

1. [#1, page 78] Let M be a metric space with metric D . Prove that if $\{x_n \mid n = 1, 2, \dots\} \subset M$ is a sequence which converges to points $x \in M$ and $y \in M$, then $x = y$.

Proof. Given $\epsilon > 0$ we will show that $D(x, y) < \epsilon$. Since $D(x, y) \geq 0$, and $D(x, y) > 0$ would contradict $D(x, y) < \epsilon$ for $\epsilon = D(x, y)$, we must have $D(x, y) = 0$, so $x = y$.

Since $x_n \rightarrow x$ there is some integer N_ϵ so that $n \geq N_\epsilon$ implies $D(x_n, x) < \epsilon/2$.

Since $x_n \rightarrow y$ there is some integer M_ϵ so that $n \geq M_\epsilon$ implies $D(x_n, y) < \epsilon/2$.

Let $n \geq \max\{N_\epsilon, M_\epsilon\}$. Then $D(x, y) \leq D(x, x_n) + D(x_n, y) < \epsilon/2 + \epsilon/2 = \epsilon$ □

2. [#4, page 78] Given distinct points x and y in a metric space M , prove that there exist open sets U and V such that $x \in U$, $y \in V$, and their closures $\overline{U} \cap \overline{V} = \emptyset$.

Proof. Let $R = D(x, y)$ and set $\epsilon = R/4$. Let $U = B(x, \epsilon)$ and $V = B(y, \epsilon)$.

The closed ball $\{z \in M \mid D(x, z) \leq \epsilon\}$ is closed, and contains $B(x, \epsilon)$, so its closure satisfies $\overline{B(x, \epsilon)} \subset \{z \in M \mid D(x, z) \leq \epsilon\}$. [They do not have to be equal - give an example.]

Likewise, the closure $\overline{B(y, \epsilon)} \subset \{z \in M \mid D(y, z) \leq \epsilon\}$. Then

$$\overline{B(x, \epsilon)} \cap \overline{B(y, \epsilon)} \subset \{z \in M \mid D(x, z) \leq \epsilon\} \cap \{z \in M \mid D(y, z) \leq \epsilon\} = \emptyset$$

where the two closed balls are disjoint by the Triangle Inequality. □

3. [#8, page 79] Let M be a metric space with metric D . Prove that the diameter of a set A in M equals the diameter of its closure, \overline{A} .

Proof. The diameter of the sets A and \overline{A} are

$$\begin{aligned} \text{diam}(A) &= \sup\{D(x, y) \mid x, y \in A\} \\ \text{diam}(\overline{A}) &= \sup\{D(x, y) \mid x, y \in \overline{A}\} \end{aligned}$$

Since $A \subset \overline{A}$ the inequality $\text{diam}(A) \leq \text{diam}(\overline{A})$ is immediate.

Suppose that $\text{diam}(A) < \text{diam}(\overline{A})$ holds. Choose ϵ so that $0 < \epsilon < \text{diam}(\overline{A}) - \text{diam}(A)$.

Now use the “three epsilon trick”. By definition of the supremum, there exists $u, v \in \overline{A}$ with $D(u, v) > \text{diam}(\overline{A}) - \epsilon/3$. This is the same as $\text{diam}(\overline{A}) < D(u, v) + \epsilon/3$. Then $u \in \overline{A}$ implies there exists $u_* \in A \cap B(u, \epsilon/3)$. Likewise, $v \in \overline{A}$ implies there exists $v_* \in A \cap B(v, \epsilon/3)$.

By the Triangle Inequality,

$$\text{diam}(\overline{A}) < D(u, v) + \epsilon/3 < \epsilon/3 + D(u_*, v_*) + \epsilon/3 + \epsilon/3 \leq \text{diam}(A) + \epsilon < \text{diam}(\overline{A})$$

which is a contradiction. □

4. [#11*, page 79] Prove that in a metric space, the closure of a countable set has cardinal number at most c . [Recall that c is the cardinal of the continuum \mathbb{R} , which equals the cardinal of the power set of the natural numbers, $\mathcal{P}(\mathbb{N})$.]

