(B(x, \delta)) \subset B(f(x), \epsilon)$.

b) **Prove:** f is continuous $\Longleftrightarrow$ $f(x_n) \to f(x)$ whenever $x_n \to x$ is a convergent sequence in X.

Solution: Assume that f is continuous, and let $x_n \to x$ be given. Set $y_n = f(x_n)$ and $y = f(x)$. We must show that $y_n \to y$, or that given any $\epsilon > 0$ there exists N so that $n \geq N$ implies that $d(y, y_n) < \epsilon$. Since f is continuous, there exists $\delta > 0$ so that $d(x, z) < \delta$ implies $d(f(x), f(z)) < \epsilon$. Since $x_n \to x$ is given, there is an integer N such that $n \geq N$ implies $d(x_n, x) < \delta$. Then by continuity, $d(f(x), f(x_n)) < \epsilon$. Thus, if $n \geq N$ then $d(y_n, y) < \epsilon$ which shows that $y_n \to y$.

The converse is shown by contradiction. Suppose there exists a sequence $x_n \to x$ such that $f(x_n)$ does not converge to $f(x)$. Then there exists $\epsilon > 0$ and a subsequence $\{x_{n_i} \mid i = 1, 2, \ldots\}$ such that $x_{n_i} \to x$ and $d(f(x), f(x_{n_i})) \geq \epsilon$. Now, $x_{n_i} \to x$ implies that for all $\delta > 0$ there exists i with $d(x, x_{n_i}) < \delta$, so f is not continuous at x.

3. Let X be an infinite set. The *Zariski topology* on X is defined by the collection of subsets

$$\mathcal{T} = \{U = X - A \mid A \subset X, \text{ A is finite } \} \cup \{\emptyset\}$$

a) Show that $\mathcal{T}$ satisfies the axioms of a topology.

Solution: The set $A = \emptyset$ is finite, so $X = X - \emptyset$ is open.

Let $\{U_\alpha \mid \alpha \in \mathcal{A}\}$ be a collection of open sets for $\mathcal{T}$. Then for each α there is a finite set $A_\alpha \subset X$ with $U = X - A_\alpha$. The by de Morgan's laws,

$$\bigcup_{\alpha \in \mathcal{A}} U_\alpha \;=\; \bigcup_{\alpha \in \mathcal{A}} (X - A_\alpha) \;=\; X - \bigcap_{\alpha \in \mathcal{A}} A_\alpha = X - A$$

where $A = \bigcap_{\alpha \in \mathcal{A}} A_\alpha$ is the intersection of finite sets, so is finite. Thus, $X - A$ is open in $\mathcal{T}$.

Let $\{U_1, \ldots, U_n\}$ be a finite collection of open sets for $\mathcal{T}$. Then for each i there is a finite set $A_i \subset X$ with $U = X - A_i$. The by de Morgan's laws,

$$\bigcap_{i=1}^{n} U_i \;=\; \bigcap_{i=1}^{n} (X - A_i) \;=\; X - \bigcup_{i=1}^{n} A_i = X - A$$

where $A = A_1 \cup \cdots \cup A_n$ is a finite union of finite sets, so finite. Thus, $X - A$ is open in $\mathcal{T}$.

b) What is the closure of the set $A = \{1/n \mid n = 1, 2, \ldots\}$ in the Zariski topology on $\mathbb{R}$?

Solution: The closure $\overline{A}$ of A is the intersection of all closed subsets in $\mathcal{T}$ containing A. The closed subsets of $\mathcal{T}$ are the complements of the open subsets in $\mathcal{T}$, so either a closed subset is a finite subset of X, or all of X. Since no finite subset contains the infinite set A, the only closed subset containing A is all of X. Thus, $\overline{A} = X$.

4. Let $K \subset \mathbb{R}^2$ be a compact subset of the plane with the metric topology. Prove that there exists a point $\xi \in K$ so that ξ is the furthest point from the origin $(0, 0)$ in K. ("Furthest" means the maximum distance from. You may assume that compact implies sequentially compact.)

Solution: We use that the function $f(x) = d((0, 0), x)$ is continuous for the metric topology. The problem is to find a point $x \in K$ so that $f(x) \geq f(y)$ for all $y \in K$.

