Math 445, Fall 2009 Exercise Set #3 Solutions

1. [#11, page 70] Let D_1 and D_2 be metrics on a single space M. Which of the following are metrics on M: $D_1 + D_2$, $\max\{D_1, D_2\}$, $\min\{D_1, D_2\}$?

Proof. In all three examples, the Non-degenerate and Symmetry conditions are essentially automatic. In building new metrics from old, it is always a question of whether the Triangle Inequality holds.

Example 1) $D = D_1 + D_2$: We have three things to check.

Non-degenerate: $D(x, y) = 0$ then both $D_1(x, y) = 0$ and $D_2(x, y) = 0$ so $x = y$ as both $D_1(x, y) \geq 0$ and $D_2(x, y) \geq 0$. Then D_1 (or use D_2) is a metric implies $x = y$.

Symmetry: $D(x, y) = D(y, x)$ as both $D_1(x, y) = D_1(y, x)$ and $D_2(x, y) = D_2(y, x)$.

Triangle: Use the triangle inequality for D_1 and D_2

$$\begin{aligned} D(x,z) = D_1(x,z) + D_2(x,z) &\leq (D_1(x,y) + D_1(y,z)) + (D_2(x,y) + D_2(y,z)) \\ &= (D_1(x,y) + D_2(x,y)) + (D_1(y,z) + D_2(y,z)) \\ &= D(x,y) + D(y,z) \end{aligned}$$

So $D = D_1 + D_2$ is a metric.

Example 2) $D = \max\{D_1, D_2\}$: We have three things to check.

Non-degenerate: $D(x, y) = 0$ then $\max\{D_1(x, y), D_2(x, y)\} = 0$ so both $D_1(x, y) = D_2(x, y) = 0$. Then D_1 (or use D_2) is a metric implies $x = y$.

Symmetry: $D(x, y) = \max\{D_1(x, y), D_2(x, y)\} = \max\{D_1(y, x), D_2(y, x)\} = D(y, x)$ as $D_1(x, y) = D_1(y, x)$ and $D_2(x, y) = D_2(y, x)$.

Triangle: We want to use the triangle inequality for D_1 and D_2 to show

$$D(x,z) = \max\{D_1(x,z), D_2(x,z)\} \ \leq \ \max\{(D_1(x,y), D_2(x,y)\} + \max\{D_1(y,z), D_2(y,z))\}???$$

The strategy is to show that $D_1(x, z) \leq D(x, y) + D(y, z)$ and the same for $D_2(x, z)$. By the triangle inequality for D_1

$$D_1(x,z) \leq D_1(x,y) + D_1(y,z) \leq \max\{(D_1(x,y), D_2(x,y)\} + \max\{D_1(y,z), D_2(y,z))\}$$

because $D_1(x, y) \leq \max\{(D_1(x, y), D_2(x, y)\}$ and $D_1(y, z) \leq \max\{(D_1(y, z), D_2(y, z)\}$. Likewise,

$$D_2(x,z) \leq D_2(x,y) + D_2(y,z) \leq \max\{(D_1(x,y), D_2(x,y)\} + \max\{D_1(y,z), D_2(y,z))\}$$

Since we have the same expression on the right hand side in both inequalities, we get

$$\max\{D_1(x,z), D_2(x,z)\} \ \leq \ \max\{(D_1(x,y), D_2(x,y)\} + \max\{D_1(y,z), D_2(y,z))\}$$

So $D = \max\{D_1, D_2\}$ is a metric.

Example 3) $D = \min\{D_1, D_2\}$: Again, Non-degenerate and Symmetry are true. Non-degenerate: $D(x, y) = \min\{D_1(x, y), D_2(x, y)\} = 0$ implies either $D_1(x, y) = 0$ or $D_2(x, y) = 0$. Then $D_1(x, y) = 0$ or $D_2(x, y) = 0$ which implies $x = y$.

Symmetry: $D(x, y) = \min\{D_1(x, y), D_2(x, y)\} = \min\{D_1(y, x), D_2(y, x)\} = D(y, x)$ as $D_1(x, y) = D_1(y, x)$ and $D_2(x, y) = D_2(y, x)$.

Triangle: The proof of the triangle inequality above for the metric $D = \max\{D_1, D_2\}$ suggests that for $D(x, y) = \min\{D_1(x, y), D_2(x, y)\}$ we should look for an example where the two estimates for $D_1(x, z)$ and $D_2(x, z)$ are realized by the minima so combining them does not work.

