You may use your textbook and class notes, but no other references.

There are 6 problems. Turn in 5 for grading.

1. Let $\{M, D\}$ be a metric space. Let $A \subset M$ be a subset and suppose $y \in \overline{A}$. Show that every continuous real-valued function $f\colon M \to \mathbb{R}$ which vanishes on A also vanishes at y.

Solution: Let $f\colon M \to \mathbb{R}$ be a continuous function and $y \in \overline{A}$. Then for every open set $U \subset M$ with $y \in U$, there is some $z \in U \cap A$. So, for each integer $n > 0$, we can choose $z_n \in B(y, 1/n) \cap A$. It is given that $f(x) = 0$ for every $x \in A$, so $f(z_n) = 0$. But also, by choice $z_n \to y$. Since f is continuous,

$$f(y) = f(\lim_{n\to\infty} z_n) = \lim_{n\to\infty} f(z_n) = \lim_{n\to\infty} 0 = 0$$

.

2. Let $\{M, D\}$ be a metric space, and suppose that there exists open sets $U, V \subset M$ such that $M = U \cup V$ and $U \cap V = \emptyset$. Prove that for if $\{x_n\} \subset U$ is a convergent sequence with limit $x_n \to u \in M$, then $u \in U$. *Use only the definitions of an open set and of a convergent sequence in your proof - do not just quote a Theorem.*

Solution: Suppose by contradiction, that $u \notin U$. Then $M = U \cup V$ implies $u \in V$. Since V is open, there exists $\epsilon > 0$ so that the open ball $B(u, \epsilon) \subset V$. Since $x_n \to u$, there exists some N_ϵ so that for all $n \geq N_\epsilon$ the points $x_n \in B(u, \epsilon)$. But $U \cap V = \emptyset$ so this implies $x_n \notin U$ for all $n \geq N_\epsilon$, which is a contradiction.

Here is another proof, using a theorem from class: As V is open, it's complement is closed. Now, $M - V = U$ so U is closed. By a theorem in class, every closed set contains all of its limit points. As $\{x_n\} \subset U$ converges, $x_n \to u$, then $u \in U$ also.

3. Let X be an infinite set.

a) Show that there exists two countably-infinite sets $A, B \subset X$ with $A \cap B = \emptyset$.

Solution: The set X is infinite, so has cardinality at least countably infinite. That is, there is an injective set map $f\colon \mathbb{N} \to X$. If we set $x_n = f(n)$ then we have a countably infinite subset $\{x_1, x_2, \ldots\} \subset X$ of distinct points.

Let $U, V \subset \mathbb{N}$ be two infinite disjoint subsets, such as $U = \{n \in \mathbb{N} \mid n \text{ is odd }\}$ and $V = \{n \in \mathbb{N} \mid n \text{ is even }\}$. Then $A = f(U)$ and $B = f(V)$ are both countably infinite subsets of X since f is injective. They are also disjoint, since $A \cap B = f(A) \cap f(B) = f(A \cap B) = \emptyset$, again because f is injective.

b) Show that there exists a countably-infinite collection $\{A_i \mid i = 1, 2, \ldots\} \subset X$ where each A_i is a countably-infinite set, and $A_i \cap A_j = \emptyset$ for $i \neq j$.

Solution: The set X is infinite, so has cardinality at least countably infinite. That is, there is an injective set map $f\colon \mathbb{N} \to X$. If we set $x_n = f(n)$ then we have a countably infinite subset $\{x_1, x_2, \ldots\} \subset X$ of distinct points.

Let $\{U_i \subset \mathbb{N} \mid i = 1, 2, \ldots\}$ be a countable collection of pairwise disjoint infinite subsets. For example, Let U_i consists of all positive integers which are a product of exactly i prime factors. Each set $A_i = f(U_i)$ is infinite since f is injective. Moreover, $A_i \cap A_j = \emptyset$ also because $U_i \cap U_j = \emptyset$ and f is injective.

4. Let $\{M, D\}$ be a metric space.

a) Prove that if M is a finite set, then the metric topology on M is discrete.

Solution: M is finite, so the set of distances $\{D(x,y) \mid x \neq y \in M\}$ is also finite. As D is a metric, 0 is not in the set, so the lower bound $\beta = \min \{D(x,y) \mid x \neq y \in M\} > 0$. Then for each $x \in M$, the ball $B(x, \beta) \subset M$ contains only the point x. If not, there is some $y \in B(x, \beta)$ with $y \neq x$ and so $D(x,y) < \beta$, which contradicts the choice of β. Thus, the point x is open. This is true for every $x \in M$, so the topology on M is discrete.

b) Give an example of a countably-infinite, complete metric space which is discrete.