Claim 1: $M = \sup\{f(x) \mid x \in K\} < \infty$. If not, then for each $n > 0$ there exists $x_n \in K$ with $f(x_n) \geq n$, and then the sequence $\{x_n\}$ has no convergent subsequence (by the Triangle Inequality.) This contradicts the assumption that K is sequentially compact.

Next, for each $n > 0$ chose $x_n \in K$ so that $f(x_n) > M - 1/n$. Then K compact implies the sequence $\{x_n\}$ has a convergent subsequence, $x_{n_i} \to x_* \in K$. Then by the continuity of f, we have

$$f(x_*) \;=\; f(\lim_{i \to \infty} x_{n_i}) \;=\; \lim_{i \to \infty} f(x_{n_i}) = M$$

Thus, $x_* \in K$ satisfies $f(x_*) = \sup\{f(x) \mid x \in K\}$ as was to be shown.

5. Let $\mathcal{T}$ be a topology on a set X.

a) **Define:** "$(X, \mathcal{T})$ is connected".

Solution: $(X, \mathcal{T})$ is connected if it is not separated. That is, if $X = U \cup V$ where U, V are open and $U \cap V = \emptyset$, then either $U = \emptyset$ or $V = \emptyset$.

b) Suppose that $f: X \to Y$ is a continuous onto map, where X and Y are topological spaces. Show that if X is connected, then Y is connected.

Solution: Suppose that $f(X)$ is not connected, then it must be separated, for the relative topology from Y. That is, there exists open disjoint sets $U, V \subset Y$ so that $f(X) \subset U \cup V$ and both $f(X) \cap U \neq \emptyset$ and $f(X) \cap V \neq \emptyset$.

As f is continuous, both $A = f^{-1}(U)$ and $B = f^{-1}(B)$ are non-empty open subsets of X. And $f(X) \subset U \cup V$ implies that $X = A \cup B$, so X is separated, which is a contradiction.

c) Suppose that $(X, \mathcal{T})$ is a connected topological space, and $f: X \to \mathbb{R}$ is a continuous map for the standard metric topology on $\mathbb{R}$. Prove that if there exists points $a, b \in X$ so that $f(a) < 0 < f(b)$, then there exists $c \in X$ such that $f(c) = 0$.

Solution: The image $f(X) \subset \mathbb{R}$ is connected by b) and contains points $f(a) < 0$ and $f(b) > 0$. Suppose that there exists no $c \in X$ with $f(c) = 0$. Then $0 \notin f(X)$, so $U = (-\infty, 0)$ and $V = (0, \infty)$ is a separation for the image $f(X)$, which contradicts that $f(X)$ is connected. Thus, there must exist $c \in X$ with $f(c) = 0$.

6. Show that a topological space X is *Hausdorff* if and only if the diagonal

$$\Delta(X) = \{(x, x) \mid x \in X\} \subset X \times X$$

is closed for the product topology. (Justify all of your claims!)

Solution: Suppose that X is Hausdorff. We must show that $\Delta(X)$ is a closed set, or that $X \times X - \Delta(X)$ is an open set. Let $(x, y) \in X \times X$ with $x \neq y$. Since X is Hausdorff, there exists open sets $U, V \subset X$ with $x \in U$, $y \in V$ and $U \cap V = \emptyset$. Then $U \times V$ is an open set for the product topology on $X \times X$. Also, $U \times V \cap \Delta(X) = \emptyset$, as $(x, x) \in U \times V$ implies that $x \in U \cap V$. Thus, $(x, y) \in U \times V \subset X - \Delta(X)$ is an open neighborhood.

Conversely, suppose that $\Delta(X)$ is a closed subset, then $X \times X - \Delta(X)$ is an open set in the product topology. Thus, for each $x \neq y$, there exists an element $U \times V$ of the basis for the product topology such that $(x, y) \in U \times V \subset X \times X - \Delta(X)$. Each of $U, V \subset X$ is open by definition. The inclusion $(x, y) \in U \times V$ implies that $x \in U$ and $y \in V$.

Finally, we claim that $U \cap V = \emptyset$. If not, then there exists $x \in U \cap V$, which implies that $(x, x) \in U \times V$ so that $U \times V \cap \Delta(X) \neq \emptyset$, contrary to choice.

Thus, we have shown there exists disjoint open neighborhoods of the distinct points $x \neq y$. Hence, X is a Hausdorff space.