Here is an example:

Let $M = \mathbb{R}^2$ be the 2-plane, but give it the distance functions, stretched by 2 along the x-axis in the first case, and the y-axis in the second.

$$D_1(\vec{x},\vec{y}) = \sqrt{4(x_1-y_1)^2 + (x_2-y_2)^2} \quad , \quad D_2(\vec{x},\vec{y}) = \sqrt{(x_1-y_1)^2 + 4(x_2-y_2)^2}$$

Consider the triangle inequality for the three points $\vec{x} = (1,0)$, $\vec{y} = (0,0)$ and $\vec{z} = (0,1)$. Then

$$D_1(\vec{x},\vec{y}) = 2 \ , \ D_1(\vec{y},\vec{z}) = 1 \ , \ D_1(\vec{x},\vec{z}) = \sqrt{4+4} = \sqrt{8}$$
$$D_2(\vec{x},\vec{y}) = 1 \ , \ D_2(\vec{y},\vec{z}) = 2 \ , \ D_2(\vec{x},\vec{z}) = \sqrt{4+4} = \sqrt{8}$$

So $D(\vec{x},\vec{z}) = \sqrt{8}$, but $D(\vec{x},\vec{y}) = 1$ and $D(\vec{y},\vec{z}) = 1$, so $\sqrt{8} \not\leq 1+1$. This is not a metric. □

2. [#14, page 71] Let M be a metric space in which the distance function assumes only the values $0, 1, 3$. Define $x \sim y$ to means $D(x,y) \leq 1$. Prove that $\sim$ is an equivalence relation on M. Show also that $\sim$ determines the metric D.

Proof. First we show this is an equivalence relation.

a) For $x \in M$: $D(x,x) = 0 \leq 1 \implies x \sim x$.

b) For $x, y \in M$: $x \sim y \iff D(x,y) \leq 1 \iff D(y,x) \leq 1 \iff y \sim x$

c) For $x, y, z \in M$: $x \sim y$ & $y \sim z \implies D(x,y) \leq 1$ & $D(y,z) \leq 1 \implies D(x,z) \leq 2$. But $D(x,z) \leq 2$ implies that $D(x,z) \leq 1$ as the metric does not take the value 2, hence $x \sim z$.

Conversely, we define D given $\sim$. First, set $D(x,x) = 0$ as this must be true for all metrics. Next, if $x \not\sim y$ then set $D(x,y) = 3$. Finally, if $x \neq y$ but $x \sim y$ then set $D(x,y) = 1$. Then D is non-degenerate, as the only case when $D(x,y) = 0$ is for $x = y$. Symmetry, $D(x,y) = D(y,x)$, follows from symmetry of $\sim$. Finally, to show the Triangle Inequality for a triple of points $x, y, z \in M$, break it down into cases: $x \sim y \sim z$, $x \sim y \not\sim z$ and its permutations, and $x \not\sim y \not\sim z \not\sim z$. Then plug in and check. □

3. [#1, page 74] Let M, D be a metric space. Prove that:

a) For every $x \in M$, the complement $V_x = M - \{x\}$ is open. [Points are closed.]

b) For any set $X \subset M$, then X is the intersection of open sets. [The problem is to find enough open sets. A finite number will not suffice, unless X is itself open.]

Proof. a) We must show that $V_x = M - \{x\}$ is open. Let $y \in V_x$ then $x \neq y$ so $D(x,y) > 0$. Let $R = D(x,y)/2$. Then $x \notin B(y,R)$ by definition, so $B(y,R) \subset V_x$. This shows that every point of V_x contains an open ball neighborhood in V_x which implies that V_x is open.

b) Let $X \subset M$, then we want to find open sets U_α such that $X = \bigcap V_\alpha$.

The hint in class was that for $V_x = M - \{x\}$ and $V_y = M - \{y\}$ then $V_x \cap V_y = M - \{x, y\}$.

So, take the intersection of all open sets V_z for $z \notin X$, to get $X = \bigcap\limits_{z \in M - X} V_z$. □

4. [#2, page 74] Let $x, y \in M$ be distinct points in a metric space M, D. Prove that there exists disjoint open sets $U, V \subset M$ with $x \in U$ and $y \in V$.

Proof. Given $x \neq y$ then $D(x,y) > 0$. Set $R = D(x,y)/2 > 0$. Then $B(x,R) \cap B(y,R) = \emptyset$. □

5. [#5, page 74] Let $M = \mathbb{R}$ be the real line, with the metric $D(x, y) = |x - y|$. Prove that there are no isolated points in $\mathbb{R}$. [A point $x \in M$ is *isolated* if there exists an open set U such that $U \cap M = \{x\}$.]