Proof. For each $u \in \bar{A}$, there exists a convergent sequence $\{x_n \mid n = 1, 2, \dots\} \subset A$ with $x_n \rightarrow u$.

By Problem #1) above, a convergent sequence $\{y_n\} \subset A$ has a unique limit point, so the number of points of \bar{A} is at most the cardinality of the set of convergent sequences in A .

The set of convergent sequences $\{x_n\} \subset A$ is a subset of the set of all sequences in A .

A sequence $\{x_n\}$ can be viewed as a function $F_x: \mathbb{N} \rightarrow A$, where $F_x(n) = x_n \in A$.

The graph of a function $F: \mathbb{N} \rightarrow A$ is a subset of the product $\mathbb{N} \times A$. Thus, the set of all sequences has cardinality bounded above by the cardinality of the power set $\mathcal{P}(\mathbb{N} \times A)$, which has the cardinality c of the continuum. (Both A and \mathbb{N} are countable, and \mathbb{N} is countably infinite.)

Combining these statements, we have that the cardinality of \bar{A} is bounded above by the cardinality of $\mathcal{P}(\mathbb{N} \times A)$; that is, there are at most c points which are the limits of such sequences. \square

5. [#12*, page 79] Prove that the following statements are equivalent for a metric space M :

- (a) Every subset of M is either open or closed;
- (b) At most one point of M is not isolated.

Proof. We first prove that (b) implies (a). If every point of M is isolated, then every point is open, so every subset is a union of open sets, hence is open. [Since every point in a metric space is also closed, this means there are subsets which are both open and closed.]

Next, suppose there is one non-isolated point, say $u \in M$. If $A \subset M$ with $u \notin A$ then A is a union of isolated points, so is open. If $A \subset M$ with $u \in A$ then we claim $A = \bar{A}$ is its own closure, hence A is closed. This follows, since the closure of A is the union of A with its limit points. A limit point is not isolated, so u is the only possible limit point in M . As $u \in A$, it contains all of its limit points, so $A = \bar{A}$.

Next, show the converse, that (a) implies (b). We show this by contradiction. Suppose that M has more than one non-isolated point, say $u \neq v \in M$. Choose a sequence $\{a_n \mid n = 1, 2, \dots\} \subset M - \{u, v\}$ with $a_n \rightarrow u$. We can also assume that each a_n satisfies $D(u, a_n) \leq D(u, v)/2$.

Set $A = \{a_n \mid n = 1, 2, \dots\} \cup \{v\}$. Then A is not closed, as u is a limit point of A , but $u \notin A$. Also, A is not open, as every open ball $B(v, \epsilon) \cap M$ contains a point other than v , since v is a limit point. For all $\epsilon < D(u, v)/2$ we have $B(v, \epsilon) \cap A = \emptyset$. So, $B(v, \epsilon) \not\subset A$. Thus A is not open. \square

6. [#13*, page 79] Let M be a metric space in which the closure of every open set is open. Prove that M is discrete. That is, show that every point of M is an open set.

Proof. First, this condition is very strange, as the closure of an open set U is closed! How can it be open? The closure \bar{U} is also open means its complement is closed, which means that every convergent sequence $\{z_n\}$ in the complement of \bar{U} with $z_n \rightarrow z$, then z must also lie in the complement.

We give a proof by contradiction. Assume that M is not discrete, so there exists a non-isolated point $u \in M$. We will construct an open set U as a union of open balls, such that the closure \bar{U} contains a limit point of the complement of \bar{U} . This will show the complement of \bar{U} is not closed. [Note - the construction below gets complicated, so it is likely there is a simpler example.]

