

1. [#1, page 93] Prove that the space $X = \mathcal{B}(\mathbb{N}, \mathbb{R})$ of bounded sequences with the “sup norm” is complete. [This is Example 7, page 119 of Appendix 1 in the text.]

The metric is defined by, for $x = \{a_n\} \in \mathcal{B}(\mathbb{N}, \mathbb{R})$ and $y = \{b_n\} \in \mathcal{B}(\mathbb{N}, \mathbb{R})$

$$D(x, y) = \sup_{n \in \mathbb{N}} |a_n - b_n|$$

Proof. A sequence $x = \{a_n \mid n = 1, 2, \dots\}$ can be written in function notation as $x: \mathbb{N} \rightarrow \mathbb{R}$ where $x(n) = a_n$. A sequence of sequences $\{x_i \mid i = 1, 2, \dots\}$ assigns to each index i the sequence $x_i = \{a_n^{(i)} \mid n = 1, 2, \dots\}$.

Let $\{x_i \mid i = 1, 2, \dots\} \subset \mathcal{B}(\mathbb{N}, \mathbb{R})$ be a Cauchy sequence in $\mathcal{B}(\mathbb{N}, \mathbb{R})$. We show that it converges to a sequence x_* and that x_* is bounded, so that $x_* \in \mathcal{B}(\mathbb{N}, \mathbb{R})$.

To define the sequence x_* we need to define $x_*(m)$ for each $m \geq 1$. The claim is that for m fixed, the sequence $\{x_i(m) \mid i = 1, 2, \dots\}$ is Cauchy. Given $\epsilon > 0$, there exists N_ϵ such that $i, j \geq N_\epsilon$ implies $D(x_i, x_j) < \epsilon$. By the definitions, this implies the Cauchy estimate

$$|x_i(m) - x_j(m)| \leq \sup_{n \in \mathbb{N}} |x_i(n) - x_j(n)| = D(x_i, x_j) < \epsilon$$

Set $x_*(m) = \lim_{i \rightarrow \infty} x_i(m)$. This defines the limit sequence $x_* = \{x_*(n) \mid n = 1, 2, \dots\}$.

We next must show that x_* is a bounded sequence. Fix $\epsilon > 0$ say $\epsilon = 1$. Then by the Cauchy condition on $\{x_i\}$ there exists N_1 such that $i, j \geq N_1$ implies $D(x_i, x_j) < 1$. Pick any $i_1 \geq N_1$. Let $C = \|x_{i_1}\| = D(0, x_{i_1}) < \infty$. Then for all $j \geq N_1$ we have $D(x_{i_1}, x_j) < 1$ which implies by the Triangle Inequality that

$$C - 1 = \|x_{i_1}\| - 1 < \|x_j\| < \|x_{i_1}\| + 1 = C + 1$$

Thus, for each m we have $C - 1 < |x_j(m)| < C + 1$ for all $j \geq N_1$. Thus, the limit $x_*(m)$ satisfies $C - 1 \leq x_*(m) \leq C + 1$. This implies $\|x_*\| \leq C + 1$ so $x_* \in \mathcal{B}(\mathbb{N}, \mathbb{R})$. \square

2. [#5, page 93] If every countable closed subset of a metric space M is complete, prove that M is complete.

Proof. First, a closed subset $F \subset M$ of a complete metric space is complete. [Note - we proved this in class, but no problem to show it again.] Let $\{x_n\} \subset F$ be a Cauchy sequence, then M complete implies there is a limit $x_n \rightarrow x_* \in M$. Since F is closed, it contains all of its limit points, so $x_* \in F$. Thus, every Cauchy sequence in F has a limit in F , showing that F is complete.

We show the converse by contradiction: assume that M is not complete, then we construct a countable closed subset $F \subset M$ which is not complete. Let $\{x_n\} \subset M$ be a Cauchy sequence for which there is no limit point. This means that every $y \in M$ is not a limit point of $\{x_n\}$, so for some $\epsilon_y > 0$ the intersection $B(y, \epsilon_y) \cap \{x_n\}$ is a finite set.

Claim: the set $F = \{x_n\}$ is closed. Let $y \in M - F$. Let $\epsilon_y > 0$ be such that the intersection $B(y, \epsilon_y) \cap \{x_n\}$ is a finite set. Since $y \notin F$ the lower bound $\delta_y = \min\{D(y, x_n) \mid n = 1, 2, \dots\} > 0$. Thus, $B(y, \delta_y) \subset M - F$. This shows that $M - F$ is open.