Proof. Let $x \in \mathbb{R}$ and let $U \subset \mathbb{R}$ be an open set with $x \in U$. Then there exists some $\delta > 0$ such that $B(x, \delta) \subset U$ as U is open. The set $B(x, \delta)$ is just the interval $(x - \delta, x + \delta)$. Consider the open interval $(x, x + \delta)$ then there exists a real number [or rational number, or irrational number] between x and $x + \delta$, so U contains some point besides x. [This also shows that no point in the rational numbers $\mathbb{Q}$ is isolated, and ditto for the irrational numbers.] $\square$

6. [#8, page 74] Let x be a point of as metric space M. Prove that the following two statements are equivalent:

a) x is not isolated.

b) Every neighborhood of x contains an infinite number of points of M.

Proof. First we show $a) \Longrightarrow b)$.

Let $U \subset M$ be a neighborhood of x. Then there exists some $\delta > 0$ such that $B(x, \delta) \subset U$. Assuming that x is not isolated, then the open ball $B(x, \delta)$ contains some point y_1 other than x. So $y_1 \in B(x, \delta) - \{x\}$. Let $\delta_1 = D(x, y_1) > 0$. Then the open ball $D(x, \delta_1)$ is an open neighborhood of X, and $y_1 \notin B(x, \delta_1)$ by the choice of δ_1. So there exists $y_2 \in B(x, \delta_1)$ with $y_2 \neq x$, and also $y_2 \neq y_1$.

This gives an inductive procedure: Assume points $\{y_1, y_2, \ldots, y_n\}$ have been chosen, which satisfy:

(1) $\delta_\ell = D(x, y_\ell) > 0$ for $1 \leq \ell \leq n$
(2) $\delta > \delta_1 > \delta_2 > \cdots > \delta_n$

Then $\{y_1, \ldots, y_n\} \cap B(x, \delta_n) = \emptyset$ and x not isolated implies there exists $y_{n+1} \in B(x, \delta_n)$ with $y_{n+1} \neq x$. Set $\delta_{n+1} = D(x, y_{n+1})$ and the induction proceeds. The collection $\{y_1, y_2, \ldots\} \subset B(x, \delta) \subset U$ is an infinite set of distinct points of M.

Next we show $b) \Longrightarrow a)$.

Let $U \subset M$ be any open neighborhood of x. Then by $b)$ it contains an infinite number of points, so contains some point of M besides x. Thus, x is not isloated. $\square$

7. [#9, page 74] Let M be an infinite metric space. Prove that M contains an open set U such that both U and its complement $M - U$ are infinite.

Proof. We consider two cases.

Suppose that all points of M are isolated. Then every point of M is an open set, so any collection of points if open. Since M is an infinite set, there is a countably infinite subset of distinct points, $\{x_1, x_2, x_3, \ldots\}$. Let $U = \{x_2, x_4, x_6, \ldots\}$ be the subset of all points in the list with even index. This is open, and the complement has an infinite number of points.

The test case gives an idea - look for a countable sequence of disjoint non-empty open balls, instead of just isolated points, then take the union of the even balls in the sequence.

Suppose that M has a non-isolated point $x \in M$. Pick $\delta_0 > 0$, then the construction used above in the proof of Problem 6, $a) \Longrightarrow b)$, gives an infinite subset of distinct points $\{y_1, y_2, \ldots\} \subset B(x, \delta_0)$ whose distances $\delta_\ell = D(x, y_\ell)$ satisfy

$$\delta_0 > \delta_1 > \delta_2 > \cdots > \delta_n > \cdots$$

For each $n \geq 1$, let $\lambda_n = \min\{(\delta_{n-1} - \delta_n), (\delta_n - \delta_{n+1})\}$. This number is chosen so that

$$B(y_n, \lambda_n) \cap \{y_1, y_2, \ldots\} = \{y_n\} \text{ for all } n \geq 1$$

[Draw a picture - it follows by Triangle Inequality.] Each disk $B(y_n, \lambda_n)$ is an open subset of M, so the union of any collection of them is also an open set. Take U to be the union of all the open balls with even index,

$$U = B(y_2, \lambda_2) \cup B(y_4, \lambda_4) \cup \cdots \cup B(y_{2\ell}, \lambda_{2\ell}) \cup \cdots$$

This is open, and the complement contains the infinite set $\{y_1, y_3, \ldots, y_{2\ell-1}, \ldots\}$. $\square$