Math 445, Fall 2009 Exercise Set #4 Turn in: September 30

1. [#1, page 78] Let M be a metric space with metric D. Prove that if $\{x_n \mid n = 1, 2, \ldots\} \subset M$ is a sequence which converges to points $x \in M$ and $y \in M$, then $x = y$.

2. [#4, page 78] Given distinct points x and y in a metric space M, prove that there exist open sets U and V such that $x \in U$, $y \in V$, and their closures $\overline{U} \cap \overline{V} = \emptyset$.

3. [#8, page 79] Let M be a metric space with metric D. Prove that the diameter of a set A in M equals the diameter of its closure, $\overline{A}$.

4. [#11*, page 79] Prove that in a metric space, the closure of a countable set has cardinal number at most c. [Recall that c is the cardinal of the continuum $\mathbb{R}$, which equals the cardinal of the power set of the natural numbers, $\mathcal{P}(\mathbb{N})$.]

5. [#12*, page 79] Prove that the following statements are equivalent for a metric space M:

(a) Every subset of M is either open or closed;
(b) At most one point of M is not isolated.

[Hint: Draw an example of a set with exactly one limit point.]

6. [#13*, page 79] Let M be a metric space in which the closure of every open set is open. Prove that M is discrete. That is, show that every point of M is an open set.

7. [#14*, page 79] Prove that in a metric space, every open set is the union of a countable number of closed sets. Deduce from this that every closed set is the intersection of a countable number of open sets.

8. [#16, page 79] Prove that a metric space is discrete if and only if every convergent sequence is ultimately constant.

9. [#17, page 79] Prove that a metric space is discrete if and only if it has no limit points.

10. [#19*, page 79] If a metric space M has only countably many open sets, prove that M is finite.