Solution: The best example is just the positive integers $\mathbb{N} \subset \mathbb{R}$, with the natural metric $D(m,n) = |n - m|$. Of course, $D(n,m) \geq 1$ if $m \neq n$, so the ball $B(n, 1/2) \subset \mathbb{N}$ contains just the point n.

c) Give an example of a countably-infinite, complete metric space which is not discrete.

Solution: Start with $X = \{\frac{1}{n} \mid n = 1, 2, \ldots\} \subset \mathbb{R}$. This is a countable set and is discrete, but is not complete as it does not include the limit point 0. Take as the example $M = X \cup \{0\})$ so that now M is closed and discrete.

5. Let $\{f_n \colon [0,1] \to \mathbb{R} | n = 1, 2, \ldots\}$ be a sequence of continuous real-valued functions. Assume the sequence is monotonically increasing: for each $0 \leq x \leq 1$ we have $f_n(x) \leq f_{n+1}(x)$ for all $n \geq 1$. Suppose that the sequence $\{f_n\}$ converges pointwise to a continuous function $f \colon [0,1] \to \mathbb{R}$. That is, for each $0 \leq x \leq 1$, $f_n(x) \to f(x)$. Prove that the sequence $\{f_n\}$ converges uniformly to f.

Solution: Given $\epsilon > 0$ we must show there is N_ϵ so that $n \geq N_\epsilon$ implies that

$$D(f, f_n) = \sup_{0 \leq x \leq 1} |f(x) - f_n(x)| < \epsilon$$

There are three steps.

Step 0: Suppose this is false, then there is some $\epsilon > 0$ so that for every n, there exists $0 \leq x_n \leq 1$ such that $|f(x_n) - f_n(x_n)| \geq \epsilon$. Let $X = \{x_n \mid n = 1, 2, \ldots\} \subset [0,1]$ be the set of these points.

Step 1: The set X has a limit point. At a limit point $x_* \in [0,1]$ the functions f_n are "behaving badly" in that they jump up by at least ϵ. This gives a contradiction in Step 2.

X is an ordered infinite set; we use the idea from the solution of problem 2 in homework #1.

If X has no limit point, then every x_n is isolated in X. We show this leads to a contradiction.

Let $y_1 \in [0,1]$ be the least upper bound of X, $y_1 = lub\ (X)$. Then $y_1 = x_{n_1}$ for some n_1, as otherwise x_{n_1} is a limit point of X.

Since y_1 is isolated in X, then $y_2 = lub\ (X - \{y_1\}) < y_1$. Again, we must have $y_2 = x_{n_2}$ for some n_2 or else y_2 is a limit point of X.

Continue in this way to obtain a sequence of points in X, $y_1 > y_2 > y_3 > \ldots$ where each $y_i = x_{n_i}$ for some n_i. Since each $y_i \in [0,1]$ the sequence is bounded below by 0. Thus, $x_* = glb\ (\{y_1, y_2, y_3, \ldots\}) \geq 0$ is a limit point of the descending sequence $\{y_i\}$. So $x_* \in [0,1]$ is a limit point of X, which is a contradiction.

Step 2: We have there exists $x_* \in [0,1]$ which is a limit point of X. Thus, there exists a sequence $\{y_i\} \subset X$ with $y_i \to x_*$. For each i there is n_i so that $y_i = x_{n_i}$.

Recall that by the choice of the x_n we have $|f(y_i) - f_n(y_i)| \geq \epsilon$ for all i.

The trouble is happening at the point x_* so we consider the graphs of the functions near this point. Also, we are given that $f_n(x_*) \to f(x_*)$ so we use that too. And we use the same $\epsilon > 0$.

Let N be an integer so that $|f(x_*) - f_N(x_*)| < \epsilon/3$. Both functions f_n and f are continuous on $[0,1]$ so we can choose $\delta_1 > 0$ so that

$$|\xi - x_*| < \delta_1| \quad \Longrightarrow \quad |f(\xi) - f(x_*)| < \epsilon/3$$

and $\delta_2 > 0$ so that

$$|\xi - x_*| < \delta_2| \quad \Longrightarrow \quad |f_N(\xi) - f_N(x_*)| < \epsilon/3$$

Let $\delta = \min\{\delta_1, \delta_2\}$. Combining these three estimates, we get that for all $|\xi - x_*| < \delta$ then

$$\begin{aligned} |f(\xi) - f_N(\xi)| &= |(f(\xi) - f(x_*)) + (f(x_*) - f_N(x_*)) + (f_N(\xi) - f_N(x_*))| \\ &\leq |(f(\xi) - f(x_*))| + |(f(x_*) - f_N(x_*))| + |(f_N(\xi) - f_N(x_*))| \\ &< \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon \end{aligned}$$

The other fact we are given is that $f_n(x) \leq f_{n+1}(x)$ for all $n \geq 1$ and all $0 \leq x \leq 1$. Combine this with the above gives, for all $n > N$ and all $|\xi - x_*| < \delta$, then $|f(\xi) - f_n(\xi)| < \epsilon$.

