The following five problems are all related. Each one builds on the previous ones.

1. [#10, page 78] Let $A, B \subset M$ be subsets of a metric space M. Definethe distance between the sets to be

$$D(A, B) = \inf\{D(a, b) \mid a \in A\ ,\ b \in B\}$$

a) Suppose that $B = \{x\}$ consists of a single point. Prove that $D(A, B) = 0$ if and only if $x \in \overline{A}$.

b) Give an example in the Euclidean plane of two closed subsets, $A, B \subset \mathbb{R}^2$, such that $A \cap B = \emptyset$ and yet $D(A, B) = 0$. [Hint: the sets A and B cannot be bounded.]

2. [#2, page 82] Let $u \in M$ be a fixed point in a metric space M. The function $f(x) = D(u, x)$ maps M into the real numbers, $f \colon M \to \mathbb{R}$. Prove that f is continuous.

3. [#3, page 82] Let $A \subset M$ be a fixed subset of a metric space M. The function $f(x) = D(A, x)$ maps M into the real numbers, $f \colon M \to \mathbb{R}$. Prove that f is continuous.

4. [#4, page 82] Let $A \subset M$ be a *closed* subset and y a point in a metric space M, with $y \notin A$. Prove that there exists a continuous real-valued function on M which vanishes on A but not at y.

5. Let $A \subset M$ be a subset and y a point in a metric space M. Suppose that $y \notin \overline{A}$. Prove that there exists a continuous real-valued function $f \colon M \to [0, \infty)$ which vanishes on A but not at y.