Math 445, Fall 2009 Exercise Set #5 Solutions

1. [#10, page 78] Let $A, B \subset M$ be subsets of a metric space M. Define the distance between the sets to be

$$D(A,B) = \inf\{D(a,b) \mid a \in A\ ,\ b \in B\}$$

a) Suppose that $B = \{x\}$ consists of a single point. Prove that $D(A,B) = 0$ if and only if $x \in \overline{A}$.

b) Give an example in the Euclidean plane of two closed subsets, $A, B \subset \mathbb{R}^2$, such that $A \cap B = \emptyset$ and yet $D(A,B) = 0$. [Hint: the sets A and B cannot be bounded.]

Proof. a) Suppose that $x \in \overline{A}$. For every $\epsilon > 0$ the intersection $B(x,\epsilon) \cap A \neq \emptyset$ so there exists $y \in B(x,\epsilon) \cap A$ with $D(x,y) < \epsilon$. Thus, $D(A,\{x\}) < \epsilon$ for all $\epsilon > 0$, so $D(A,\{x\}) = 0$.

Conversely, suppose that $D(A,\{x\}) = 0$. Then by the definition of the infimum, for every $\epsilon > 0$ there exists some $y \in A$ with $D(y,x) < \epsilon$. This implies $B(x,\epsilon) \cap A \neq \emptyset$ for all $\epsilon > 0$, hence for every open set U with $x \in U$ we have $A \cap U \neq \emptyset$, hence $x \in \overline{A}$.

b) There are lots of ways this can happen; here is one: Let $A = \{(x,0) \mid x \in \mathbb{R}\} \subset \mathbb{R}^2$ be the x-axis. Let $B = \{(x,e^x) \mid x \in \mathbb{R}\}$ be the graph of the exponential function. Both sets are closed in $\mathbb{R}^2$ as their complements are open. [obvious?] For each integer $n \geq 1$, set $a_n = (-n,0) \in A$ and $b_n = (-n,e^{-n}) \in B$. Then $D(a_n,b_n) = e^{-n} \to 0$, so $D(A,B) = 0$. □

2. [#2, page 82] Let $u \in M$ be a point in a metric space M. The function $f(x) = D(u,x)$ maps M into the real numbers, $f\colon M \to \mathbb{R}$. Prove that f is continuous.

Proof. We use the $\epsilon - \delta$ criteria for continuity. For each $x_0 \in M$ then we show $f(x) = D(u,x)$ is continuous at x_0. Let $\epsilon > 0$ be given. Translating the condition for continuity into this case, we need to find $\delta > 0$ such that, if $D(x_0,x) < \delta$ then the distance from u to x is within ϵ of the distance from u to x_0. The claim is that we can take $\delta = \epsilon$ then this follows from the triangle inequality. [A picture might help.] Assume that $D(x_0,x) < \epsilon$ then

$$f(x) = D(u,x) \leq D(u,x_0) + D(x_0,x) < f(x_0) + \epsilon$$

so $f(x) - f(x_0) < \epsilon$. Similarly,

$$f(x_0) = D(u,x_0) \leq D(u,x) + D(x,x_0) < f(x) + \epsilon$$

so $f(x_0) - f(x) < \epsilon$, hence we have $|f(x) - f(x_0)| < \epsilon$. □

3. [#3, page 82] Let $A \subset M$ be a fixed subset of a metric space M. The function $f(x) = D(A,x)$ maps M into the real numbers, $f\colon M \to \mathbb{R}$. Prove that f is continuous.

Proof. Again, we use the $\epsilon - \delta$ criteria for continuity. The difference from the case in Problem 2 above is that the distance to the set A is the infimum over the distances to all points of A. The idea is to pick points $u \in A$ close to x_0 and $v \in A$ close to x, and make the estimates as above, but with some "room for error". This will be provided by using a "two epsilon" trick.

Let $x_0 \in M$ then we show $f(x) = D(A, x)$ is continuous at x_0. Let $\epsilon > 0$ be given, then take $\delta = \epsilon/2$. Let $D(x_0, x) < \epsilon/2$ and choose $u \in A$ so that $D(u, x_0) < D(A, x_0) + \epsilon/2$.

$$\begin{aligned} f(x) = D(A, x) &= \inf\{D(y, x) \mid y \in A\} \\ &\leq D(u, x) \\ &\leq D(u, x_0) + D(x_0, x) \\ &< D(A, x_0) + \epsilon/2 + \epsilon/2 \\ &= f(x_0) + \epsilon \end{aligned}$$

so $f(x) - f(x_0) < \epsilon$. Similarly, choose $v \in A$ so that $D(v, x) < D(A, x) + \epsilon/2$. Then

$$\begin{aligned} f(x_0) = D(A, x_0) &= \inf\{D(y, x_0) \mid y \in A\} \\ &\leq D(v, x_0) \\ &\leq D(v, x) + D(x, x_0) \\ &< D(A, x) + \epsilon/2 + \epsilon/2 \\ &= f(x) + \epsilon \end{aligned}$$

so $f(x_0) - f(x) < \epsilon$, hence we have $|f(x) - f(x_0)| < \epsilon$. ☐

4. [#4, page 82] Let $A \subset M$ be a *closed* subset and y a point in a metric space M, with $y \notin A$. Prove that there exists a continuous real-valued function on M which vanishes on A but not at y.

Proof. Since A is closed, the complement $M - A$ is open, so there exists $\epsilon > 0$ so that the open ball $B(y, \epsilon) \subset M - A$. But this means that for all $u \in A$, the distance $D(u, y) \geq \epsilon$. Hence, $D(A, y) > 0$. Define $f(x) = D(A, x)$, which is continuous by Problem 3, and $f(y) \geq \epsilon > 0$ by the above. The property $f(u) = 0$ for $u \in A$ is obvious. ☐

5. Let $A \subset M$ be a subset and y a point in a metric space M. Suppose that $y \notin \overline{A}$. Prove that there exists a continuous real-valued function $f\colon M \to [0, \infty)$ which vanishes on A but not at y.

Proof. Since $y \in M - \overline{A}$, we can use Problem 4 for the closed set $\overline{A}$ to define a continuous function $f(x) = D(\overline{A}, x)$ which vanishes on $\overline{A}$, so also vanishes on A, but $f(y) > 0$. ☐