This is a problem though, as $y_i \to x_*$ and $n_i \to \infty$ so for i large enough, we have both $|y_i - x_*| < \delta$ and $n_i > N$ so $|f(y_i) - f_n(y_i)| \geq \epsilon$. This is a contradiction. The conclusion is that the convergence $f_n \to f$ is uniform.

6. a) Let $X \subset [0,1] \times [0,1] \subset \mathbb{R}^2$ be a subset of the plane contained in the unit square. Show that if X is infinite, then X has a limit point.

Solution: We will show that X contains a Cauchy sequence. Then as the square is a closed subset of the plane, it contains the limit of the Cauchy sequence, hence contains a limit point of X.

The construction of a Cauchy sequence in X uses the "Pigeon Hole Principle" repeatedly.

Let $S_0 = [0,1] \times [0,1]$ be the given square.

We divide S_0 into four subsquares, as follows:

$S_{1,1} = [0,1/2] \times [0,1/2]$; $S_{1,2} = [0,1/2] \times [1/2,1]$; $S_{1,3} = [1/2,1] \times [0,1/2]$; $S_{1,4} = [1/2,1] \times [1/2,1]$
The set X is infinite, so its intersection with at least one of these squares must contain an infinite number of points. We call this square S_{1,i_1} where $i_1 \in \{1,2,3,4\}$.

Now repeat for the square S_{1,i_1} dividing it into four subsquares, with corners at rational points with denominator $\frac{1}{4} = \frac{1}{2^2}$. One of these four squares must intersect the set $S_{1,i_1} \cap X$ in an infinite number of points. Call this square S_{1,i_1,i_2} where $i_2 \in \{1,2,3,4\}$. Note that $S_{1,i_1,i_2} \subset S_{1,i_1} \subset S_0$.

Repeat this process indefinitely. The n^{th} square will be labeled $S_{1,i_1,i_2,\ldots,i_n}$ and it has size $\frac{1}{2^n}$ along its edges. Thus, the set $S_{1,i_1,i_2,\ldots,i_n}$ has diameter $\sqrt{2}/2^n$.

Then for each n, chose a point $x_n \in S_{1,i_1,i_2,\ldots,i_n}$. Draw a picture to see the basic idea, which becomes much clearer with a picture.

<u>Claim</u>: $\{x_n\}$ is a Cauchy sequence. Let $\epsilon > 0$ be given. Choose N_ϵ so that $N_\epsilon > 2^n/\sqrt{2}$. Suppose that $n > m \geq N_\epsilon$ then

$$x_m, x_n \in S_{1,i_1,i_2,\ldots,i_n} \subset S_{1,i_1,i_2,\ldots,i_{N_\epsilon}}$$

Since the diameter of $S_{1,i_1,i_2,\ldots,i_{N_\epsilon}}$ is $\sqrt{2}/2^{N_\epsilon} < \epsilon$ we have $d(x_n, x_m) < \epsilon$. This holds for all $n > m \geq N_\epsilon$ so the sequence $\{x_n\}$ is Cauchy.

Let $x_* = \lim\limits_{n\to\infty} x_n$ then $x_* \in S_0 = [0,1] \times [0,1]$ is a limit point of the set X.

b) Let $X \subset \mathbb{R}^2$ be a subset of the plane. Show that if X is uncountable, then X has a limit point.

Solution: Assume the solution to part a). Then divide the plane $\mathbb{R}^2$ into unit squares, whose vertices are located at the integer points (m,n). Label the squares by the bottom left corner, so $S(0,0)$ is the unit square with corners at $(0,0), (0,1), (1,0), (1,1)$.

In general, $S(m,n)$ is the unit square with corners at $(m,n), (m,n+1), (m+1,n), (m+1,n+1)$.

The plane $\mathbb{R}^2$ is the union of all such squares. Suppose that $S(m,n) \cap X$ is finite for all pairs (m,n) of integers. Then the set X must be countable. The hypothesis is that X is uncountable, so for at least one pair (m,n) the intersection $S(m,n) \cap X$ is infinite (in fact it must be uncountable). Then by part a) the set $S(m,n) \cap X$ must have a limit point, which is also a limit point of X.