Inductively choose a sequence $A = \{a_n \mid n = 1, 2, \dots\} \subset M - \{u\}$ such that $a_n \rightarrow u$, and so that $D(u, a_n) < \frac{1}{2}D(u, a_{n-1})$. That is, a_n is less than half the distance to u than the previous point a_{n-1} . Since a convergent sequence has a unique limit point by Problem #1) the set A has a unique limit point u , hence $\bar{A} = A \cup \{u\}$.

Set $\epsilon_n = \frac{1}{2}D(u, a_n)$. Then by choice,

$$D(u, a_{n+1}) < \frac{1}{2}D(u, a_n) = \epsilon_n \implies \epsilon_{n+1} < \frac{1}{2}\epsilon_n$$

Also, $D(u, a_{n+1}) < \frac{1}{2}D(u, a_n)$ implies that $D(a_n, a_{n+1}) > \frac{1}{2}D(u, a_n) = \epsilon_n$ by the Triangle Inequality. Since $\epsilon_n < \epsilon_{n-1}$ we also have that $D(a_{n-1}, a_n) > \epsilon_{n-1} > \epsilon_n$. Combining these estimates we get that

$$B(a_n, \epsilon_n/2) \cap A = \{a_n\} \quad \& \quad B(a_m, \epsilon_m/2) \cap B(a_n, \epsilon_n/2) \quad \text{for all } m \neq n \geq 1$$

In other words, by choosing the sequence $\{a_n\}$ to converge to u exponentially fast, we are able to construct a sequence of open metric balls about the points of the sequence which are all disjoint.

Consider the set of points in the sequence with even index, $B = \{a_{2n} \mid n = 1, 2, 3, \dots\}$. Then u is also a limit point of B . Define the open set U to be the union of the open balls around these points with even index:

$$U = B(a_2, \epsilon_2/2) \cup B(a_4, \epsilon_4/2) \cup \dots \cup B(a_{2n}, \epsilon_{2n}/2) \cup \dots$$

The claim is that the closure \bar{U} is not open, or that the complement $M - \bar{U}$ contains a convergent sequence whose limit lies in \bar{U} .

First, note that $u \in \bar{U}$ as $u \in \bar{B}$. Also, u is the limit of the odd terms in the sequence $\{a_n\}$ as well, so we just need to show that $\{a_{2n-1} \mid n = 1, 2, 3, \dots\} \cap \bar{U} = \emptyset$.

Suppose that there exists a sequence $\{b_\ell\} \subset U$ with $b_\ell \rightarrow a_m$. We claim that m must be even. As the sequence converges, there exists ℓ_0 so that $\{b_\ell \mid \ell \geq \ell_0\} \subset B(a_m, \epsilon_m/2)$. But $\{b_\ell\} \subset U$ and $U \cap B(a_n, \epsilon_n/2) = \emptyset$ for n odd implies that m is even. It follows that no odd term a_{2n-1} is a limit point of U . Hence \bar{U} cannot be open. \square

7. [#14*, page 79] Prove that in a metric space, every open set is the union of a countable number of closed sets. Deduce from this that every closed set is the intersection of a countable number of open sets.

Proof. The second part follows from the first by de Morgan's laws.

Let $U \subset M$ be an open set, and $F = M - U$ its complement which is closed. Notice that U open implies that for each $x \in U$ choose $\epsilon_x > 0$ such that $B(x, \epsilon_x) \subset U$. Then for any point $y \in F$, the distance $D(x, y) \geq \epsilon_x$ and so $D(x, F) \geq \epsilon_x$.

For each $n \geq 1$, define

$$V_n = \bigcup_{y \in F} B(y, 1/n)$$

Then each V_n is open as it is a union of open sets, and $F \subset V_n$ for all n .

Finally, note that given any $x \in U$, if n satisfies $1/n < \epsilon_x$ then $x \notin V_n$.

