Math 445, Fall 2009 Exercise Set #3 Turn in: September 21

1. [#11, page 70] Let D_1 and D_2 be metrics on a single space M. Which of the following are metrics on M: $D_1 + D_2$, $\max\{D_1, D_2\}$, $\min\{D_1, D_2\}$?

2. [#14, page 71] Let M be a metric space in which the distance function assumes only the values $0, 1, 3$. Define $x \sim y$ to means $D(x, y) \leq 1$. Prove that $\sim$ is an equivalence relation on M. Show also that $\sim$ determines the metric D.

3. [#1, page 74] Let M, D be a metric space. Prove that:
a) For every $x \in M$, the complement $V_x = M - \{x\}$ is open. [Points are closed.]
b) For any set $X \subset M$, then X is the intersection of open sets. [The problem is to find enough open sets. A finite number will not suffice, unless X is itself open.]

2. [#2, page 74] Let $x, y \in M$ be distinct points in a metric space M, D. Prove that there exists disjoint open sets $U, V \subset M$ with $x \in U$ and $y \in V$.

5. [#5, page 74] Let $M = \mathbb{R}$ be the real line, with the metric $D(x, y) = |x - y|$. Prove that there are no isolated points in $\mathbb{R}$. [A point $x \in M$ is *isolated* if there exists an open set U such that $U \cap M = \{x\}$.]

6. [#8, page 74] Let x be a point of as metric space M. Prove that the following two statements are equivalent:
a) x is not isolated.
b) Every neighborhood of x contains an infinite number of points of M.

7. [#9, page 74] Let M be an infinite metric space. Prove that M contains an open set U such that both U and its complement $M - U$ are infinite.