Now F is closed and contains a Cauchy sequence $\{x_n\}$ which has no limit point, so F is not complete, a contradiction. \square

3. [#6, page 93] Let $\{M, d\}$ be a metric space. If for every $u \in M$ and $\epsilon > 0$, the closed ball $D(u, \epsilon) = \{y \in M \mid d(u, y) \leq \epsilon\}$ is complete, prove that $\{M, d\}$ is complete.

Proof. Let $\{x_n\} \subset M$ be a Cauchy sequence. Set $\epsilon = 1$, then by the Cauchy condition there exists N_1 such that $m, n \geq N_1$ implies $D(x_m, x_n) < 1$. In particular, for all $n \geq N_1$ we have $D(x_{N_1}, x_n) < 1$. That is,

$$\{x_n \mid n \geq N_1\} \subset B(x_{N_1}, 1) \subset D(x_{N_1}, 1)$$

It is given that $D(x_{N_1}, 1)$ is complete, so there exists $x_* \in D(x_{N_1}, 1)$ such that $x_n \rightarrow x_*$. But then $x_n \rightarrow x_* \in M$ also, so M is complete. \square

4. [#11, page 93] In the space of real numbers \mathbb{R} , give an example of a descending sequence of non-empty closed sets with empty intersection. That is, find $F_1 \supset F_2 \supset \dots$ where each $F_n \subset \mathbb{R}$ is closed, and $\bigcap_{n=1}^{\infty} F_n = \emptyset$.

Proof. There are lots of examples, but no examples that are *bounded*. For example, let $F_n = [n, \infty)$. The complement $\mathbb{R} - F_n = (-\infty, n)$ is open, so F_n is closed. \square

5. [#12, page 93] The following functions are continuous from the real numbers to the real numbers. Which are *uniformly* continuous?

a) $f(x) = x^2$ b) $g(x) = |x|$ c) $h(x) = \frac{1}{1+x^2}$

Proof. a) $f(x) = x^2$ is not uniformly continuous. Given any $\delta > 0$ and $x > 0$ consider

$$|f(x+\delta) - f(x)| = |(x+\delta)^2 - x^2| = 2x\delta \rightarrow \infty$$

That is, for $\epsilon > 0$ there is no $\delta > 0$ so that $|f(x+\delta) - f(x)| < \epsilon$ for all $x \in \mathbb{R}$.

b) $g(x) = |x|$ is uniformly continuous. Given $\epsilon > 0$ set $\delta = \epsilon$. Then $|x - y| < \delta$ implies (by the Triangle Inequality) that

$$|g(x) - g(y)| = ||x| - |y|| \leq |x - y| < \delta = \epsilon$$

c) $h(x) = \frac{1}{1+x^2}$ is uniformly continuous. Given $x, y \in \mathbb{R}$, consider

$$|h(x) - h(y)| = \left| \frac{1}{1+x^2} - \frac{1}{1+y^2} \right| = \frac{|y^2 - x^2|}{(1+x^2)(1+y^2)} = \frac{|y-x| \cdot |x+y|}{(1+x^2)(1+y^2)}$$

The claim is that there is a finite upper bound

$$C = \sup \left\{ \frac{|x+y|}{(1+x^2)(1+y^2)} \mid x, y \in \mathbb{R} \right\} < \infty$$

Then for $\epsilon > 0$ set $\delta = \epsilon/C$. To show that C is finite is just standard calculus. The quotient is bounded by $|x+y|$ for $|x| \leq 1$ or $|y| \leq 1$. And if $|x| \geq 1$ and $|y| \geq 1$ then the quotient is bounded by $|x+y|/(x^2+y^2) \leq 1$. \square

6. [#13, page 93] Let \mathbb{R} be the real line with the standard metric, $d(x, y) = |x - y|$. Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be a function which has derivative $f'(x)$ for every $x \in \mathbb{R}$. Assume there exists $K \geq 0$ such that $|f'(x)| \leq K$ for all $x \in \mathbb{R}$. Prove that f is uniformly continuous. [Hint: Mean Value Theorem]

Proof. Let $x < y \in \mathbb{R}$ then the Mean Value Theorem states there exists $x < \xi < y$ for which $f(y) - f(x) = f'(\xi) \cdot (y - x)$. Then

$$|f(y) - f(x)| = |f'(\xi) \cdot (y - x)| \leq K \cdot |y - x|$$

Given $\epsilon > 0$ set $\delta = \epsilon/K$. \square