Let $K_n = M - V_n$ be the complement of V_n . Each F_n a closed subset. Since $F \subset V_n$ we have $K_n \subset U$ for each n . Finally, for each $x \in U$ if $n > 1/\epsilon_x$ then $x \in K_n$. Thus, each every point of U is contained in some K_n hence $U = \bigcup_{n=1}^{\infty} K_n$. \square

8. [#16, page 79] Prove that a metric space is discrete if and only if every convergent sequence is ultimately constant.

Proof. The metric space M is discrete means, by definition, that every subset of M is open. In particular, for each point $y \in M$ the singleton set $\{y\}$ is open.

Now suppose that $x_n \rightarrow x$ is a convergent sequence. Then $\{x\}$ is open implies there is some $\epsilon > 0$ such that $B(x, \epsilon) \subset \{x\}$. That is, $\{y \in M \mid D(x, y) < \epsilon\} = \{x\}$. [This means that there is no point of M within distance ϵ of x .] Now $x_n \rightarrow x$ means there exists N_ϵ an integer such that $n \geq N_\epsilon$ implies $D(x, x_n) < \epsilon$. But $x_n \in B(x, \epsilon)$ implies $x_n = x$ as the open ball consist of the singleton, $\{x\}$.

Conversely, suppose that M is not discrete, then there is some point $x \in X$ which is not itself open. That is, x is not isolated. By the Limit Lemma, there exists a sequence $\{x_n\} \subset M$ of distinct points with $x_n \rightarrow x$ and $\epsilon_n = D(x, x_n) > 0$ so that $\epsilon_n \rightarrow 0$ monotonically. Thus, $n \neq m$ implies $x_n \neq x_m$ as their distances from x differ, so the sequence $\{x_n\}$ is not ultimately constant. \square

9. [#17, page 79] Prove that a metric space is discrete if and only if it has no limit points.

Proof. The metric space M is discrete means, by definition, that every subset of M is open. In particular, each point $x \in M$ defines an open set $\{x\}$. Then $\{x\}$ is open implies there is some $\epsilon > 0$ such that $B(x, \epsilon) \subset \{x\}$. That is, $U = \{y \in M \mid D(x, y) < \epsilon\} = \{x\}$. Thus, x is isolated.

Conversely, suppose that there exists $x \in M$ and a sequence $\{x_n\} \subset M - \{x\}$ with $x_n \rightarrow x$. Then for every open set U with $x \in U$ the intersection $U \cap \{x_n\}$ contains an infinite number of points, hence x is not isolated. That is, $\{x\}$ is not an open set. \square

10. [#19*, page 79] If a metric space M has only countably many open sets, prove that M is finite.

Proof. Suppose that M is not finite, then we show there exists uncountably many distinct open subsets. First, if M is discrete, then every subset $A \in \mathcal{P}(M)$ is open, and M infinite implies the cardinality of $\mathcal{P}(M)$ is uncountable.

If M is not discrete, then there exists a limit point $x \in M$. Then by the Limit Lemma, there is a sequence of points $\{x_n\} \subset M - \{x\}$ with $\epsilon_n = D(x, x_n) > 0$ and $\epsilon_1 > \epsilon_2 > \dots > \epsilon_n \rightarrow 0$.

Set $\delta_n = \min\{(\epsilon_n - \epsilon_{n+1})/2, (\epsilon_{n-1} - \epsilon_n)/2\}$. Then by the triangle inequality, for each $n \geq 1$,

$$B(x_n, \delta_n) \cap B(x_{n+1}, \delta_{n+1}) = \emptyset$$

This produces a countably infinite collection $\mathcal{U} = \{B(x_n, \delta_n) \mid n = 1, 2, \dots\}$ of open sets. Each subset of \mathcal{U} is a collection of open balls, so their union if an open subset of M . Distinct subsets give rise to distinct open sets in M , as the open balls are disjoint. The cardinality of $\mathcal{P}(\mathcal{U})$ is uncountable, so this produces an uncountable collection of distinct open subsets of M . \square